PRL - Private Soc.
Deposited

SHL
WITHDRAWN

PROCEEDINGS

OF

THE LONDON MATHEMATICAL SOCIETY

SECOND SERIES

VOLUME 36

LONDON:
PRINTED AND PUBLISHED FOR THE SOCIETY BY
C. F. HODGSON & SON, LTD.,
2 NEWTON STREET, KINGSWAY, W.C.2

1934

BIBL.
LONDIN.
UNIV.

CONTENTS.

Papers published in the *Proceedings*, pp. 1–551.

Index of Papers published in this Volume arranged according to the names of Authors, p. 552.

Index according to Subjects, p. 553.

Addenda and Corrigenda, p, 554,

Digitized by the Internet Archive
in 2026

https://archive.org/details/london-mathematical-society-proceedings_1934_36

PAPERS

PUBLISHED IN THE

PROCEEDINGS OF THE LONDON MATHEMATICAL SOCIETY

THE FREEDOMS OF DETERMINANTAL MANIFOLDS

By T. G. Room.

[Received 3 March, 1932.—Read 10 March, 1932.]

The object of this paper is to establish a formula for the freedom of a determinantal manifold, that is of a manifold represented by the vanishing of all the determinants of p rows in a matrix of p columns and q rows $(p \leqslant q)$, whose elements are linear functions of the coordinates in $[n]$. It is found that, with certain well defined exceptions, the freedom of such a manifold is $pq(n+1)-p^2-q^2+1$.

It is well known that determinantal manifolds can be generated by projectively related star systems, and from this point of view they have been studied during the past fifty or more years*. An excellent summary of some of the properties of a general manifold of the type has been given by Segre†, and since the publication of this summary various new results have been found, mostly for manifolds defined by particular values of p, q, and n‡. More is known now, however, than has been published, and no references could be found for many of the theorems required in the proof of the results with which this paper is concerned. It was thought desirable, therefore, to preface the discussion of these results with a general account of those parts of the theory of determinantal

* The two classical expositions are Schur, **1**, and Veronese, **2**.

† Segre, **3**.

‡ White, **4** (a rational sextic surface in [4], $p = 3$, $q = 4$). White, **5** (a rational surface in $[n]$ represented by plane curves of order n, $p = 3$, $q = n$). L. Roth, **6.1** (a quartic primal in [4], polo-form with regard to four quadrics). L. Roth, **6.2** (a quartic primal in [4] represented by a symmetrical determinant). L. Roth, **7** (a birational transformation of [4], in which corresponding points are conjugate with regard to four quadrics). L. Roth, **8** (a generalisation of **6.2** and **7** to $[n]$). L. E. Dickson, **11** (an upper limit to the freedom of a determinantal primal).

manifolds which are wanted to establish them*. In doing this, the author was faced with the difficulty that no recognised notation exists for expressing concisely the entity dual to a "space", that is, there is no symbol for the linear ∞^r family of primes dual to the linear ∞^r family of points expressed by the symbol $[r]$, and at the beginning of this paper, therefore, a notation is suggested to remedy this deficiency.

The paper consists thus of three parts:

(1) A suggested notation for dual entities.

(2) Some properties of determinantal manifolds.

(3) The freedoms of determinantal manifolds.

Throughout the paper it is assumed that the two numbers p and q satisfy the inequality $p \leqslant q$.

1. *A notation for dual entities.*

1.1. *The symbols* $[n]$ *and* $]n[$.

$[n]$, *to be read as "n-space".*

The symbol $[n]$ has the customary meaning of an ∞^n family of points linearly dependent on $n+1$ of their number, and all the lines, planes, solids, [4]'s, ..., secunda ($[n-2]$'s) and primes defined by them. If $A_0, \ldots, A_n$ are the symbols of any $n+1$ of the points, the symbol for a general point of the $[n]$ is $\sum_{\epsilon=0}^{n} x_\epsilon A_\epsilon$, and, referred to $A_0 \ldots A_n$ as simplex of reference, the point coordinates of the point are $(x_0, \ldots, x_n)$. The point equation of a general prime is

$$\sum_{\epsilon} P_\epsilon x_\epsilon = 0.$$

$]n[$, *to be read as "n-star".*

The dual of a $[n]$, which we denote by $]n[$†, is an ∞^n family of primes, linearly dependent on $n+1$ of their number, together with the stars, or spaces, defined by these primes. But whereas the symbol $[n]$ conveys a quite definite geometrical meaning independent of the space (of

* These manifolds have been one of the subjects of discussion among the geometrical circle in Cambridge for some time past, and it may be that in this account results due to members of the circle (particularly to Messrs. Du Val and Todd) have been used without acknowledgment. To such members the author wishes to take this opportunity of offering his apologies.

† The sign $\langle n \rangle$ has also been proposed, but the sign $]n[$ seems preferable, since, although its appearance is somewhat awkward, the brackets appearing not to enclose the "n" but the text on either side, there is suggested by it a family of entities radiating from some centre. The symbol $]1[$, for example, contains some suggestion of the usual blackboard representation of a pencil of rays in a plane.

higher dimension) in which the space of n dimensions lies, to obtain a definite geometrical idea of a $]n[$ it is necessary to specify the point-dimension of the space in which the star lies. Thus a $]1[$ is a "pencil", but whether of $[n-1]$'s through a $[n-2]$ in $[n]$, of lines through a point in a plane, or even of points on a line (the line itself being the containing space), is not conveyed by the symbol. If $a_0, \ldots, a_n$ are the symbols of $n+1$ independent primes, then the symbol of any prime of the $]n[$ defined by them is of the form $\sum_\epsilon X_\epsilon a_\epsilon$; if the containing space is a $[n]$, so that the $n+1$ primes $a_0, \ldots, a_n$ can be used for the simplex of reference, then $(X_0, \ldots, X_n)$ are the prime coordinates of the prime in that coordinate system*. A $]n[$ in $[n]$ consists of all the primes, pencils, sheaves, ... (*i.e.* all the $]0[$'s, $]1[$'s, $]2[$'s, ...) of the $[n]$. If $n' > n$, a $]n[$ of $[n']$ consists of all the primes which pass through a certain $[n'-n-1]$ together with all the pencils, etc., defined by these primes. The $[n'-n-1]$ will be called the base of the star. A $]n[$ of $[n']$ is the system obtained by joining a $]n[$ in $[n]$ to a $[n'-n-1]$ outside the $[n]$.

It should be noticed that in effect we are using the symbol $[n'-n-1]$ here with a meaning rather different from that assigned in the definition. The difference is illustrated by the rôles that it is possible for a line to assume in [3]. With the definitions above, a [1] in [3] is a "range" (of points on a line in [3]), and a $]1[$ in [3] is a "pencil" (of planes through a line in [3]). Neither of the symbols [1] or $]1[$ strictly stands for the line itself as an entity. In the same way the symbol $[r]$ has been used above for the space itself, and not, as is required by the definition, for the system of points composing the space. In practice, however, the symbol $[r]$ is rarely ambiguous, and in doubtful cases, when it is required to refer to a space of r dimensions, without its being thought of as a system of points, ambiguity can be avoided by calling it the "base of a $[r]$", or, more briefly, a "base-$[r]$"—as in the example the line is the base of a range†.

If α is the symbol for a base-$[r]$ in a base-$[n]$, we denote by (α) the $[r]$ of which it is the base, and by $)\alpha($ the $]n-r-1[$ of which it is the base.

1.2. *Projection and section.*

As ordinarily defined, the operations of projection and section are not strict duals; in section only one "base" is involved (that by which the

* "Prime coordinates" seems preferable to the more usual "tangential coordinates".

† On these grounds, the following terminology might be adopted: (i) *space* = a linear point system corresponding to the symbol $[.]$; (ii) *star* = a linear prime system corresponding to $].[$; (iii) *base* = the base of a space or star. But for the present purpose, at least, the disadvantages of introducing a new term (and changing the meaning of an old one) outweigh the slight gain in conciseness, and, except in § 1.2, "space" is used for both (i) and (iii).

section is made), while in projection there are the vertex of projection and the "base" on to which the projection is made. The operation of projection consists, in fact, of joining the elements concerned to the vertex of projection (in itself the operation dual to section), and then taking the section by the "base" on which the projection is made. The second part of the operation is only a matter of geometrical convenience, since the choice of the "base" has no effect on the resulting figure.

If the operations are expressed in terms of the dual elements $[r]$ and $]r[$ the discrepancy disappears, since we have the two precisely dual statements: the section of a $[n]$ in a $[n']$ by a general $[m]$ is a $[n+m-n']$*, and the projection of a $]n[$ in a $]n'[$ from a general $]m[$ is a $]n+m-n'[$. On the other hand, the section of a $]n[$ by a general base-$[m]$ is just a $]n[$ in that "base" (we are, in fact, merely defining the "base" in which the $]n[$ lies) and the projection of a $[n]$ in $[n']$ from a general base-$]m[$ is a $[n]$ in a base-$[m]$.

1.3. *Degenerate spaces and their relation.*

A $[n]$ has been defined as the aggregate of points whose symbols are $\sum_{\epsilon=0}^{n} x_\epsilon A_\epsilon$, where the A_ϵ are some given $n+1$ points. It has been assumed tacitly that these $n+1$ points are independent, *i.e.* that it is not possible to express the symbol of any of them linearly in terms of the symbols of the rest; if this condition is not satisfied, we shall call the space "degenerate". A r-ply degenerate $[n]$ consists in effect of all the points of a $[n-r]$ each counted ∞^r times.

Consider now the projective relation of a degenerate $[n]$ to another $[n]$ which may or may not be degenerate. For an understanding of the geometry of such a relation it is best to suppose that the $[n]$'s have been derived by projection from non-degenerate $[n]$'s in some space of higher dimension. Let α and β be two related non-degenerate $[n]$'s in $[m]$, and let α' and β' be their projections from a $[t-1]$ H which lies in α, but does not meet β. α' is then t-ply degenerate and β' is non-degenerate. In the relation between α and β let V of β correspond to H of α, and let V project into V' of β'; then to a point P' of α', which is the projection of any point of a $[t]$, $[P, H]$, of α, correspond all the points of a $[t]$, $[Q', V']$, of β'. The correspondence is thus between points of the degenerate $[n]\alpha'$, and $[t]$'s of the $]n-t[$, $)V'($, of β', while to a point of V' in β' corresponds any point whatsoever of α'.

* We suppose here that $n+m \geqslant n'$.

If H lies in both α and β, α' and β' are both t-ply degenerate. Let U', V' be the projections of the $[t-1]$'s U, V of α and β which correspond to H considered as a space in β and α respectively; then the relation between α' and β' is such that the two $]n-2t[$'s, $)U'($ and $)V'($, are related, and to a point P' of α' there corresponds any point of the $[t]$ in β' that corresponds to $[P', U']$ of α'. If $n < 2t$, the relation loses significance altogether, and there is no actual correspondence between the points of the two spaces at all.

The relation between α and β can always be reduced to the form in which a point $\Sigma\lambda_\epsilon A_\epsilon$ of α corresponds to $\Sigma\lambda_\epsilon B_\epsilon$ of β. Corresponding points of α' and β' are then represented by the same symbols, but there are in addition t syzygies connecting each set of base points, obtained by making vanish the symbols for some t independent points of the vertex of projection H. New base systems can be selected in α' and β', each composed of $n-t+1$ independent points, in such a way that points which correspond in the relation are

$$\lambda_0' A_0' + \ldots + \lambda_t' A_t' + \ldots + \lambda_{n-t}' A_{n-t}'$$

$$\lambda_t' B_t' + \ldots + \lambda_{n-t}' B_{n-t}' + \ldots + \lambda_n' B_n',$$

provided that $n \geqslant 2t$. In the notation above, U' is $[A_0' \ldots A_{t-1}']$ and V' is $[B_{n-t+1}' \ldots B_n']$. If $n \leqslant 2t$, the sets of parameters λ in the symbols for the two points do not overlap, and any point of one space corresponds to all points of the other.

2. *Determinantal manifolds.*

2.1. *The manifolds* $(|p, q|, [n])$, *and* $)|p, q|, [n]($.

We denote by $(|p, q|, [n])$ the manifold in $[n]$ represented by the vanishing of p-rowed minors in a matrix of p columns and q rows $(p \leqslant q)$ whose elements are linear functions of the coordinates, *i.e.* represented by the equations

$$||x_{\alpha\beta}|| = 0 \quad \left\{ \begin{array}{l} \alpha = 1, \ldots, p \text{ (along each row)}, \\ \beta = 1, \ldots, q \text{ (down each column)}, \end{array} \right\} \qquad \text{(I)}$$

where

$$x_{\alpha\beta} = \sum_\epsilon a_{\alpha\beta\epsilon} x_\epsilon \quad (\epsilon = 0, 1, \ldots, n), \qquad \text{(II)}$$

the x_ϵ being the coordinates in the $[n]$. Further, we denote by $(|p, q|_r, [n])$ the manifold represented by the equations

$$||x_{\alpha\beta}||_r = 0, \qquad \text{(III)}$$

where the suffix r means that all determinants of $r+1$ rows are to vanish,

while not all those of r rows vanish. When $p \leqslant q$, $(|p, q|, [n])$ is just $(|p, q|_{p-1}, [n])$*.

The equations of $(|p, q|, [n])$ are the result of eliminating the parameters λ from among the q equations

$$\sum_{\alpha} \lambda_\alpha x_{\alpha\beta} = 0. \qquad \text{(IV)}$$

If we suppose that $p \leqslant n$, the p equations $x_{1j} = x_{2j} = \ldots = x_{pj} = 0$ represent a base-$[n-p]$, and each set of parameters corresponds to a prime of the $]p-1[$ of which it is the base. Each set of values of λ gives a set of q primes, one prime for each number β, whose intersection, as the sets of values are varied, generates the manifold; thus $(|p, q|, [n])$ is the locus of meets of corresponding primes of q related $]p-1[$'s.

Since the section of a $]p-1[$ by any space is a $]p-1[$, and a relation between two stars is undisturbed by taking a section, the manifolds $(|p, q|, [n])$, as n varies, form a series in which each is a section of those with higher values of n; and any member of the series can be defined as the locus of meets of corresponding primes of q related $]p-1[$'s in $[n]$, whether or not $p \leqslant n$, that is, whether the stars have proper bases or are degenerate. If $n \geqslant q$, $(|p, q|, [n])$ consists of ∞^{p-1} $[n-q]$'s, so that the total dimension of the manifold is $n-q+p-1$; if $n < q$, not every set of corresponding primes meets, but only those which satisfy a certain $q-n$ conditions, that is, of the ∞^{p-1} sets, $\infty^{p-1-(q-n)}$ sets meet in points. Thus the dimension of $(|p, q|, [n])$ is in any case $n-q+p-1$†. We suppose always in what follows that n is great enough for this not to be negative.

For the dual in $[n]$ of $(|p, q|, [n])$ we use the symbol $)|p, q|, [n](.$ This manifold is defined first as that which has prime equations of the same form as the point equations I and II, and secondly as the locus of the ∞^{p-1} $[q-1]$'s that join corresponding points of q related $[p-1]$'s in $[n]$. If $n' < n$, $)|p, q|, [n'](.$ is the projection from a general base-$]n'[$ of $)|p, q|, [n](.$ In their geometrical definitions $(|p, q|, [n])$ and $)|p, q|, [n](.$ are not quite strict duals. Take, for example, $)|2, 2|, [3](.$, the locus of the joins of corresponding points of two related ranges in $[3]$, *i.e.* a quadric surface. If we consider this as an ∞^2 family of points (*i.e.* consider the joins as $[1]$'s and not merely as lines), the dual manifold consists of an ∞^2 family of planes (actually the tangent planes to a quadric), which belong to ∞^1 pencils whose bases are the meets of corresponding planes of two related pencils. That is, the meets of corresponding planes should be thought of not merely as lines, but as $]1[$'s.

* p, q, and r are used here where Segre, **3**, has $p+1$, $q+1$, and $p-h+1$. In Room, **9**, the symbol $(|p, q|_r, [n])$ is written in the simpler but less expressive form $(p, q, r, [n])$.

† In the same way the dimension of $(|p, q|_r, [n])$ is $n-(p-r)(q-r)$.

The two definitions can be brought into agreement in the general case in this way: we take $(|p, q|, [n])$ to consist of the whole ∞^{p+q-2} family of primes whose point equations are $\sum_{\alpha,\beta} \lambda_\alpha \mu_\beta x_{\alpha\beta} = 0$, these being arranged in ∞^{p-1} sets of ∞^{q-1}, $\sum_\beta \mu_\beta (\sum_\alpha \lambda_\alpha x_{\alpha\beta})$, each set forming a $]q-1[$. The locus of the bases of these $]q-1[$'s is $(|p, q|, [n])$ as originally defined. Dually, $)|p, q|, [n]($ consists of ∞^{p+q-2} points whose prime equations are $\Sigma \Lambda_\alpha M_\beta X_{\alpha\beta} = 0$; these are arranged in ∞^{p-1} sets of ∞^{q-1}, each set forming a $[q-1]$. The point dimension of $)|p, q|, [n]($ is $p+q-2$ whatever n may be, and the prime dimension of $(|p, q|, [n])$ is the same. In practice, however, we regard both as point manifolds, the dimension of the second being $n-q+p-1$.

2.2. *The two systems of stars generating* $(|p, q|, [n])$.

We have so far defined $(|p, q|, [n])$, which we shall in future denote by D, by a certain set of q related $]p-1[$'s, whose corresponding primes are represented by the equations

$$\sum_\alpha \lambda_\alpha x_{\alpha\beta} = 0; \qquad \text{(III)}$$

but equally

$$\sum_\alpha \lambda_\alpha (\sum_\beta m_\beta x_{\alpha\beta}) = 0$$

represents, for fixed m and varying λ, a $]p-1[$ related to the given q, and such that it may be used instead of any of the given q in the generation of D. Now group the terms in the last equation differently:

$$\sum_\beta \mu_\beta (\sum_\beta l_\alpha x_{\alpha\beta}) = 0$$

represents, for fixed l and varying μ, a $]q-1[$ whose base is one of the family of $[n-q]$'s of which D is composed*. This is one of an ∞^{p-1} family of stars, of which p independent members are

$$\sum_\beta \mu_\beta x_{\alpha\beta} = 0.$$

Corresponding primes of these $]q-1[$'s meet in the $[n-p]$'s which are the bases of the original family of $]p-1[$'s. These $[n-p]$'s do not lie on D (unless $p = q$), but each of them cuts D in a manifold $(|p, q-1|, [n-p])$ of dimension $n-q$, *i.e.* one greater than the dimension of the section of D by a general $[n-p]$. On the other hand, the result of eliminating the

* We suppose, for convenience, that $q \leqslant n$; the argument is easily adapted to other cases, but it is not capable of such simple geometrical form.

parameters $\mu_{\beta\gamma}$ from the $p(q-p+1)$ equations

$$\sum_{\beta} \mu_{\beta\gamma} x_{\alpha\beta} = 0 \quad (\gamma = 1, \ldots, q-p+1)$$

is just the matrix equation

$$||x_{\alpha\beta}||_{p-1} = 0. \tag{I}$$

Thus D is also the locus of the meets of corresponding $[n-q+p-1]$'s of p related $]q-1[$'s.

The ∞^{p+q-2} family of primes which form between them the ∞^{q-1} $]p-1[$'s is the same as that which forms the $\infty^{p-1}]q-1[$'s, and, since the dual of this family of primes is just the points which form $)|p, q|, [n]($, say $\mathfrak{D}$, we are able to express the two methods of generating $\mathfrak{D}$ in a more symmetrical manner than is possible for D. Thus, $\mathfrak{D}$ is the locus of the joins of corresponding points of q related $[p-1]$'s, or of corresponding points of p related $[q-1]$'s, the joins in the one case providing the bases of the point systems in the other.

2.3. *The key manifold of a series,* $(|p, q|, [pq-1])$.

When $n = pq-1$ the pq primes whose point equations are $x_{\alpha\beta} = 0$ may themselves be used for the simplex of reference, and the properties of this particular manifold of a series, which we denote by K, are in consequence very easy to obtain. The peculiarity of the dual manifold $)|p, q|, [pq-1]($, or $\mathfrak{K}$, is that it is normal*. For the manifolds $)|p, q|, [n]($ form a series in which each is obtainable as a general projection of other members of the series, and each is defined by a relation among q $[p-1]$'s. These $[p-1]$'s lie in a space of dimension $pq-1$ at most, and this is, therefore, the normal space of the series.

Just as the quadric surface, the key manifold of the $)|2, 2|, [n]($ series, may be defined first to be the locus of joins of corresponding points of two related ranges, and proved thence to be the locus of transversals of three skew lines in [3], so $)|p, q|, [pq-1]($, which has been defined as the locus of joins of q related $[p-1]$'s, is also the locus of transversal $[q-1]$'s to $q+1$ general $[p-1]$'s in $[pq-1]$.

It is easy to see that $(|p, q|_1, [pq-1])$ is the same manifold as $\mathfrak{K}$, and it follows that the locus of $[q(p-1)-1]$'s which join sets of $p-1$ of the $[q-1]$'s which generate $\mathfrak{K}$, that is, which join sets of corresponding primes ($[p-2]$'s) of the related $[p-1]$'s, is a general manifold K. Further,

* A manifold in $[n]$ is normal if there is no manifold of the same dimension and order in $[n+1]$ of which it is a projection.

the $[pq-p-1]$'s which join sets of $q-1$ of the $[p-1]$-generators of $\mathfrak{K}$ are the bases of the $]p-1[$'s which generate K*.

2.4. *Spaces on* $(|p, q|, [n])$.

If $n \geqslant pq$, the pq primes $x_{\alpha\beta} = 0$ from which $(|p, q|, [n])$ is constructed have a common $[n-pq]$, and the manifold is a cone consisting of the join of this space to a key manifold $(|p, q|, [pq-1])$†. Thus, of the members of the series of manifolds $(|p, q|, [n])$ for varying n, only those for which $n < pq$ need be considered.

$(|p, q|, [n])$, or D, is the $[n]$-section of K, and consequently any space on it is the section of some space on K. But the families of spaces on K are easily enumerated; the join of any $p-i-1$ independent $[q-1]$-generators of $\mathfrak{K}$ to any i independent $[p-1]$-generators is a space which lies on K, and any space on K must be obtainable in this way, or must lie inside a space of this type. We denote any space of this family by h_i; h_0 then represents the join of $p-1$ $[q-1]$-generators of $\mathfrak{K}$, *i.e.* the meet of a set of corresponding primes of the $]p-1[$'s which generate K. The

* Segre, **3**, 828; also Room, **9**, 208. Further to a remark in the latter paper: "there exist quadrics with respect to which the $[q-1]$'s of a manifold $\mathfrak{K}$ are the polars of the $[pq-q-1]$'s of the manifold K whose equations belong to the same matrix". We can take the prime equations of a $[q-1]$ of $\mathfrak{K}$ to be

$$\sum_{\alpha} \Lambda_\alpha X_{\alpha\beta} = 0$$

and, with the same coordinate system, the point equations of a $[pq-q-1]$ of K to be

$$\sum_{\alpha} \lambda_\alpha x_{\alpha\beta} = 0.$$

It is not difficult to see that the members of these two families of spaces can be paired in such a way that they are mates in any correlation of the family

$$\rho X_{ij} = \sum_{\alpha\beta} c_{i\alpha} d_{j\beta} x_{\alpha\beta} \quad (\alpha, i = 1, \ldots, p; \ \beta, j = 1, \ldots, q).$$

This family of correlations depends on the p^2 parameters c and the q^2 parameters d, and thus has freedom p^2+q^2-2. Included in it is a family of polarities with respect to quadrics, the conditions that a correlation should be such a polarity being

$$c_{i\alpha} = c_{\alpha i} \quad \text{and} \quad d_{j\beta} = d_{\beta j}.$$

The freedom of this family of quadrics is $\frac{1}{2}p(p+1)+\frac{1}{2}q(q+1)-2$. There are also two families of polarities with respect to null systems, one of freedom $\frac{1}{2}p(p-1)+\frac{1}{2}q(q+1)-2$, and the other of freedom $\frac{1}{2}p(p+1)+\frac{1}{2}q(q-1)-2$. The conditions satisfied by the constants in the first case are

$$\left.\begin{aligned} c_{\alpha i} + c_{i\alpha} &= 0 \quad \text{including} \quad c_{ii} = 0 \\ d_{\beta j} - d_{j\beta} &= 0 \end{aligned}\right\},$$

and in the second

$$\left.\begin{aligned} c_{\alpha i} - c_{i\alpha} &= 0 \\ d_{\beta j} + d_{j\beta} &= 0 \quad \text{including} \quad d_{jj} = 0 \end{aligned}\right\}.$$

† The equivalent of this for the dual manifold has already been remarked, namely that $)|p, q|, [n]($ is normal in $[pq-1]$.

dimension of the spaces h_i is $m = q(p-i-1)+i(i+1)-1$, and the freedom of these $[m]$'s on K

$$= \text{the freedom of } [p-i-2]\text{'s in } [p-1]$$

$$\text{plus the freedom of } [i-1]\text{'s in } [q-1]$$

$$= (p-i-1)(i+1)+i(q-i).$$

Any space on D is the section of some h_i on K, and the freedom of spaces of a particular dimension m' and type can easily be found by using the usual formulae for the number of conditions imposed on a $[m]$ in $[pq-1]$ in order that it shall meet a $[n]$ in a $[m']$. One result of particular interest is that the spaces of highest dimension on D are derived from spaces h_0 on K, and that, if there are other spaces of the same dimension on D, they have smaller freedom than those derived from h_0, except

(i) when $p = q$: sections of h_0 and h_{p-1} which are two equivalent systems on K always have the same dimension and freedom on D, a rather trivial case; and

(ii) when $p = q = n$: of the spaces of any of the families h_i on K, ∞^{p-3} are cut by any $[q]$ in lines*.

Taking $(|4, 4|, [n])$ as an example, we find that there are on the key manifold, $n = 15$, two ∞^3 families of $[11]$'s, h_0 and h_3, and two ∞^7 families of $[9]$'s, h_1 and h_2. ∞^1 members of each of these four families are cut by any $[4]$ in lines, *i.e.* there are on $(|4, 4|, [4])$ four ∞^1 families of lines equivalent in pairs. Similarly on the cubic surface in $[3]$, $(|3, 3|, [3])$,

* *Cf.* Roth, **6.1**. The general result about the n systems of ∞^{n-3} lines on $(|n, n|, [n])$ is due to Roth. His paper is mainly concerned with a special case of $(|4, 4|, [4])$, although his results are true for a general manifold of that type. Roth generates the manifold $(|n, n|, [n])$ as a polar form with respect to ∞^{n-1} quadrics, which is equivalent to having the coefficients in the equations

$$x_{\alpha\beta} = \sum_{\epsilon} a_{\alpha\beta\epsilon} x_{\epsilon} \quad (\alpha, \beta = 1, \ldots, n;\ \epsilon = 0, \ldots, n) \qquad \text{(II)}$$

restricted by the conditions $a_{\alpha\beta\epsilon} = a_{\alpha\epsilon\beta}$ $(\beta, \epsilon = 1, \ldots, n)$.

In the case of the cubic surface $n = 3$, these conditions can always be satisfied, but this is not so for higher values of n, as is well known, and as can easily be seen by working out the freedoms of the manifold generated by polarities, and of the "general" manifold.

$$\text{The freedom of the first} \leqslant \text{the freedom of } \infty^{n-1} \text{ system of quadrics in } [n]$$

$$+ \text{the freedom of a prime in } [n]$$

$$= \tfrac{1}{2}n(n^2+n+4),$$

while that of the second (§3.4) is $n^3 - n^2 + 1$. In **6.2** (and **7**) Roth discusses the quartic determinantal primal in $[4]$ for which $a_{\alpha\beta\epsilon} = a_{\beta\alpha\epsilon}$.

there are three finite families of lines, a and b (in the usual notation) being derived from h_0 and h_2, which are ∞^2 families of [5]'s in [8], and c from h_1, which is an ∞^4 family of [4]'s in [8]. The cubic surface has a property not shared by any other determinantal manifold, namely that the whole family of lines on it may be split up into sets a, b, and c in several ways (thirty-six in fact). $(|4, 4|, [4])$ illustrates the position in regard to other manifolds; on it the two pairs of families of lines are quite distinct, one forming surfaces of order 20, and the other surfaces of order 140. To this property is due the further peculiarity that a cubic surface is capable of seventy-two independent projective generations, while no other determinantal locus, except a plane curve, is capable of more than two (§ 3 . 4).

In the same way we may discuss spaces having exceptional intersections with D, all of which must arise as sections of spaces specially related to K. To obtain these special spaces we join $q-j-1$ independent $[q-1]$-generators of $\mathfrak{K}$ to j independent $[p-1]$-generators. In particular, if $n > p+\alpha$, we find that the only $[n-p+\alpha]$'s cutting $(|p, q|, [n])$ in manifolds $(|p, q-1|, [n-p+\alpha])$ are the sections of the $[pq-p-1]$'s which join sets of $q-1$ of the $[p-1]$-generators, *i.e.* the bases of the $]p-1[$'s which generate the manifold*.

2 . 5. *The normal space of* $(|p, q|, [n])$.

It is to be proved that

$(|p, q|, [n])$ *is normal, provided that* $(p \leqslant) q \leqslant n$†.

If the general prime section of a manifold is normal, then the manifold itself must be normal, so that it will be sufficient to prove the theorem

* If a $[n]$ cuts the base of a $]p-1[$ in a base-$[n-p+\alpha]$, it cuts the $]p-1[$ itself in a "degenerate" $]p-1[$, which consists of all the primes of what is really a $]p-\alpha-1[$ each counted ∞^α times.

† The theorem is also true for values of $n < q$, but, except for the case proved in § 3 . 7 (mentioned below), we do not need the result in the subsequent work. As examples of these additional values in which the theorem is true we have:

(i) $(|q, q|, [2])$ is a general plane curve, and therefore normal in the plane; since it is the general plane section of $(|q, q|, [n])$, it follows that the primal $(|q, q|, [n])$ is normal in $[n]$ provided that $n \geqslant 2$.

(ii) The surface $(|q-1, q|, [4])$ is without double points, and is therefore normal, and thus all manifolds $(|q-1, q|, [n])$ are normal for $n \geqslant 4$.

(iii) It is proved in § 3 . 7 that the curve $(|3, q|, [q-1])$ is normal.

It seems probable that all manifolds $(|p, q|, [n])$ are normal with the trivial exceptions $n = q-p+1$ (when the manifold reduces to a finite number of points), and $p = 1$ (in which case the manifold is just a $[n-q]$ of $[n]$).

in the case $q = n$. In two cases the theorem is already known to be true:

(i) $(|2, q|, [q])$ is the normal rational curve in $[q]$,

(ii) $(|3, q|, [q])$ is a rational surface represented by all plane curves of order q through $\frac{1}{2}q(q+1)$ fixed points, and is therefore normal*.

We prove the general theorem by induction. Suppose that

$$D'' \equiv (|p-2, q|, [q]) \quad \text{and} \quad D' \equiv (|p-1, q|, [q])$$

are normal in $[q]$, and that $D \equiv (|p, q|, [q])$ can be obtained as the projection of some manifold Δ in $[q+1]$. D is generated by the corresponding $[p-1]$'s of an ∞^{p-1} family of related $]q-1[$'s and D' by the corresponding $[p-2]$'s of an ∞^{p-2} family of related $]q-1[$'s; there is thus on D a family of manifolds D' of freedom $p-1$ at least†. Any two of these manifolds D' have common a manifold D''. A general manifold of the family D'' does not lie in a space of less than q dimensions. Any manifold on Δ in $[q+1]$ which is to project into a manifold D' on D, since D' is normal, must lie in a $[q]$. Any two of these manifolds on Δ therefore meet in a manifold which lies in $[q-1]$, unless Δ lies wholly in a $[q]$ of the $[q+1]$. But this intersection is to project into a manifold D'', which is not contained in a space of less than q dimensions. Thus D is normal in $[q]$.

It may be proved similarly (and with probably a similar extension of the range of values of n for which the theorem is true) that

$$(|p, q|_1, [n]) \textit{ is normal if } n \geqslant q(p-1),$$

the cases that form the basis of the induction being

(i) $(|2, q|_1, [q])$, the normal rational curve in $[q]$,

(ii) $(|3, q|_1, [2q])$, the rational surface represented by all plane curves of order q through $\frac{1}{2}q(q-1)$ fixed points (*cf.* § 2.7).

2.6. *The representation of* $(|p, q|, [q])$ *on* $[p-1]$.

If the dimension of the space is such that a set of corresponding primes of the q related $]p-1[$'s generating a manifold E meets in general in a point, *i.e.* if $n = q$, then we can establish a representation of E by way of a correspondence between points of a $[p-1]$ and the primes of one of the $]p-1[$'s. Algebraically, when the equations are written in the parametric form

$$\sum_{\alpha} \lambda_\alpha x_{\alpha\beta} = 0 \quad (\alpha = 1, \ldots, p;\ \beta = 1, \ldots, q), \quad \text{(IV)}$$

* White, 5.

† *Cf.* twisted cubics of one system on a cubic surface in [3] ($p = q = 3$).

each set of parameters λ corresponds to a point of the locus, so that a representation is obtained by using these parameters as coordinates in $[p-1]$. The nature of the representation is clear when the actual coordinates in $[q]$ are substituted for the $x_{\alpha\beta}$ by means of equations (II). The equations then appear as

$$\sum_{\alpha,\epsilon} \lambda_\alpha a_{\alpha\beta\epsilon} x_\epsilon = 0 \quad (\epsilon = 0, \ldots, q). \tag{V}$$

The section of E by a prime

$$\Sigma P_\epsilon x_\epsilon = 0$$

is represented by the primal Π in $[p-1]$ whose equation is obtained by eliminating the x_ϵ from among the equations (V) and that of the prime, viz.:

$$\left| \begin{matrix} \sum_\alpha a_{\alpha\beta\epsilon}\lambda_\alpha \\ P_\epsilon \end{matrix} \right| = 0 \qquad \begin{cases} \beta = 1, \ldots, q \text{ (down each column)}, \\ \epsilon = 0, \ldots, q \text{ (along each row)}. \end{cases}$$

This is a determinant of $q+1$ rows, the elements of all except one of which are linear functions of the coordinates in the $[p-1]$, while those of the last are constants. The equation is, therefore, that of a determinantal primal $(|q, q|, [p-1])$. All the primals Π pass through the manifold Θ whose equations are

$$\| \sum_\alpha a_{\alpha\beta\epsilon}\lambda_\alpha \| = 0,$$

i.e. a manifold $(|q, q+1|, [p-1])$.

The main features of the representation are:

(i) The ∞^{p-3} lines which are the intersections of exceptional sets of corresponding primes of the $]p-1[$'s are represented by the points of Θ. If among these lines there is a $[m]$, it is represented by a point of the manifold $(|q, q+1|_{p-m}, [p-1])$ which lies on Θ, and is multiple on it*.

(ii) A point of the manifold $(|p, q|_r, [q])$ which lies on E, if there are any such points, is represented by a $[p-r-1]$ which is the base of an exceptional $]q-1[$ of the family generating Θ, and cuts Θ in a manifold $(|q, q|, [p-r-1])$. In particular, when $r = p-2$, we get a locus of multiple points on E, represented by the q-secant lines of Θ.

(iii) A prime section of E is represented by a determinantal q-ic primal Π through Θ.

(iv) The section of E by a $[q-m]$ is represented by a manifold $(|q-m+1, q|, [p-1])$, the residual intersection of m primals Π through Θ.

---

* The multiplicity of a point of $(|p, q|_r, [n])$ on $(|p, q|, [n])$ is $\binom{q-r}{p-r-1}$. Segre, 3, 828.

(v) To a $[m]$ in the $[p-1]$ corresponds a $(|m-1, q|, [n])$ on E^*.

There is an important consequence of the fact that $(|p, q|, [q])$ is normal, viz. that the system of primals through Θ is complete, *i.e. all* q-ic *primals through* Θ *are determinantal*†.

2.7. *The representation of* $(|p, q|_1, [pq-q])$ on $[p-1]$.

Two primals Π meet residually in a manifold $(|q-1, q|, [p-1])$, say Θ', and there is a locus associated with E, which is represented by the primals of order q through Θ'. In effect this locus is $(|p, q|_1, [pq-q])$, say E', as we proceed to prove. We may write the equations of this manifold parametrically as

$$\rho x_{\alpha\beta} = \xi_\alpha \eta_\beta \quad \left\{ \begin{matrix} (\alpha = 1, \ldots, p) \\ (\beta = 1, \ldots, q) \end{matrix} \right\}, \qquad \text{(VI)}$$

where the $x_{\alpha\beta}$, being linear functions of coordinates in $[pq-q]$, are connected by $q-1$ linear identities, say

$$\sum_{\alpha,\beta} b_{\alpha\beta\gamma} x_{\alpha\beta} \equiv 0 \quad (\gamma = 1, \ldots, q-1).$$

The parameters of points of the manifold satisfy the equations

$$\sum_{\alpha,\beta} b_{\alpha\beta\gamma} \xi_\alpha \eta_\beta = 0, \qquad \text{(VII)}$$

which can just be solved for the ratios of the η when the ξ are given; thus there is a (1, 1) correspondence between the sets of values of the ξ, *i.e.* the points of a $[p-1]$, and the points of E'.

The points of a prime section of E' satisfy an additional linear relation, say

$$\Sigma P_{\alpha\beta} \xi_\alpha \eta_\beta = 0,$$

and we get, on the elimination of the η among this equation and the equations (VII), the equation of the locus in $[p-1]$, which represents a prime section of E'. The equation is

$$\begin{vmatrix} \sum_\alpha b_{\alpha\beta\gamma} \xi_\alpha \\ \sum_\alpha P_{\alpha\beta} \xi_\alpha \end{vmatrix} = 0 \quad \begin{cases} \beta = 1, \ldots, q \text{ (along each row)}, \\ \gamma = 1, \ldots, q-1 \text{ (down each column)}. \end{cases}$$

This is a manifold of the same type as Π. Exceptionally, a set of values

* In White, **5**, the special case of $p = 3$, q general is discussed. In Roth, **6.1**, the representation of $(|4, 4|, [4])$ is worked out in detail for a slightly special case. Most of the work, however, applies equally to the general case.

† *E.g.*, if a quartic surface in [3] has on it a curve $(|4, 5|, [3])$, a 10-ic of genus 11, it is determinantal, *cf.* Roth, **6.1**.

of ξ may be such that ∞^1 sets of ratios of the η satisfy the equations (VII), and we get then a line on E'; such a line is represented by a point of the locus Θ' whose equations are

$$\| \sum_{\alpha} b_{\alpha\beta\gamma} \xi_\alpha \| = 0.$$

Θ' is thus a manifold $(|q-1, q|, [p-1])$. Again, since E' is normal, *every* q-ic *primal through* Θ' *is determinantal**. The details of the representation of E' are similar to those for E. From the two representations, it follows that on any primal Π there are ∞^{q-1} manifolds Θ' residual to any Θ, and ∞^{pq-q-1} manifolds Θ residual to any Θ'.

2.8. *A geometrical description of these representations.*

We can get a quite simple geometrical picture of these two representations by considering the key manifold K of the series $(|p, q|, [pq-1])$. K is the locus of $[pq-q-1]$'s joining corresponding primes, $[p-2]$'s, of q related $[p-1]$'s, and E is its $[q]$-section. Each of the $[pq-q-1]$'s meets a general $[q]$ in a point, and any one of the $[p-1]$'s in a prime, thus, as the spaces trace E point by point on the $[q]$, they trace a representation of E by primes in the $[p-1]$. The representation obtained above is just the dual of this. The representation of the points of E' by points of a $[p-1]$ is somewhat simpler. Each point of E' is the section by the $[pq-q]$ in which it lies of a $[q-1]$ of the system joining corresponding points of q related $[p-1]$'s in $[pq-1]$, so that, as the $[q-1]$'s trace E', they also trace its representation on each of these $[p-1]$'s.

As an example of this construction, take the $(|3, 3|, [n])$ series. The key manifold $)|3, 3|, [8](,$ or $\mathfrak{K}$, consists of two ∞^2 families of planes, those of one family being the joins of corresponding points in a relation among those of the other, while $(|3, 3|, [8])$, or K, is generated by the ∞^2 family of $[5]$'s which join corresponding lines in this relation. E', or $(|3, 3|_1, [6])$, is the sextic Del Pezzo surface. The planes of one of the families on $\mathfrak{K}$ all meet a fixed plane ϖ of the other family in points, and in general meet a $[6]$ in points, viz. those of a manifold E'. Three of the planes meet the $[6]$ in lines, and meet ϖ in the three base points of the representation.

E, or $(|3, 3|, [3])$, is the ordinary cubic surface. This time the $[5]$'s of one of the families on K meet ϖ in lines, whilst in general they meet a $[3]$ in points, which form the manifold E. This gives a representation of the points of E by the lines of ϖ. Six of the $[5]$'s meet the $[3]$ in lines, and give in ϖ the base lines of the representation.

* *E.g.*, all quartic surfaces in $[3]$ through a sextic curve of genus three are determinantal.

3. *The freedom of* $(|p, q|, [n])$.

We denote by f_S the freedom of the star systems that generate $(|p, q|, [n])$, by f_D the freedom of the manifold D itself, and by $f(p, q, n)$ the number $pq(n+1)-p^2-q^2+1$. In §§3.1 and 3.2 we find that $f_S=f(p, q, n)$ for all relevant values of p, q, and n, with one trivial exception, and we are left then with the problem of determining in what circumstances $f_D=f_S$. When $n \geqslant p$ two causes may operate to make the numbers differ:

(i) There may be an infinity of different families of ∞^{q-1} $[n-p]$'s which will serve as bases of the $]p-1[$'s in the generation of D; in this case $f_D<f_S$; and

(ii) There may be an infinity of different systems of relations among the $]p-1[$'s when the bases of the stars are assigned, all of which give D; in this case also $f_D<f_S$.

Thus, provided that $n \geqslant p$, we have $f_D \leqslant f_S$. In §§3.3 and 3.4 we show that there is actually equality between the numbers*.

When $n<p$ the position is more obscure geometrically, but we can still show that f_D cannot be greater than f_S. When D is defined by a system of equations, f_D is the number of independent free constants in the equations. The definition of D by star systems can be reduced to a system of equations in which the only constants appearing are those in the equations of the stars, whose freedom is f_S. It may happen that the way in which these constants appear reduces the number of them that need be independent, *i.e.* that $f_D<f_S$, but it certainly cannot increase that number.

3.1. *The freedom of the system of stars generating* $(|p, q|, [n])$: *algebraic method.*

Let $(|p, q|, [n])$ be determined by the set of q $]p-1[$'s whose equations are

$$\sum_{\alpha} \lambda_\alpha x_{\alpha\beta} = 0, \tag{I}$$

where

$$x_{\alpha\beta} = \sum_{\epsilon} a_{\alpha\beta\epsilon} x_\epsilon. \tag{II}$$

* This is not a result that is necessarily to be expected. The cubic surface in [3] is an example in which there might quite easily not have been equality, for, when we are given the bases of the sheaves that are to generate the surface—viz., the points of the surface itself—there are seventy-two ways in which the sheaves may be related so that their corresponding planes meet in points of the surface. There is no *a priori* reason why, in other cases, there should not be an infinity of systems of relations.

Each one of the q equations (I) may be divided through by a constant without altering the system, so that the number of effective (non-homogeneous) constants is $pq(n+1)-q$. Each of the stars is a member of an ∞^{q-1} family and it may be replaced by any other member, so that $\infty^{q(q-1)}$ sets of equations (I) and (II) represent the same star system, and the freedom of the star systems is reduced on this account by $q(q-1)$. This result is expressed algebraically by the fact that any set of q equations

$$\sum_{\alpha} \lambda_{\alpha} \sum_{\beta} d_{\beta\beta'} x_{\alpha\beta} = 0 \quad (\beta' = 1, \ldots, q), \qquad (I')$$

where the $d_{\beta\beta'}$ are arbitrary*, represents the same system of stars.

The equations have so far been constructed so that a certain set of values of λ corresponds to each prime of any of the $]p-1[$'s, but any homographic replacement of the parameters will not change the relation among the $]p-1[$'s, so that the freedom of the star systems may be reduced by a further p^2-1. This corresponds to a replacement of the equations (I') by the system

$$\sum_{\alpha', \alpha, \beta} \lambda_{\alpha'} c_{\alpha\alpha'} d_{\beta\beta'} x_{\alpha\beta} = 0 \quad (\alpha' = 1, \ldots, p). \qquad (I'')$$

Thus the freedom of the star system represented by equations (I) and (II) is $pq(n+1)-p^2-q^2+1 = f(p, q, n)$†.

Strictly we should say that the freedom is at least this, because, while the initial number of arbitrary constants, $pq(n+1)-q$, cannot be reduced, there is a possibility that the freedom of the product set $c_{\alpha\alpha'} d_{\beta\beta'}$ may be less than the sum of the freedoms of the constituent sets. There is, in fact, one very simple class of "manifolds" for which the formula is not valid, namely those for which $p = q$, $n = 1$. For these $f(p, q, n) = 1$, while $(|q, q|, [1])$ is a set of q points on a line. The set is "general", since any q points on the line can have their equation put in the form

$$\begin{vmatrix} x+a_1x' & & & & \\ & x+a_2x' & & & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & & x+a_qx' \end{vmatrix} = 0,$$

(x, x') being the coordinates on the line, and this is the equation of a

* A factor may be divided through every set of q constants having the same suffix β', thus reducing the effective number of constants from q^2 to q^2-q.

† This method is substantially that used by L. E. Dickson, **11** (in Theorem 6, page 177), in proving that the quantity here called f_S, for the case $p = q$, is equal to $q^2(n-1)+1$, and that f_D is not greater than this. The same method is used by Jessop, *Quartic surfaces* (Cambridge, 1916), 161, to find the freedom of the determinantal quartic surface.

$(|q, q|, [1])$. Thus $f_D = q$ and f_S cannot be less than this. Finding that the formula does not hold in this simple case raises the question of its validity in other cases. To answer this question we turn to the geometry.

3.2. *The freedom of the system of stars generating* $(|p, q|, [n])$: *geometric method.*

Suppose that $n \geqslant p-1$, so that a $]p-1[$ consists of the primes through a base-$[n-p]$, or, in the case of equality, all the primes of a base-$[p-1]$. In calculating f_S account has to be taken of three factors:

N_B: the freedom of the bases of the stars. These bases are $[n-p]$'s in $[n]$, and are q in number, so that $N_B = qp(n-p+1)$;

R_B: the freedom of possible replacements of the bases. Each base is an arbitrary member of an ∞^{q-1} family, so that $R_B = q(q-1)$;

N_C: the freedom of the relations among the $]p-1[$'s. When the bases of two $]p-1[$'s are given, the freedom of projective relations between them is p^2-1; the whole system of relations among the q stars is defined when one of them is related to the remaining $q-1$, thus $N_C = (q-1)(p^2-1)$.

Then
$$f_S = N_B - R_B + N_C$$
$$= f(p, q, n).$$

If $n < p-1$ we cannot apply this process directly, since the symbols N_B and R_B lose their geometrical significance. The ∞^{q-1} family of $]p-1[$'s by which D is generated may, however, include some members which have proper bases. If then we use q of these special members to generate D, we shall be able to calculate by geometrical methods the numbers $N_B - R_B$, and N_C. But, whatever process of calculation is adopted (*i.e.* whatever the specialization of the bases and relations), we are bound to obtain for $N_B - R_B + N_C$ the value $f(p, q, n)$, and this then is the freedom of the star system generating D. Thus the limitations on the validity of the expression are that we shall be able

(i) to find q independent $]p-1[$'s which have proper bases (no matter how degenerate the $]p-1[$'s may be), and

(ii) to effect a proper relation among these $]p-1[$'s.

We can restrict the range of our inquiries by observing that, if the formula holds for certain values of p, q, n, then it holds for all higher

values of n and all smaller values of p that are relevant, *i.e.* for which $n-q+p-1 \geqslant 0$. The critical cases are:

(i) $p=q$, $n=1$. D consists of q points on a line. These points must be taken as the bases of the q $]q-1[$'s, and we get $N_B-R_B=q^*$. Each point itself is, however, all there is of the "star", and the freedom of the relation among them, N_C, has no geometrical significance. If the usual formula for the freedom of the manifold is accepted, we find that the "freedom" of the relation among these degenerate $]q-1[$'s is $-(q-1)$. *Cf.* foot-note *, p. 21.

(ii) $p=q$, $n=2$. D is a plane curve of order q, and we take for the bases of the $]q-1[$'s any q of its points, so that $N_B-R_B=q$. We then have to relate these q $]q-1[$'s, each of which consists of the lines of a pencil counted ∞^{q-2} times. This is a definite geometrical operation whose freedom, N_C, can be calculated without the necessity of applying any corrections. We should get for $N_B-R_B+N_C$, the freedom of the star systems generating D (not of D itself), the value q^2+1.

Thus the only case of failure is the one already noticed, viz. $(|q, q|, [1])$, and the freedom of the star systems generating $(|p, q|, [n])$ is $f(p, q, n)$, except when $p=q$, $n=1$.

Although it is not necessary to calculate N_B-R_B and N_C in order to prove the validity of the result, the process is perhaps not without its interest, and we carry it out here for the case $n<p<2n$. The equations of the base (if it exists) of one of the $]p-1[$'s generating D are

$$\sum_{\beta} b_\beta x_{\alpha\beta}=0,$$

each set of ratios of b_β corresponding to a different $]p-1[$. These p linear equations in $[n]$ will represent a point if the coefficients b_β satisfy some $p-n$ conditions, so that of the ∞^{q-1} family of $]p-1[$'s $\infty^{n+q-p-1}$ members will have points as their bases. That is, each point of the $[n]$ is the base of ∞^{q-p-1} (degenerate) $]p-1[$'s†, and $N_B-R_B=-q(q-p-1)$.

We have now to find the freedom of a relation among q degenerate $]p-1[$'s, and the process will be rather more intelligible if it is worked through first for simple degeneration. Further, a clearer picture is

---

* We might take each star to consist of the points of the line itself, *i.e.*, to be a degenerate $]q-1[$ composed of points of a $]1[$ in $[1]$. Then $N_B=0$, and N_C is a geometrically definable positive number, but the difficulty is presented by R_B.

† If $p=q$, not all the points of the $[n]$ are vertices of $]p-1[$'s, but only those of the primal D. This is the geometrical significance of each point of space being a vertex of "∞^{-1}" stars.

obtained if spaces are used instead of stars. Suppose $\alpha, \beta, \gamma, \delta, \ldots$* to be a family of q related non-degenerate $[m]$'s, and that each is projected from a point of itself, so that we get a family of related simply-degenerate $[m]$'s, $\alpha', \beta', \gamma', \delta', \ldots$. Let α be projected from A, and suppose that B_1, C_1, D_1 are the corresponding points of β, γ, δ, and define, similarly, B, C, D, and $A_2, \ldots, D_3$. Let also the projection of A_2 from A be a point A_2' of α', and denote similarly all the projected points by accented letters.

If the spaces were not degenerate, the relation of α' to $\beta', \gamma', \delta', \ldots$ would determine the relation among all the spaces; this is not true in the case of degenerate spaces. We find first the freedom of a relation between α' and β'. The variable elements of the relation are: the points A_2' and B_1', and the relation between the two $]m-2[$'s, $)A_2'($, and $)B_2'($. Thus the freedom of the relation between α' and β' is

$$\{2(m-1)\}+\{(m-1)^2-1\} = m^2-2,$$

and thence the freedom of that connecting α' with $\beta', \gamma', \delta', \ldots$ is $(q-1)(m^2-2)$. Next take the relation between β' and γ'. Corresponding to the line $A_2'A_3'$ of α' there is a certain line through B_1' in β', and on this line there lies the point B_3'; A_3' and B_3' are the projections of the points which correspond to C of γ. Thus B_3' and similarly C_2' lie on lines which are determined by the relation connecting α' with β' and γ'. To define completely the relation between β' and γ', we have to relate the $]m-2[$'s, $)B_3'($, and $)C_2'($, and the relation must be such that the relation which has already been established between $)B_3'B_1'($ and $)C_2'C_1'($ is preserved. Thus the freedom of the relation between β' and γ' is equal to twice the freedom of a point on a line plus the freedom of a relation between two $]m-2[$'s in which a relation between two assigned $]m-3[$'s is given; it is equal, that is, to $2+m-1$. Hence the freedom of the relation connecting β' with $\gamma', \delta', \ldots$ is $(q-2)(m+1)$. We can show now that the relation between any pair of the spaces is determined; for, taking γ' and δ', corresponding to $A_3'A_4'$ in α' and $B_3'B_4'$ in β', there are certain lines through C_1' and C_2' respectively in γ', and these must meet in C_4'. That is, the known relations determine C_4' and similarly D_3'. Then, in the relation between $)C_4'($ and $)D_3'($, the relations between $)C_4'C_1'($ and $)D_3'D_1'($, and between $)C_4'C_2'($ and $)D_3'D_2'($ are known, and these are sufficient to determine completely the relation between $)C_4'($ and $)D_3'($. Thus, when α' is related to $\beta', \gamma', \delta', \ldots$, and β' is related to $\gamma', \delta', \ldots$, the relation between any other pair of the spaces is determined. Collecting the terms, we find, for the freedom of a

* Following §1.1 we should write (α), etc., for these, but no confusion is caused by omitting the brackets.

relation among q simply-degenerate $[m]$'s, the value

$$(q-1)\,m^2+(q-2)\,m-q.$$

If $t-1<m-t$, the whole process above is applicable with slight modifications to the case when the elements A, ... from which α, ... are projected are not points, but $[t-1]$'s, so that α', ... lie in $[m-t]$'s and are t-ply degenerate $[m]$'s. The numbers we get are: the freedom of a relation between α' and β' is equal to twice the freedom of a $[t-1]$ in a $[m-t]$ plus the freedom of a relation between two $]m-2t[$'s, that is, it is equal to $(m+1)^2-2t(m+1)-1$*; and the freedom of the relation between β' and γ' is equal to twice the freedom of a $[t-1]$ in a $[2t-1]$ plus the freedom of a relation between two $]m-2t[$'s in which an assigned pair of $]m-3t[$'s satisfy a known relation, that is, it is equal to $t(m+1)$.

The relation between any pair is established when α' has been related to β', γ', δ', ... and β' to γ', δ', ..., so that the freedom of the relation connecting $\alpha'\beta'\gamma'\delta'\ldots$ is $(q-1)(m^2-2mt+2m-2t)+(q-2)\,t(m+1)$. To apply the formula to $(|p, q|, [n])$ we take $m=p-1$, $t=p-n$, which gives $N_C=(q-1)(p^2-1)-pq(p-n)$. We have already

$$N_B-R_B=-q(q-p-1),$$

and thence

$$N_B-R_B+N_C=f(p, q, n).$$

3.3. *The freedom of a manifold* $(|p, q|, [n])$ *for which* $n\geqslant p+q-2$.

It is possible, on the basis of the second definition given in § 2.3 of the key manifold $\mathfrak{K}$ or $)|p, q|, [pq-1]($, to find $f_{\mathfrak{K}}$ without having to use explicitly the freedom of the projective relation; $\mathfrak{K}$ we saw was the locus of transversal $[q-1]$'s to $q+1$ $[p-1]$'s, the $[p-1]$'s being in general position in $[pq-1]$, and general members of an ∞^{q-1} family on $\mathfrak{K}$. Thus

$$f_{\mathfrak{K}}=(q+1)\{p(pq-p)-(q-1)\}=(p^2-1)\,(q^2-1)$$
$$=f(p, q, pq-1).$$

Further $\mathfrak{D}$, or $)|p, q|, [n]($, is a projection of $\mathfrak{K}$, so that, if the space of projection does not meet $\mathfrak{K}$, we can use the simple formula that connects $f_{\mathfrak{D}}$ and $f_{\mathfrak{K}}$, viz.

$$f_{\mathfrak{D}}=f_{\mathfrak{K}}-(n'-n)\,m,$$

n' being the dimension of the space in which $\mathfrak{K}$ lies, and m that of the space

* If $t-1=m-t$, the projective relation disappears altogether, and any point of α' corresponds to any point of β'. The formula gives -1 for the freedom of such a relation. On the other hand, there are identifiable elements in a mutual relation among three such degenerate $[m]$'s.

in which it is normal*, that is, $n' = m = pq-1$. This gives

$$f_{\mathfrak{D}} = f(p, q, n),$$

provided that $n \geqslant p+q-2$. Since the freedoms of a manifold and of its dual are necessarily equal, we get the same value for f_D, with n restricted in the same way.

We can also arrive at this result by showing that D determines uniquely both the bases of the system of stars that generates it and the relation among them. $(|p, q|, [n])$ is, in fact, the locus of $[n-q]$'s which are the meets of corresponding primes of related $]p-1[$'s whose bases are determined by D (§ 2.4), *i.e.* D defines a family of $[n-q]$'s and a family of $[n-p]$'s in which any $[n-q]$ lies with every $[n-p]$ in a prime. If $n < p+q-2$, a $[n-q]$ and a $[n-p]$ lie in more than one prime (they lie together in a space of dimension less than $n-1$), but if $n \geqslant p+q-2$ any $[n-q]$ and any $[n-p]$ determine uniquely the prime containing them both. That is, if $n \geqslant p+q-2$, the relation between two of the $]p-1[$'s is determined by joining their bases to the $[n-q]$-generators of D, and thus

$$f_D = f_S = f(p, q, n).$$

3.4. *The freedom of* $(|p, q|, [n])$ *when* $p \leqslant q \leqslant n$.

If $p \leqslant q \leqslant n$, D consists of ∞^{p-1} $[n-q]$'s, and defines a system of $[n-p]$'s which are the bases of the $]p-1[$'s by which it may be generated. These systems are necessarily unique (with the trivial exception of the case of $p = q$, when either may be replaced by the other), since they are sections of definite systems on or associated with K; but they do not, as they stand, define the relation among the $]p-1[$'s, since the join of a $[n-p]$ to a $[n-q]$ will not be a prime if $n < p+q-2$. It may, therefore, be possible to find more than one system of relations among the $]p-1[$'s (whose bases are given) which is such that sets of corresponding primes meet in the $[n-q]$'s; we proceed to show that the freedom of these systems of relations, which is $f_S - f_D$, is zero.

If we can establish the uniqueness of the system of relations for any given n, then it is established for all higher values†; it will be sufficient,

* Room, **10**, formula 2.21, 534.

† The section of a system of stars generating a manifold D is a system of stars generating the section D' of the manifold, *i.e.*, each star system generating D gives rise to a system generating D'. Conversely, there may be star systems generating D' which are not sections of star systems generating D'. For example, if D is the cubic determinantal primal in $[4]$, $(|3, 3|, [4])$, and D' is its prime section, there are two systems of stars generating D (§ 3.3), and seventy-two generating D'.

therefore, to establish it for the manifold $(|p, q|, [q])$. We prove:

(i) *If $p < q$, then $(|p, q|, [q])$ can be generated in only one way as the locus of meets of corresponding primes of ∞^{q-1} related $]p-1[$'s.*

(ii) *If $p = q$, then $(|q, q|, [q])$ can be generated in two such ways (except when $p = q = 3$, in which case there are seventy-two ways).*

We observe first that $(|2, q|, [q])$, the normal rational curve in $[q]$, can be generated projectively in only one way*, and secondly that $(|3, q|, [q])$, which is the rational surface represented by plane curves of order q through $\frac{1}{2}q(q+1)$ points, has on it, when $q > 3$, only one ∞^2 system of normal rational curves, namely that represented by the lines of the plane†, and therefore is capable of only one projective generation.

Denote by E the manifold $(|p, q|, [q])$. Besides being the locus of meets of corresponding primes of ∞^{q-1} related $]p-1[$'s, E is the locus of meets of corresponding $[p-1]$'s of ∞^{p-1} related $]q-1[$'s. The bases of the $]q-1[$'s are just the points of the manifold, and consequently are completely determined when the manifold is given. We have to show, therefore, that the $]q-1[$'s with arbitrary p points of E as their bases can be related in only one way (or two, or seventy-two ways) so that their corresponding $[p-1]$'s meet only in points of E.

Suppose that there are N ways in which the stars may be related. The $[n-p]$'s that are the meets of corresponding primes of the $]q-1[$'s of one of these systems are the bases of the stars in the corresponding generation by primes of related $]p-1[$'s. Corresponding primes of a set of corresponding $]1[$'s in these $]p-1[$'s generate a normal rational curve. There are thus N different ∞^{2p-4} families of normal rational curves on E. Each of these families of related $]p-1[$'s gives a different representation of E on $[p-1]$ of the type considered in § 2.6, in which the lines of the $[p-1]$ correspond to the particular family of normal rational curves on E. A family of normal rational curves on E obtained from a star system other than that by which the representation is effected will be represented by a family of rational curves in $[p-1]$, say of order s, which, since prime sections of E are represented by primals of order q through Θ, meet Θ in $q(s-1)$ points. Thus N is finite, unless one of these families of curves is of greater freedom than $2p-4$, say of freedom $2p-4+i$. Use now the representation of E corresponding to this family of normal rational curves;

* The $(q-1)$-secant $[q-2]$'s of the curve form a single ∞^{q-1} system, and these are the only possible bases of the $]1[$'s.

† *Cf.* White, 5.

we obtain in $[p-1]$ ∞^{2p-4+i} curves among which there are included the ∞^{2p-4} lines of the space. This is not possible*, and thus N must be finite.

Suppose next that $p<q$, and that E has two distinct projective generations $\mathcal{G}$ and $\mathcal{G}'$, giving two families of normal rational curves r and r'. There is on E only one ∞^{p-3} family of lines h, represented, whether the representation is made *via* $\mathcal{G}$ or $\mathcal{G}'$, by the points of Θ. Any line h is the meet of corresponding primes in both systems $\mathcal{G}$ and $\mathcal{G}'$, and thus forms part of any curve r or r' which passes through any of its points. Now take the representation via $\mathcal{G}$ and suppose that in it the curves r' are represented by curves ρ' of order s. Any ρ' must meet Θ in $q(s-1)$ points, and thus any r' meets $q(s-1)$ lines h, and must consist therefore in part of these lines. The only possibility is $s=1$, and then $\mathcal{G}$ and $\mathcal{G}'$ coincide.

If $p=q=n$ the position is only slightly different, there being in this case $q+1$ ∞^{q-3} families of lines (§ 2.4), h_0, h_1, ..., h_q, but of these, except when $q>3$, only h_q is equivalent to h_0, and we have, therefore, just one projective generation corresponding to h_0 and one to h_q.

3.5. *The freedom of* $(|p, q|, [n])$ *when* $p=q-1$.

So far all the manifolds considered have been generated by stars which have proper geometrical bases. We have now to consider those for which $n<q$ (and therefore also $n<p$) whose algebraic specification is the same as in the other cases, but whose geometric specification in terms of $[n-p]$'s and $[n-q]$'s lacks actuality. The process of § 3.4 is inapplicable, but another is available. We make use of the connection between the freedoms of a rational manifold and of its representation†. The manifold $(|p, q|, [q])$, or E, can be birationally represented by primals Π through a manifold $(|q, q+1|, [p-1])$, or Θ (§ 2.6), Θ corresponding to a locus of lines, say T on E. The freedoms are connected by the equation

$$f_E = f_\Theta - p^2 + (q+1)^2 - {}_E f_T.$$

The locus T on E arises from the exceptional sets of corresponding primes of the $]p-1[$'s generating E which meet in lines, and thus its freedom on E

* Consider, *e.g.*, $(|4, 4|, [4])$, represented on [3] by quartic surfaces through a curve of order ten and genus eleven (Roth, **6.1**). The family of rational quartics derived from the star system by which the representation is effected is represented by the lines of the [3]. If this family of lines is part of a family of conics, for example, there will have to be some fixed locus, which meets all the lines, lies on all the quartic surfaces, and is such that together with any line it is included in the system of conics.

† Room, **10**, § 1.3, 256.

is the same as that of star systems which generate E, namely 0. Thus

$$f_\Theta = f(p, q, q) + p^2 - (q+1)^2 + 0$$
$$= f(q, q+1, p-1).$$

That is, the freedom of $(|q-1, q|, [n])$, when $n < q$, is $f(q-1, q, n)$.

3.6. *The freedom of* $(|p, q|, [n])$ *when* $p < q$.

To show that $f(p, q, n)$ is the freedom of the manifold for all relevant values of p, q, and n, except possibly when $p = q$, we have now only to show that, if the freedom of D or $(|p, q|, [n])$ is $f(p, q, n)$, then the freedom of D or $(|p-1, q|, [n])$ is $f(p-1, q, n)$, since we have now established the result for $p = q-1$. D' can be defined by $p-1$ related $]q-1[$'s, and D by relating another $]q-1[$ to this system; thus the greatest possible difference between the freedoms of D' and D is the difference between the freedoms of the star systems generating them. From this the required result follows, since the freedom of a manifold cannot exceed the freedom of the star systems generating it.

Thus, for all values of p, q, and n for which $(|p, q|, [n])$ exists, and $p < q$, the freedom of the manifold is $f(p, q, n)$*.

3.7. *Note on the freedoms of determinantal curves.*

We can compare the freedoms given by the formula in the cases where the locus is a curve of known genus with the known values of the freedom of such curves. For a curve of order m and genus π in $[n]$,

$$f = m(n+1) - (\pi-1)(n-3),$$

provided that
$$m \geqslant n + \frac{n}{n+1}\pi.$$

(i) When $n = 3$, a curve of order m has freedom $4m$. $(|q-1, q|, [3])$ is a curve of order $m = \frac{1}{2}q(q-1)$. Also $f(q-1, q, 3) = 2q(q-1) = 4m$.

(ii) By §2.6, Θ, or $(|3, n+1|, [n])$, is in (1, 1) correspondence with $(|n+1, n+1|, [2])$, which is a general plane curve of order $n+1$. For Θ

* $(|2, q|, [q-1])$ would appear at first to be an exception, since it consists of q arbitrary points in $[q-1]$, and these points have freedom $q(q-1)$, while $f(2, q, q-1) = q^2-3$. But $(|2, q|, [q-1])$ consists, in addition to the q points, of the bases of the ∞^{q-1} family of $]1[$'s by which it is generated. These bases are secunda in the $[q-1]$, and they form a system analogous to the tetrahedral complex of lines in [3]. The difference between the freedom of the simplex formed from the q points and that of $(|2, q|, [q-1])$ corresponds to the number of independent invariants of the system of primes joining any of these secunda to the vertices of the simplex.

we have $m=\frac{1}{2}n(n+1)$, $\pi=\frac{1}{2}n(n-1)$, and Θ has $\frac{1}{2}(n+1)(n+4)-8$ moduli, while a general curve of the same genus has $\frac{3}{2}(n^2-n-2)$ moduli*. Thus, applying the formula above,

$$f_\Theta=\tfrac{1}{2}n(n+1)^2-\tfrac{1}{2}(n^2-n-2)(n-3)-\tfrac{3}{2}(n^2-n-2)+\tfrac{1}{2}(n^2+5n-12)$$
$$=2n^2+4n-6=f(3,\ n+1,\ n).$$

(iii) Θ', or $(|3,\ n+1|_1,\ [2n+1])$, is also in (1, 1) correspondence with $(|n+1,\ n+1|,\ [2])$, and for Θ', $m=\frac{1}{2}(n+1)(n+2)$, $\pi=\frac{1}{2}n(n-1)$, and the numbers of moduli are the same as in (ii)†. Thus

$$f_{\Theta'}=\tfrac{1}{2}(n+1)(n+2)(2n+2)-\tfrac{1}{2}(n^2-n-2)(2n-2)-\tfrac{3}{2}(n^2-n-2)$$
$$+\tfrac{1}{2}(n^2+5n-12)$$
$$=5n^2+10n-3=f(3,\ n+1,\ 2n+1).$$

Further, the projection of Θ' from any $n+1$ of its points is Θ, as is easily seen by considering the representations of the rational surfaces $(|3,\ n+1|,\ [n+1])$ and $(|3,\ n+1|_1,\ [2n+2])$, and the points of projection on Θ' are not "identifiable" on Θ. Thus, since the projection is made from a $[n]$ of an ∞^{n+1} family, and

$$f_{\Theta'}-f_\Theta=3n^2+6n+3=(n+1)(2n+2)+(n+1)n+n+1,$$

Θ' must be normal in $[2n+1]$ and Θ must be normal in $[n]$‡.

3.8. *The freedom of* $(|p,\ q|,\ [n])$ *when* $p=q$.

The only remaining case is $p=q$, $n<q$, and here we do not expect to get the freedom to be $f(p,q,n)$ in all cases, since we know already that a plane curve may be represented by a determinant whose elements are linear functions of the coordinates, *i.e.* that $(|q,\ q|,\ [2])$ is a general plane curve of order q, and that, while $f(q,\ q,\ 2)=q^2+1$, the freedom of the plane curve itself is only $\frac{1}{2}q(q+3)$§. We find, however, in fact, that, except when $n=2$ or 1, the freedom of $(|q,\ q|,\ [n])$ is $f(q,\ q,\ n)$. We prove the result for $n=3$,

* Except when $n=2$; Θ is then the elliptic cubic.

† Except when $n=2$; Θ' is then the normal elliptic sextic.

‡ Room, **10**, formula 2.2, p. 534, $f_{\Theta'}-f_\Theta=(n+1)k+h+u$, $[k-1]$ being the normal space of Θ', h being the number of conditions of specialisation of the space of projection, and u the freedom of the set of points on Θ corresponding to the points of projection on Θ'.

§ In fact, any plane curve can be represented (in a finite number of ways) by a *symmetrical* determinant. A. C. Dixon, *Proc. Camb. Phil. Soc.*, 11 (1902), 350; Grace, *Journal London Math. Soc.*, 2 (1927), 182.

‖ L. E. Dickson, **11**, found that the freedom of $(|q,\ q|,\ [n])$ was at most $f(q,\ q,\ n)$ (*cf.* foot-note †, p. 17). Roth, **8**, finds this value for the freedom of $(|q,\ q|,\ [n])$ (in his notation V^q_{n-1}) by the same method as Dickson, but does not point out that the method proves only that the freedom cannot exceed this number; this, however, is clear, since immediately afterwards he quotes the result that the freedom of the curve $(|q,\ q|,\ [2])$ is $\frac{1}{2}q(q+3)$.

and it then follows by induction or by a similar argument that it is true for higher values.

Using the notation of § 2 . 6, we denote by Π the surface $(|q, q|, [3])$, and by Θ any of the curves $(|q, q+1|, [3])$ on it. Then

$$f_\Pi = f_\Theta + {}_\Theta f_\Pi - {}_\Pi f_\Theta.$$

We have found that $f_\Theta = f(q, q+1, 3)$, and, since every surface of order q through Θ is determinantal and these surfaces represent prime sections of the threefold $(|4, q|, [q])$, ${}_\Theta f_\Pi = q$. We have now to find ${}_\Pi f_\Theta$. Two surfaces Π through Θ meet further in a curve Θ' of the type $(|q-1, q|, [3])$. All q-ic surfaces through Θ' are determinantal, and represent prime sections of a threefold $(|4, q|_1, [3q])$, E' in the notation of § 2 . 7. Thus the freedom of the curves Θ on Π *which are residual to a given* Θ' is the freedom of secunda in a given prime of $[3q]$, namely $3q-1$. This is not at first sight the same as ${}_\Pi f_\Theta$, since there might be a family of curves of the same type as Θ of larger freedom which includes among its members those residual to a given Θ'. But the curves Θ represent secundum sections of E, and these secundum sections are normal (§ 3 . 7), and therefore form a complete system on any prime section of E'. Thus the curves Θ form a complete system on Π and

$${}_\Pi f_\Theta = 3q-1$$

and

$$f_\Pi = f(q, q+1, 3) + q - (3q-1)$$
$$= f(q, q, 3).$$

It is now possible to see the reason for the failure of the formula in the case of $n = 2$. Consider the representation in the manner of § 2 . 6 of the normal rational surface $(|3, q|, [q])$*, say E. Prime sections of E are represented by plane curves Π of order q, whose symbol is $(|q, q|, [2])$, through a set of $\frac{1}{2}q(q+1)$ points $(|q, q+1|, [2])$ which we call Θ. The residual intersection of two curves Π is a set of $\frac{1}{2}q(q-1)$ points $(|q-1, q|, [2])$, or Θ'. On the other hand, curves Π through Θ' represent prime sections of a surface $(|3, q|_1, [2q])$, say E'. We use, as before, the equation

$$f_\Pi = f_\Theta + {}_\Theta f_\Pi - {}_\Pi f_\Theta,$$

in which we know that

$$f_\Theta = f(q, q+1, 2) = q(q+1)$$

[Θ is, in fact, $\frac{1}{2}q(q+1)$ general points in the plane], and that ${}_\Theta f_\Pi$ is equal to the

* *Cf.* White, **5**, 2.

freedom of primes in $[q]$, *i.e.* q. But now consider ${}_{\Pi}f_{\Theta}$; this is the freedom of sets of $\frac{1}{2}q(q+1)$ points on Π. It is *not* the freedom of sets of points Θ on a curve Π residual to a given set Θ' of $\frac{1}{2}q(q-1)$ points on the curve, *i.e.* it is not the freedom of secunda in a given prime of $[2q]$. Or we may express it thus: a general set of $\frac{1}{2}q(q+1)$ points on a curve Π through Θ' represents a general set of $\frac{1}{2}q(q+1)$ points on a prime section of E', whereas a set of points Θ on Π residual to Θ' represents the points of E' which lie in a secundum in the prime.

Summing up, we have the result:

The freedom of the manifold represented by the vanishing of the determinants of order p in a matrix of p columns and q rows ($p \leqslant q$), whose elements are linear functions of the coordinates in $[n]$, is

$$pq(n+1)-p^2-q^2+1,$$

except when $p = q$, $n \leqslant 2$.

References.

1. Schur, *Math. Annalen*, 18 (1881), 1–33.
2. Veronese, *Math. Annalen*, 19 (1882), 161–234 (215).
3. Segre, *Encyk. d. Math. Wiss.*, III C 7 (1912), 825.
4. White, *Proc. Camb. Phil. Soc.*, 21 (1923), 216–227.
5. White, *ibid.*, 22 (1925), 1–10.

6.1. Roth, *Proc. London Math. Soc.* (2), 30 (1929), 118–126.

6.2. Roth, *ibid.*, 305–318.

7. Roth, *Proc. Camb. Phil. Soc.*, 25 (1929), 268–271.
8. Roth, *Journal London Math. Soc.*, 5 (1930), 291–295.
9. Room, *Journal London Math. Soc.*, 4 (1929), 206–209.
10. Room, *Proc. Camb. Phil. Soc.*, 27 (1931), 518–537.
11. Dickson, *Trans. American Math. Soc.*, 22 (1921), 167–179.

A CONTRIBUTION TO THE THEORY OF GROUPS OF PRIME-POWER ORDER

By P. HALL.

[Received and read 10 March, 1932.]

Introduction.

The pages which follow represent the first stages of an attempt to construct a systematic general theory of groups of prime-power order. It is widely recognised, I believe, that the astonishing multiplicity and variety of these groups is one of the main difficulties which beset the advance of finite-group-theory. And hitherto the greater part of the very extensive literature dealing with them has been devoted to special classes of p-groups* having particular properties which allow them to be treated in more or less detail. Thus the groups of order p^4 are well known, and those of order p^5 have been determined first by Bagnera and later more concisely by Schreier. Much is known also, thanks chiefly to the American school, of those p-groups which possess "relatively large" Abelian subgroups. Again, the Sylow subgroups of the better-known insoluble groups, such as the symmetric and linear modular groups, provide examples of p-groups whose remarkable structure has been closely investigated. With researches such as these, the present paper is not immediately concerned. I have tried, on the contrary, to discover some of the more obvious features which underlie the structure of the most general p-group. Starting from first principles—that is, from the classical results enumerated for convenience of reference in §1.1—the theory which follows has been developed as a self-contained account of the subject.

We have been guided by the idea that the structure of a group should be exhibited as far as possible by the inter-relations of its *characteristic subgroups*. A subgroup is characteristic, according to Frobenius, if it is transformed into itself by every automorphism of the group; thus

* A group of order a power of the prime p is called a p-group.

characteristic subgroups represent what one would naturally call *invariant** features of the group structure.

There are various principles which may be used to derive characteristic subgroups from others already known. For example, if H and K are any two characteristic subgroups of G, then the *join* $\{H, K\}$ and the *meet* $H \wedge K$ are also both characteristic subgroups of G. Again, a characteristic subgroup of a characteristic subgroup of G is characteristic in G. And if K is characteristic in G, then the subgroup of G which "corresponds" in the multiple isomorphism between G and G/K to any characteristic subgroup of G/K is characteristic in G.

Of great importance in the sequel is another mode of generating characteristic subgroups—*commutation*. The subgroup which is generated by all the commutators $P^{-1} Q^{-1} PQ$ of an element P of H with an element Q of K will be denoted by (H, K) and will be called the commutator subgroup of H with K. If H and K are characteristic in G, so is (H, K). An example of this operation is the *derived group* (G, G) of G. More generally we may define the i-th derived group $\theta_i = \theta_i(G)$ of G by the rules

$$\theta_0 = G, \quad \theta_1 = (G, G),$$

$$\theta_{i+1} = (\theta_i, \theta_i) \quad (i = 1, 2, 3, \ldots).$$

In a p-group, the derived series $\theta_0, \theta_1, \theta_2, \ldots$ ends with the identity E. Of even greater importance, however, than the derived series is the series of groups $H_1, H_2, H_3, \ldots, H_i = H_i(G), \ldots$ defined by the rules

$$H_1 = G, \quad H_2 = \theta_1 = (G, G),$$

$$H_{i+1} = (H_i, G) \qquad (i = 1, 2, 3, \ldots),$$

which we shall call the *lower central series* of G. In a p-group, this series (like the derived series) ends with the identity; but, conversely, we can say here that, if G is a group of finite order whose lower central series ends with the identity, then G is the direct-product of a number of groups of prime-power order†. Thus, apart from trivialities, the termination in E of the lower central series is a characteristic property of p-groups. The "length" of the lower central series of G, that is, the greatest integer c for which $H_c > E$, is called the *class* of G. The class of a p-group is, roughly speaking, a measure of the extent to which the group is non-

* Were it not that the word invariant is used by some writers for the weaker notion of self-conjugate.

† This theorem is substantially equivalent to the theorem VII of Burnside's *Theory of groups*, 2 ed. (1911), 166.

Abelian; Abelian groups are of class 1 and, conversely, groups of class 1 are Abelian.

Finally, we must mention a third series of characteristic subgroups of G (this time an ascending series as opposed to the descending series θ and H) which is analogous in many of its properties to the lower central series. This is the *upper central series* $Z_0, Z_1, Z_2, \ldots$ defined by the rules $Z_0 = E$, $Z_1 =$ the central of G, and for $i = 2, 3, \ldots$, Z_i/Z_{i-1} is the central of G/Z_{i-1}. In a p-group, the upper central series terminates in G and this is also a characteristic property of the direct products of groups of prime-power order.

We shall now describe the results contained in the present paper, mentioning, as occasion requires, their relation to previous work.

Section 1 is of an elementary character and is not essential to the remainder of the work. It deals with the circle of ideas of Burnside's minimal basis theorem. Of this, we give a very simple proof* in § 1.2. Next, in § 1.3, and based on Burnside's theorem, we give a discussion of the group of automorphisms of any p-group G and show in particular that, if G is of order p^n and has d independent generators, then the order of its group of automorphisms must divide the number

$$p^{d(n-d)}(p^d-1)(p^d-p)\ldots(p^d-p^{d-1}).$$

This limit for the order is actually attained by groups of arbitrary high class (we do not prove this in the present paper). Previous work† based on the use of characteristic series had given (apart from special assumptions about G) a limit whose prime-to-p factor was of an essentially higher order of magnitude in any complicated group and whose power-of-p factor was $p^{\frac{1}{2}n(n-1)}$.

In the remaining sections of § 1, we develop a new method of dealing with the so-called *Anzahl* theorems of p-group theory. This method is based on an *enumeration principle* (§ 1.4) which may be described as the p-group analogue of the well-known principle‡ of enumeration in the elementary theory of numbers due to Sylvester. To illustrate the formula, we prove in § 1.5 three of the more important known Anzahl theorems by a uniform process. In particular, we give a proof of the extremely interesting new theorem of Kulakoff§. It may be well to mention here

* The original proof of Burnside, *Proc. London Math. Soc.* (2), 13 (1913), 6, is rather difficult to follow at one point.

† *Cf.* the chapter on p-groups in the book *Finite groups* (1916), by Miller, Blichfeldt, and Dickson.

‡ J. J. Sylvester, *Comptes rendus*, 96 (1883), 463.

§ A. Kulakoff, *Math. Annalen*, 104 (1931), 778.

that the results of § 4 (*cf.* in particular the remark at the end of the proof of Theorem 4.51) contain implicity a number of Anzahl theorems of a considerably more far-reaching character than the classical results just referred to—proved there, however, only for regular p-groups. For formal enunciations and for the corresponding results for the general p-group, we must refer to a subsequent paper.

Section 2 is devoted to a discussion of commutator subgroups, and in particular to the properties and mutual relations of the derived series θ and the two central series H and Z already defined. These relations have already been dealt with by the late Prof. Burnside in two papers published in these *Proceedings**; he did not, however, for the most part arrive at "best possible" results. We have attempted to generalize, as far as possible, this work of Burnside, at the cost, it is true, of considerable elaboration of the proofs. Thus, Burnside showed that every element of Z_{i+1} is permutable with every element of θ_i, or, more concisely, that $(\theta_i, Z_{i+1}) = E$. We improve this to $(\theta_i, Z_{2^i}) = E$. More precisely, we show that $\theta_i \leqslant H_{2^i}$ and $(H_j, Z_j) = E$ for all i and j respectively. Moreover, both of these results are "best possible". Again, for the order p^n of a p-group whose δ-th derived group $\theta_\delta > E$, Burnside gives the limits $n \geqslant \frac{1}{2}(\delta+1)(\delta+2)$ and $n \geqslant 3\delta$. We shall improve this to $n \geqslant 2^\delta + \delta$, though it is unlikely that this result is best possible. These and many similar results will be found proved in § 2.5, while in (2.4) we recall known definitions and elementary theorems. The proofs of the theorems of (2.5) depend on the use of a general law of commutation, (2.3), to which we attach some importance. Section 2.1 is devoted to explaining our notation (adopted from formal logic) for repeated commutators, the understanding of which is essential to the intelligibility of most of our subsequent work. In (2.2) we prove the (crucial) result that this notation may also be legitimately employed for commutator subgroups. The theorems of (2.8) are needed in § 4.

The most important result of § 2 from the point of view of subsequent applications is probably Theorem 2.53; for on its validity depends the usefulness of the formula which is proved in § 3. This formula, which holds in any group (in the sense explained in § 3.1), is an expression for $(PQ)^x$ as a product of powers of P, Q, and the various commutators of P and Q arranged (roughly speaking) in order of complication. For the details we must refer the reader to (3.1).

* *Proc. London Math. Soc.* (2), 11 (1912), 225, and 13 (1913), 6. In the first of these papers, Burnside also discusses the representations of p-groups by linear substitutions. With his important results in this direction we shall not be concerned here.

In § 4 we use the formula of § 3 to develop a theory of what we have called *regular* p-groups. (I hope, in a later paper, to deal with irregular groups on similar principles.) For the definition of regular the reader is referred to (4.1). It results from the formula of § 3 that any p-group of class less than p is regular; and, in particular, that any group of order p^n with $n \leqslant p$ is regular. Thus, if we confine our attention to the groups of order p^n for some particular n, then only a finite number of these groups are irregular. In this sense, then, regular groups are the rule, irregular ones the exception, since they occur only for the first few values of p.

The theory which is developed in § 4 shows that most of the classical theory of Abelian groups as given, *e.g.* in Burnside's *Theory of groups* (2nd edition), is valid for the more general class of regular groups. Thus in (4.2) we prove that in a regular group the order of a product can never exceed the orders of all the factors; and, dually, that the product of any number of p^α-th powers is itself a p^α-th power. It follows that, in a regular group, the p^α-th powers form a characteristic subgroup $\mathfrak{V}_\alpha$ and the elements whose orders divide p^β form a characteristic subgroup Ω_β; just as in an Abelian group. In § 4.3 we introduce invariants having precisely the same meaning as the invariants of an Abelian p-group. Finally, in (4.5), we prove an analogue of the fundamental *Basis theorem* of Abelian group theory, viz. that in any regular p-group G, whose invariants are $\mu_1, \mu_2, \ldots, \mu_\omega$, elements $P_1, P_2, \ldots, P_\omega$ of orders $p^{\mu_1}, p^{\mu_2}, \ldots, p^{\mu_\omega}$ respectively, can always be found such that any element P of G is uniquely expressible in the form

$$P = P_1^{x_1} P_2^{x_2} \ldots P_\omega^{x_\omega}$$

with

$$0 \leqslant x_i < p^{\mu_i}$$

for $i = 1, 2, \ldots, \omega$, the order p^n of G being given by

$$n = \mu_1 + \mu_2 + \ldots + \mu_\omega.$$

It is hardly necessary to add that the proofs of these theorems usual when G is Abelian no longer apply in the general case owing to the non-commutability of the elements. We have therefore been compelled to devise new ones.

The results mentioned hitherto are all generalizations of well-known properties of Abelian groups. In § 4.4 we prove a system of theorems for which Abelian groups provide only trivial analogies. These concern the relations between the "*order-power*"-*structure*—as determined by the invariants or by the characteristic series $\mathfrak{V}$ and Ω—and the "*commutator*"-*structure*—as determined by the characteristic series θ, H, and Z—of a

regular group. These relations are all derived from Theorems 4.23 and 4.27, and perhaps the most interesting of them are the identities

$$H_\beta(\mathfrak{V}_\alpha) = \mathfrak{V}_{\alpha\beta}(H_\beta),$$

and

$$\theta_\beta(\mathfrak{V}_\alpha) = \mathfrak{V}_{2^\beta\alpha}(\theta_\beta).$$

We also prove, *e.g.*, that $(\mathfrak{V}_\alpha, \Omega_\alpha) = E$, *i.e.* that every element of order p^α or less is permutable with every p^α-th power.

It may be well to mention that §§4.4 and 4.5 are quite independent of one another.

1. *Minimal bases. Anzahl theorems.*

1.1. *Summary of elementary results.* If G is any p-group, then:

1.11. The central Z of G is greater than E.

1.12. G is soluble and the factor-groups of a chief-series are all of order p.

1.13. If K is any self-conjugate subgroup of G, of order p, then $K \leqslant Z$.

1.14. If L is any proper subgroup of G, and N the normalizer of L in G, then $L < N$.

1.15. Every subgroup of index p in G is self-conjugate.

1.16. Every proper subgroup of G is contained in some subgroup of index p.

1.17. Every self-conjugate subgroup K of G of order greater than or equal to p contains some self-conjugate subgroup of G of order p.

1.18. A group of order p^2 is necessarily Abelian.

1.19. Every self-conjugate subgroup of G of index p^2 contains the derived group of G.

1.2. *Burnside's basis theorem.*

In the present section, we give a new and simplified proof of the fundamental theorem of Burnside already referred to, on the possible modes of generation of a p-group; it shows, I think, that this theorem must be reckoned along with the elementary results of (1.1).

Definition. G being any p-group, a subgroup of index p in G will be called a *maximum subgroup* of G. The common part D of all the

maximum subgroups of G will be called the *principal* subgroup of G. Any subgroup M of G satisfying $D \leqslant M \leqslant G$ will be called a *major subgroup* of G.

THEOREM 1.21. *D is a characteristic subgroup of G and G/D is Abelian and of type* (1, 1, ..., 1). *The subgroup M ($< G$) is a major subgroup of G if and only if* (i) *it is self-conjugate in G, and* (ii) *G/M is Abelian of type* (1, 1, ..., 1).,

Proof. Since every maximum subgroup is self-conjugate (1.15), their common part D is also self-conjugate; and, since every maximum subgroup contains the derived group (1.19), so does D, and hence G/D is Abelian. If P is any element of G, and M_1 any maximum subgroup, then P^p belongs to M_1. Thus P^p belongs to every maximum subgroup of G, and so it must also belong to D. Hence the Abelian group G/D contains no element of order p^2, and so it is of type (1, 1, ..., 1). And it follows that, for any major subgroup $M(< G)$, G/M is Abelian and of type (1, 1, ..., 1). Conversely, if M is self-conjugate with G/M Abelian and of type (1, 1, ..., 1), it can clearly be expressed as the common part of a number of maximum subgroups of G. Hence it will contain D, and so it will be a major subgroup of G.

Since any automorphism of G must permute among themselves the maximum subgroups of G, it follows that D is a characteristic subgroup of G.

The invariant d. We shall denote by p^d the index in G of the principal subgroup of G. The number $d = d(G)$ is an important arithmetical invariant of the group, whose significance is made clear by Burnside's basis theorem (1.2).

Definition. Any set of elements $P_1, P_2, \ldots, P_r$ of the group G which "generate" G, *i.e.* which are such that $\{P_1, P_2, \ldots, P_r\} = G$, will be called a *basis* of G. This is in accordance with current usage in ideal-theory. Clearly a given set of elements of G is a basis of G if and only if there is no *proper* subgroup of G containing all the members of the set.

THEOREM 1.2 (Burnside). *Let G be any p-group and D the principal subgroup of G. The order of G/D being p^d, let $P_1, P_2, \ldots, P_d$ be any set of representatives in G of any minimal basis of the Abelian group G/D. Then $\{P_1, P_2, \ldots, P_d\} = G$, i.e. $P_1, P_2, \ldots, P_d$ is a basis of G. A basis chosen in this way will be called a* minimal basis *or M.B. of G. Then every basis of G possesses at least d members, and contains among its members the elements of some minimal basis of G.*

Proof. Suppose that $\{P_1, P_2, \ldots, P_d\} \neq G$, therefore is less than G. Then, by (1.16), $P_1, P_2, \ldots, P_d$ all belong to some maximum subgroup M_1 of G. But, since $M_1 \geqslant D$, this implies that $P_1D, P_2D, \ldots, P_dD$ all belong to M_1. Hence $P_1D, P_2D, \ldots, P_dD$ cannot be a basis of G/D; and this is a contradiction.

Conversely, let $\{P_1, P_2, \ldots, P_r\} = G$. Then

$$\{P_1D, P_2D, \ldots, P_rD\} = G.$$

Hence, among the r expressions $P_1D, P_2D, \ldots, P_rD$, there must occur the members of some minimal basis of G/D. If we suppose these to be

$$P_{i_1}D, P_{i_2}D, \ldots, P_{i_d}D,$$

then, by definition, $P_{i_1}, P_{i_2}, \ldots, P_{i_d}$ is a minimal basis of G; and incidentally $r \geqslant d$.

1.3. In the present section we shall be concerned with the *group of automorphisms* A of the p-group G. We shall obtain a limitation for its order in terms of the order of G and the invariant d defined in (1.2).

Definition. Two M.B.'s of G will be called *equivalent*:

$$P_1, P_2, \ldots, P_d \sim Q_1, Q_2, \ldots, Q_d,$$

whenever there is an automorphism of G which changes each element P_i of the first M.B. into the corresponding element Q_i of the second. Two M.B.'s, it will be noted, are considered to be the same only if they consist of the same elements given in the same order. Thus, altogether, G obviously has exactly

$$f(p, n, d) = p^{d(n-d)}(p^d-1)(p^d-p)\ldots(p^d-p^{d-1}) \tag{1.31}$$

distinct M.B.'s.

The relation of equivalence just defined clearly satisfies the three usual requirements; it is symmetrical, reflexive, and transitive. The M.B.'s of G must fall, therefore, into a number of distinct *classes*, two M.B.'s being equivalent if and only if they belong to the same class.

Let $B_1, B_2, \ldots, B_a$ denote the distinct M.B.'s of any one of these classes. Then the group A permutes these a M.B.'s transitively among themselves. But it also permutes them *regularly*. For, suppose there to be two automorphisms I and I' of G, both of which changed B_i into B_j. Then $I'I^{-1}$ would change B_i into itself, which means that it would change each of the d elements of B_i into itself. Since these d elements form a basis of G, $I'I^{-1}$ must therefore change every element of G into itself. Hence $I' = I$.

A being both transitive and regular on the a symbols, it results that a is precisely the order of A. And (incidentally) also that every class of M.B.'s contains exactly a members. Hence, if b is the number of classes of M.B.'s, we have

$$ab = f(p, n, d). \tag{1.32}$$

The order of the group of automorphisms A of G must, therefore, in any case divide the number $f(p, n, d)$.

The number b has a very simple interpretation. It is equal (in a certain sense) to the *number of distinct types of generating relations* by means of which G may be defined; it being understood that the generating elements are to be d in number, *i.e.* are to form an M.B. By this we mean, more precisely, that the generating relations satisfied by two different M.B.'s—say $P_1, P_2, \ldots, P_d$ and $Q_1, Q_2, \ldots, Q_d$—are of the same type if and only if the identity of any power-product of the P's,

$$f(P_1, P_2, \ldots, P_d) = E,$$

implies the identity of the corresponding power-product of the Q's, *i.e.*

$$f(Q_1, Q_2, \ldots, Q_d) = E,$$

and conversely.

The limit $f(p, n, d)$ for the order of the group of automorphisms of G is actually attained in the trivial case $d = n$ of the Abelian groups of type $(1, 1, \ldots, 1)$. In general, a may be expected to be large, if G can be defined by a "highly symmetrical" system of generating relations. There exists in actual fact an extensive series of non-trivial groups for which $a = f(p, n, d)$, and which may be defined as follows. $U_{p,d,k,l}$ is to denote the group generated by the d elements $P_1, P_2, \ldots, P_d$ subject only to the d relations

$$P_\alpha^{p^l} = E \quad (\alpha = 1, 2, \ldots, d)$$

together with the relations which express that every k-fold repeated commutator of the P's (for definition, *cf.* 2.1) shall be the identity. It follows from the results of the present paper* that, provided only that $k \leqslant p$, the group $U_{p,d,k,l}$ so defined has the maximum number of automorphisms, viz. $f(p, n, d)$. (The calculation of n, *i.e.* of the order p^n of U, seems to be a rather difficult problem.)

The method by which (1.32) was proved may be made to give a more precise result, which concerns the structure of A. If, for instance, A_1 denotes the self-conjugate subgroup of A, consisting of all the auto-

* For a formal proof, we must refer the reader to a subsequent paper.

morphisms of G which give rise to the identical automorphism of G/D, then it is clear that *the order of* A_1 *must divide* $p^{d(n-d)}$. That is, *an automorphism of* G *which multiplies each element of* G *by an element of* D *is of order a power of* p. The factor-group A/A_1 must naturally be isomorphic with a subgroup of the group of isomorphisms of G/D which is of order

$$(p^d-1)(p^d-p)\dots(p^d-p^{d-1}).$$

If G is known to possess a characteristic major subgroup other than G and D, then a further limitation on A/A_1 can easily be written down.

1.4. *An enumeration principle.*

Notation. We shall denote by A_{p^α} an Abelian group of order p^α and type $(1, 1, \dots, 1)$; and by $\phi_{\alpha,\rho}^{(p)} = \phi_{\alpha,\rho}$ the number of subgroups A_{p^ρ} contained in any given A_{p^α} $(0 \leqslant \rho \leqslant \alpha)$.

It is well known* that $\phi_{\alpha,\rho} = \phi_{\alpha,\alpha-\rho}$, and also that

$$\phi_{\alpha,\rho} = \frac{(p^\alpha-1)(p^{\alpha-1}-1)\dots(p^{\alpha-\rho+1}-1)}{(p^\rho-1)(p^{\rho-1}-1)\dots(p-1)}.$$

We shall begin by proving the formula

$$\phi_{\alpha+1,\rho} = \phi_{\alpha,\rho} + p^{\alpha-\rho+1}\phi_{\alpha,\rho-1} \quad \text{for} \quad 1 \leqslant \rho \leqslant \alpha. \tag{1.41}$$

Proof. Let A be any $A_p^{\alpha+1}$ and B any subgroup of index p in A, so that B is an A_{p^α}. Then $\phi_{\alpha+1,\rho}$ is equal to the number of subgroups A_{p^ρ} of A, and of these $\phi_{\alpha,\rho}$ are contained in B. It remains to show that the number *not* contained in B is equal to $p^{\alpha-\rho+1}\phi_{\alpha,\rho-1}$. Let H then be an A_{p^ρ} of A not contained in B and let $K = H \wedge B$ (the common part or meet of H and B). Then, since A/B is of order p and $K \neq H$, it follows that H/K is also of order p. Hence K is one of the $\phi_{\alpha,\rho-1}A_p^{\rho-1}$'s of B. And the number of H's which contain a given K is equal to the number of A_{p^1}'s H/K of the $A_p^{\alpha-\rho+2}A/K$ which are not contained in the given subgroup $A_p^{\alpha-\rho+1} = B/K$ of A/K. But this number is

$$(p^{\alpha-\rho+2}-p^{\alpha-\rho+1})/(p-1) = p^{\alpha-\rho+1}.$$

Thus each K belongs to $p^{\alpha-\rho+1}$ H's and the total number of H's is therefore $p^{\alpha-\rho+1}\phi_{\alpha,\rho-1}$.

It is obvious from (1.41) that

$$\phi_{\alpha+\beta,\rho} \equiv \phi_{\alpha,\rho} \pmod{p^{\alpha-\rho+1}} \tag{1.42}$$

for all $\beta \geqslant 0$. Since $\phi_{\rho,\rho} = 1$, this shows that all the ϕ's are $\equiv 1 \pmod{p}$.

* Burnside, *Theory of groups*, 2 ed., 111.

And if $0 < \rho < \alpha$, we have $\phi_{\alpha,\rho} \equiv 1+p \pmod{p^2}$, because

$$\phi_{\alpha,1} = \phi_{\alpha,\alpha-1} = 1+p+p^2+\ldots+p^{\alpha-1}.$$

Another immediate consequence of (1.41) is the important identity

$$(1.43)\quad (x-1)(x-p)(x-p^2)\ldots(x-p^{\alpha-1})$$
$$= x^\alpha - \phi_{\alpha,1}x^{\alpha-1} + p\phi_{\alpha,2}x^{\alpha-2} - p^3\phi_{\alpha,3}x^{\alpha-3}+\ldots$$
$$+(-1)^\rho p^{\frac{1}{2}\rho(\rho-1)}\phi_{\alpha,\rho}x^{\alpha-\rho}+\ldots+(-1)^\alpha p^{\frac{1}{2}\alpha(\alpha-1)},$$

for $\alpha = 1, 2, \ldots,$ *and all* x.

Proof. For $\alpha = 1$, the result is trivial. We may argue, therefore, by induction over α. Supposing (1.43) to be true as it stands, multiply both sides by $x-p^\alpha$. The coefficient of $x^{\alpha-\rho+1}$ on the right-hand side then becomes $(-1)^\rho p^{\frac{1}{2}\rho(\rho-1)}\phi_{\alpha,\rho} - p^\alpha.(-1)^{\rho-1}p^{\frac{1}{2}(\rho-1)(\rho-2)}\phi_{\alpha,\rho-1}$; and, by (1.41), this is equal to $(-1)^\rho p^{\frac{1}{2}\rho(\rho-1)}\phi_{\alpha+1,\rho}$. This is true for $1 \leqslant \rho \leqslant \alpha$. Further, the coefficient of $x^{\alpha+1}$ is obviously equal to 1 and the constant term is $(-1)^{\alpha+1}p^{\frac{1}{2}\alpha(\alpha+1)}$. The formula then holds for $\alpha+1$ if it holds for α. It must, therefore, be true generally.

We now come to the main result of the present section, which consists in an application of the formula (1.43) adapted to the discussion of "Anzahl" theorems.

THEOREM 1.4. (Enumeration Principle.) *Let G be any p-group and let M_α denote a typical major subgroup of index p^α in G; $0 \leqslant \alpha \leqslant d = d(G)$. Let C be any class whose members are elements or subgroups or (more generally) sets of elements of G; and let each member of C belong to at least one maximum subgroup M_1 of G. Let $n(M_\alpha)$ denote the number of members of C which belong to M_α. Then*

$$n(M_0) - \sum_{(M_1)} n(M_1) + p\sum_{(M_2)} n(M_2) - p^3\sum_{(M_3)} n(M_3)+\ldots$$
$$+(-1)^\alpha p^{\frac{1}{2}\alpha(\alpha-1)}\sum_{(M_\alpha)} n(M_\alpha)+\ldots+(-1)^d p^{\frac{1}{2}d(d-1)}n(M_d) = 0,$$

the sum $\sum_{(M_\alpha)}$ *being taken over the* $\phi_{d,\alpha}$ *subgroups* M_α *of* G.

Naturally, $M_d = D =$ the principal subgroup of G, and $M_0 = G$. The formula, therefore, provides an expression for the *total* number of members of C, viz. $n(G) = n(M_0)$, in terms of the numbers $n(M_\alpha)$ $(\alpha > 0)$, giving the numbers of members of C contained in the *proper* major subgroups of G.

Proof of (1.4). Let γ be any member of C. Then the proof consists in showing that the number of times that γ is "counted" in the left-hand side

of (1.4) is zero, for all γ; it being understood that, if $n_\alpha(\gamma)$ is the number of M_α's which contain γ, then the term $(-1)^\alpha p^{\frac{1}{2}\alpha(\alpha-1)} \sum_{M_\alpha} n(M_\alpha)$ contributes just $(-1)^\alpha p^{\frac{1}{2}\alpha(\alpha-1)} n_\alpha(\gamma)$ to the "count" of γ.

Let M_ρ be the common part of all the major subgroups of G which contain γ. M_ρ is itself a major subgroup of G, so that our notation is justified. Moreover, since γ is contained (by hypothesis) in some M_1, we must have $\rho > 0$. The number $n_\alpha(\gamma)$ is clearly equal to the number of M_α's of G which contain M_ρ, and this is $\phi_{\rho,\alpha}$ for $\alpha \leqslant \rho$ and 0 for $\alpha > \rho$. Hence the number of times γ is counted is equal to

$$1-\phi_{\rho,1}+p\phi_{\rho,2}-p^3\phi_{\rho,3}+\ldots+(-1)^\alpha p^{\frac{1}{2}\alpha(\alpha-1)}\phi_{\rho,\alpha}+\ldots+(-1)^\rho p^{\frac{1}{2}\rho(\rho-1)};$$

ρ being greater than 0, we see that this number vanishes by taking in (1.43) the values $x = 1$ and $\alpha = \rho$.

1.5. *"Anzahl" theorems.*

In the present section we shall endeavour to prove the more important known "Anzahl" theorems of p-group theory by a uniform method based on the enumeration principle of (1.4).

THEOREM 1.51. *The number of subgroups of order p^m of a p-group of order p^n is congruent to* 1 (mod p) *for* $0 \leqslant m \leqslant n$.

Proof. The theorem is trivial for $n = 1$, so that we may employ induction over n. Excluding the trivial case $m = n$, we may apply (1.4) with C equal to the set of all subgroups of order p^m of G (of order p^n). This gives

$$n(M_0) \equiv \sum_{(M_1)} n(M_1) \pmod{p}.$$

Since $m \leqslant n-1$, we have $n(M_1) \equiv 1 \pmod{p}$ by hypothesis. Hence $n(M_0) = \phi_{d,1}$ the number of M_1's of G; and, since $\phi_{d,1} \equiv 1 \pmod{p}$, the result follows.

THEOREM 1.52. *If G is a non-cyclical p-group of odd order p^n, i.e. $p > 2$ and $d > 1$, and if $0 < m < n$, then the number of subgroups of G of order p^m is congruent to* $1+p \pmod{p^2}$. (Kulakoff's theorem*.)

Proof. For $m = n-1$, the number in question is equal to

$$\phi_{d,1} = 1+p+p^2+\ldots+p^{d-1},$$

and this is congruent to $1+p \pmod{p^2}$, since $d > 1$. Suppose then that $m < n-1$. We may apply (1.4) with C (as before) equal to the set of all sub-

* A. Kulakoff, *Math. Annalen*, 101 (1931), 779.

groups of order p^m of G. This gives $n(M_0) \equiv \sum_{(M_1)} n(M_1) - p \sum_{(M_2)} n(M_2) \pmod{p^2}$. But, since $m \leqslant n-2$, each $n(M_2) \equiv 1 \pmod{p}$, by (1.51). And, since the total number of M_2's of G, viz. $\phi_{d,2} \equiv 1 \pmod{p}$, we have

$$n(M_0) \equiv \sum_{(M_1)} n(M_1) - p \pmod{p^2}.$$

Now the theorem is trivial for $n = 2$, the non-cyclical group of order p^2 having $1+p$ subgroups of order p. We may, therefore, apply induction over n, and assume that it holds for each M_1 of G.

There are now two cases to distinguish, according to whether G possesses only non-cyclical M_1's or not.

First, let $d(M_1) \geqslant 2$ for each M_1 of G. Then, since $m < n-1$, we have $n(M_1) \equiv 1+p \pmod{p^2}$ for each M_1 by hypothesis. The total number of M_1's being $\phi_{d,1} = 1+p+p^2+\ldots+p^{d-1}$ is also congruent to $1+p \pmod{p^2}$, since $d > 1$. Hence $n(M_0) \equiv (1+p)^2 - p \equiv 1+p \pmod{p^2}$, and so the result holds for G.

Secondly, suppose that G possesses a cyclical M_1. We must then have $d = 2$ (since in any case $d > 1$). Such groups are, of course, well known; p being odd, they all possess exactly p cyclical M_1's, the remaining M_1 being non-cyclical. This fact is most easily proved by considering the particular case $n = 3$. There are then two types of group of the kind in question, viz. the Abelian group of type (2, 1) and the non-Abelian group of order p^3 possessing elements of order p^2; in both cases, there are just p cyclical subgroups of order p^2, p being odd. If now $n > 3$, let Δ be a self-conjugate subgroup of G of index p in D. To a cyclical M_1 of G will correspond a cyclical M_1 of G/Δ. But also conversely, since, if G possesses a cyclical M_1, D must be cyclical, and so a cyclical M_1 of G/Δ will correspond to a cyclical M_1 of G. Hence G has p cyclical M_1's, each giving $n(M_1) = 1$; and for the remaining non-cyclical M_1 we have, by hypothesis, $n(M_1) \equiv 1+p \pmod{p^2}$. Therefore

$$\sum_{(M_1)} n(M_1) \equiv p \,.\, 1 + 1 + p,$$

and so $n(M_0) \equiv 1+2p-p \equiv 1+p \pmod{p^2}$.

THEOREM 1.53. *If G is a p-group of order p^n with $p > 2$ and $d > 1$, then the number of cyclical subgroups of G of order p^m is $\equiv 0 \pmod{p}$ for $1 < m < n$.* (Miller*.)

Proof. We may suppose that $m < n-1$, since we have shown during the proof of 1.52 that, if G has a cyclical M_1, it will have just p cyclical M_1's.

* G. A. Miller, *Proc. London Math. Soc.* (2, 2 (1904), 142–143.

The theorem is therefore true for $n = 3$, and we may apply induction over n, and suppose it to hold for all proper subgroups of G. Applying (1.4) as before, we have $n(M_0) \equiv \sum_{(M_1)} n(M_1)$. If M_1 is non-cyclical, then, since $m < n-1$, we have, by hypothesis, $n(M_1) \equiv 0 \pmod{p}$. But, if there is a cyclical M_1, then, as we have seen, there are just p of them, and for each of these p cyclical M_1's we have $n(M_1) = 1$. Hence, in any case, $n(M_0) \equiv 0 \pmod{p}$.

THEOREM 1.61. *The number of subgroups of index p^α in G $(0 \leqslant \alpha \leqslant d)$ is congruent to $\phi_{d,\alpha} \pmod{p^{d-\alpha+1}}$; the number of non-major subgroups of index p^α is therefore congruent to $0 \pmod{p^{d-\alpha+1}}$.*

Proof. Since $\phi_{d,\alpha} \equiv \phi_{\delta,\alpha} \pmod{p^{\delta-\alpha+1}}$ for all $d \geqslant \delta$, by (1.42), Theorem (1.61) is equivalent to the statement that, if $d(G) \geqslant \delta$, then the number of subgroups of index p^α in G is congruent to $\phi_{\delta,\alpha} \pmod{p^{\delta-\alpha+1}}$. Using the induction argument as in the previous two cases [(1.61) being obvious when $D = E$, since then the number in question is by definition exactly $\phi_{d,\alpha}$], we have, from (1.4),

$$n(M_0) \equiv \sum_{(M_1)} n(M_1) - p \sum_{(M_2)} n(M_2) + \ldots + (-1)^{\alpha-1} p^{\frac{1}{2}\alpha(\alpha-1)} \sum_{(M_\alpha)} n(M_\alpha).$$

By hypothesis, for $0 < \beta \leqslant \alpha$, $n(M_\beta)$ being the number of subgroups of index $p^{\alpha-\beta}$ in M_β is congruent to $\phi_{d-\beta,\alpha-\beta} \pmod{p^{(d-\beta)-(\alpha-\beta)+1} = p^{d-\alpha+1}}$, since

$$d(M_\beta) \geqslant d - \beta.$$

Hence $\pmod{p^{d-\alpha+1}}$ we have

$$n(M_0) \equiv \phi_{d,1}\phi_{d-1,\alpha-1} - p\phi_{d,2}\phi_{d-2,\alpha-2} + \ldots + (-1)^{\alpha-1} p^{\frac{1}{2}\alpha(\alpha-1)} \phi_{d,\alpha}\phi_{d-\alpha,0}.$$

But, by the enumeration principle (1.4), the right-hand side is simply the number of subgroups of index p^α of an $A_p{}^d$, and so is equal to $\phi_{d,\alpha}$. (We have supposed that $\alpha > 0$.)

2. *Commutator subgroups.*

2.1. *Notation for commutators.*

It will be essential for our subsequent work to be able to express repeated commutators in a form more compact than the usual bracket notation; and the multiple-stop notation of formal logic is here extremely appropriate.

We write

$$P^{-1}Q^{-1}PQ = (P, Q) = (Q, P)^{-1}, \tag{2.11}$$

so that

$$PQ = QP.(P, Q), \tag{2.12}$$

and

$$Q^{-1}PQ = P.(P, Q), \tag{2.13}$$

where P and Q are any two elements of a given group. Further, in place of $\big((P, Q), R\big)$ we shall write (P, Q, R); and, more generally, we define by recurrence the symbol

$$(P_1, P_2, \ldots, P_n) = \big((P_1, P_2, \ldots, P_{n-1}), P_n\big). \tag{2.14}$$

An expression such as (2.14) will be called a *simple n-fold commutator*; and the elements $P_1, P_2, \ldots, P_n$ will be called its *components*.

Before defining the more complicated commutators, it will be convenient to prove two useful formulae, viz.

$$(P, QR) = (P, R)(P, Q)(P, Q, R) \tag{2.15}$$

and

$$(PQ, R) = (P, R)(P, R, Q)(Q, R). \tag{2.16}$$

This may, of course, be done by direct calculation. For, by (2.13), we have $(P, Q)(P, Q, R) = R^{-1}(P, Q)R = R^{-1}P^{-1}Q^{-1}PQR$, and so the right-hand side of (2.15) is equal to

$$P^{-1}R^{-1}PR.R^{-1}P^{-1}Q^{-1}PQR = P^{-1}R^{-1}Q^{-1}PQR = (P, QR).$$

Again, by (2.13),

$$(P, R)(P, R, Q) = Q^{-1}(P, R)Q = Q^{-1}P^{-1}R^{-1}PRQ,$$

and so the right-hand side of (2.16) is equal to

$$Q^{-1}P^{-1}R^{-1}PRQ.Q^{-1}R^{-1}QR = Q^{-1}P^{-1}R^{-1}PQR = (PQ, R).$$

Complex commutators. Let $P_1, P_2, \ldots, P_r$ be any r elements of a group G. We shall define by recurrence what we mean by a *complex commutator of weight w in the components* $P_1, P_2, \ldots, P_r$. The complex commutators of weight 1 are the elements $P_1, P_2, \ldots, P_r$ themselves. Supposing the complex commutators of all weights less than w to have been defined already, then those of weight w consist of all the expressions of the form (S, T), where S and T are any complex commutators of weights w_1 and w_2 in the components $P_1, P_2, \ldots, P_r$ respectively, such that $w_1 + w_2 = w$.

Thus every complex commutator of weight greater than 1 is the commutator of two complex commutators of less weight. The weight of a complex commutator is, of course, always relative to the choice of components; these must be specified before the weight can be determined. For example, (P, Q, R) is of weight 3 in the three components P, Q, and R; but of weight 2 in the two components (P, Q) and R. It will usually happen, however, that the components in terms of which the weight of a commutator is to be measured are obvious from the context and do not need to be mentioned explicitly.

Notation for complex commutators. The simplest case of a non-simple commutator is the commutator of two simple commutators:

$$\big((P_1, P_2, \ldots, P_r), \ (Q_1, Q_2, \ldots, Q_s)\big).$$

This we denote by $(P_1, P_2, \ldots, P_r; \ Q_1, Q_2, \ldots, Q_s)$. The commutator of this expression by $(R_1, R_2, \ldots, R_t)$ will be written

$$(P_1, P_2, \ldots, P_r; \ Q_1, Q_2, \ldots, Q_s; \ R_1, R_2, \ldots, R_t).$$

And so on.

With the help of commas, semi-colons, triple stops, quadruple stops, and so on (to as high a multiplicity as may be necessary), we can write down a linear expression for the most general complex commutator (S, T) in terms of its components $P_1, P_2, \ldots, P_r$. The rule will be: write, first, the linear expression for S, omitting the right-hand bracket; to the right of it write the linear expression for T, omitting the left-hand bracket; and between them place a stop of higher order than any stop occurring in T and of an order at least as high as any occurring in S—the order of the stop between S and T being as small as is consistent with these two conditions. Thus,

$$\big((R_1, R_2, \ldots, R_t), \ (P_1, P_2, \ldots, P_r; \ Q_1, Q_2, \ldots, Q_s)\big)$$

would be written

$$(R_1, R_2, \ldots, R_t \cdot;\cdot \ P_1, P_2, \ldots, P_r; \ Q_1, Q_2, \ldots, Q_s).$$

2.2. *Commutator subgroups; the substitution principle.* With the notation of the preceding section, let $f(P_1, P_2, \ldots, P_r)$ be the symbol of a complex commutator in the components $P_1, P_2, \ldots, P_r$; and let $K_1, K_2, \ldots, K_r$ be any r subgroups of the group G—not necessarily distinct. Then, by the symbol

$$f(K_1, K_2, \ldots, K_r) \tag{2.21}$$

we shall denote the least subgroup of G which contains (or the subgroup generated by) all the expressions which can be obtained from the symbol $f(K_1, K_2, \ldots, K_r)$ by substituting in it for each $K_i (i = 1, 2, \ldots, r)$ in all possible ways an arbitrary element of K_i, the same element of K_i not being necessarily substituted for two different occurrences of K_i in f. [Owing to this last clause, the generators of $f(K_1, K_2, \ldots, K_r)$ are not necessarily all of the form $f(P_1, P_2, \ldots, P_r)$.]

For example, (H, K) is the group generated by all elements of the form (P, Q) with P in H and Q in K. From (2.11) it follows that

$$(H, K) = (K, H). \tag{2.22}$$

The equation $(H, K) = E$ is equivalent to the statement that every element of H is permutable with every element of K.

2.23. If the subgroups $K_1, K_2, \ldots, K_r$ are all self-conjugate in G, then $f(K_1, K_2, \ldots, K_r)$ is also self-conjugate in G, and the generators of the definition fall into a number of complete sets of conjugates in G.

We leave the proof to the reader.

The use of the terms weight and component which has been described for the complex commutators may also be applied, *mutatis mutandis*, to the commutator subgroups $f(K_1, K_2, \ldots, K_r)$. This is justified by the following very important principle.

THEOREM 2.24. *If we have identically**

$$f(P_1, P_2, \ldots, P_r) \equiv \phi(Q_1, Q_2, \ldots, Q_s),$$

where, for $i = 1, 2, \ldots, s$, $Q_i \equiv Q_i(P_1, P_2, \ldots, P_r)$ *is a complex commutator in the components* $P_1, P_2, \ldots, P_r$ *(not all of which need actually occur in each* Q_i*); then, for any* r *self-conjugate subgroups* $K_1, K_2, \ldots, K_r$ *of the group* G*, the two commutator subgroups* $f(K_1, K_2, \ldots, K_r)$ *and* $\phi(L_1, L_2, \ldots, L_s)$*, where* $L_i = Q_i(K_1, K_2, \ldots, K_r)$ *for* $i = 1, 2, \ldots, s$*, are the same.*

Proof. Let $f(P_1, P_2, \ldots, P_r) \equiv (R, S)$, where

$$R \equiv f_1(P_1, P_2, \ldots, P_r) \equiv \phi_1(Q_1, Q_2, \ldots, Q_s)$$

and $S \equiv f_2(P_1, \ldots, P_r) = \phi_2(Q_1, \ldots, Q_s)$. Further, let $H_1 = f_1(K_1, \ldots, K_r)$ and $H_2 = f_2(K_1, \ldots, K_r)$; $H_1' = \phi_1(L_1, \ldots, L_s)$ and $H_2' = \phi_2(L_1, \ldots, L_s)$. Then it will suffice to prove that

$$\text{(i) } H_1 = H_1', \quad \text{(ii) } H_2 = H_2', \quad \text{(iii) } f(K_1, K_2, \ldots, K_r) = (H_1, H_2),$$
$$\text{(iv) } \phi(L_1, L_2, \ldots, L_s) = (H_1', H_2').$$

* That is, if f is the expression obtained by substituting in ϕ for the Q_i's their expressions in terms of the P_i's, the multiplicities of the stops being raised if necessary.

We may apply induction over the weight of f, since the theorem is trivial when that weight is 1. Suppose then that the theorem holds for all complex commutators of weight less than that of f. Then it must hold for both f_1 and f_2; this gives the results (i) and (ii). There remain (iii) and (iv), and both of these are instances of the general result (2.24) for the particular case in which $s = 2$, $Q_1 = R$, $Q_2 = S$, and

$$\phi(Q_1, Q_2, \ldots, Q_s) \equiv (R, S).$$

Hence we need prove only (iii).

Now, since $f(P_1, P_2, \ldots, P_r) \equiv (R, S)$, it follows that the generators (by definition) of $f(K_1, K_2, \ldots, K_r) = K$ are precisely the commutators (R_i, S_j), where R_i goes through the generators of H_1 and S_j independently through those of H_2. The K_i's being self-conjugate in G, both the R_i's and the S_j's form a number of complete sets of conjugates in G, *i.e.* for any P of G, $P^{-1} R_i P$ is an $R_{i'}$ and $P^{-1} S_j P$ is an $S_{j'}$. On the other hand, the generators of (H_1, H_2) are precisely the commutators (U, V), where U is any product of R_i's and V any product of S_j's. It is clear then that $(H_1, H_2) \geqslant K$, since every generator of K is at the same time a generator of (H_1, H_2). To prove the required result that $(H_1, H_2) = K$, therefore, we need only show that each of the commutators (U, V) can be expressed as a product of commutators (R_i, S_j); for then each (U, V) will be contained in K, and so $(H_1, H_2) \leqslant K$.

Let U be a product of ρ R_i's and V a product of σ S_j's. If $\rho = \sigma = 1$, then (U, V) is itself an (R_i, S_j). We may apply induction, therefore, over $\rho + \sigma$. Suppose, first, that $\rho > 1$, and write $U = \bar{U} R_a$, where $\bar{U}$ is a product of $\rho - 1$ R_i's. Then $(U, V) = (\bar{U} R_a, V) = R_a^{-1}(\bar{U}, V) R_a . (R_a, V)$ by (2.16). Now (R_a, V) is a (U, V) having $1 + \sigma$ in place of $\rho + \sigma$, and, since $\rho > 1$, it is by hypothesis a product of (R_i, S_j)'s. Similarly, $(\bar{U}, V)$ is a (U, V) having $\rho - 1 + \sigma$ in place of $\rho + \sigma$; and so, by hypothesis, it is also a product of (R_i, S_j)'s. But the R_i's and S_j's form a number of complete sets of conjugates in G, and so $R_a^{-1}(R_i, S_j) R_a$ is also an $(R_{i'}, S_{j'})$ with $R_{i'} = R_a^{-1} R_i R_a$, $S_{j'} = R_a^{-1} S_j R_a$. Hence $R_a^{-1}(\bar{U}, V) R_a$ is a product of (R_i, S_j)'s and, therefore, finally so is (U, V), assuming that $\rho > 1$. If, however, $\rho = 1$, we may assume that $\sigma > 1$ and write $V = \bar{V} S_j$. Then $(U, V) = (U, \bar{V} S_j) = (U, S_j) . S_j^{-1}(U, \bar{V}) S_j$ by (2.15), and the argument proceeds as before. Both (U, S_j) and $(U, \bar{V})$ are products of (R_i, S_j)'s by the induction hypothesis; and so is $S_j^{-1}(U, \bar{V}) S_j$ therefore, and hence also (U, V). This proves the result in every case.

As an illustration (the simplest possible one) of (2.24) we may consider the group $(G, G; G, G)$. It is, by definition, generated by all the commutators of the form $(P, Q; R, S)$, where P, Q, R, S are arbitrary

elements of G. By (2.24), it is the same as (G_1, G_1), where $G_1 = (G, G)$ is the first derived group of G; in other words, $(G, G; G, G)$ is the second derived group of G. Similarly, for example, $(G, G; G, G; G, G; G, G)$ is the third derived group of G, and so on. Thus we may express the i-th derived group of G as a complex commutator subgroup of weight 2^i in the single component G.

As another illustration, let $(L, M, \ldots)$ be any commutator subgroup whose first two components are the groups L and M self-conjugate in G. Then $(L, M) = (M, L)$ is a component of $(L, M, \ldots)$ and, by (2.24), therefore, we have $(L, M, \ldots) = \big((L, M), \ldots\big) = \big((M, L), \ldots\big) = (M, L, \ldots)$.

2.3. THEOREM 2.3. *If G is any group and L, M, N are any three self-conjugate subgroups of G (not necessarily distinct), then each of the three groups $A = (L, M, N)$, $B = (M, N, L)$, and $C = (N, L, M)$ is contained in the join of the other two.*

Proof. Let $D = \{A, B\}$. Then (by symmetry) it will suffice to prove that $C \leqslant D$. Now C is generated by elements of the form

$$(Z, X, Y) = (Y; Z, X)^{-1},$$

where X, Y, and Z are arbitrary elements of L, M, and N respectively. Hence we need prove only that every element of the form

$$(Y; Z, X) = (Y, Z^{-1}X^{-1}ZX)$$

lies in D.

Now, by (2.15), we have both

$$(Y, ZX) = (Y, X)(Y, Z)(Y, Z, X)$$

and

$$(Y, ZX) = (Y, XX^{-1}ZX) = (Y, X^{-1}ZX)(Y, X)(Y, X, X^{-1}ZX).$$

But (Y, Z, X) belongs to $(M, N, L) = B$; while

$$(Y, X, X^{-1}ZX) = (Y, X, Z'),$$

say, since N is self-conjugate, and so belongs to

$$(M, L, N) = (L, M, N) = A;$$

for, since $(L, M) = (M, L)$ and L, M, and N are self-conjugate, it follows from (2.24) that $(M, L, N) = \big((M, L), N\big) = \big((L, M), N\big) = (L, M, N)$.

Both (Y, Z, X) and $(Y, X, X^{-1}ZX)$ then belong to D; and therefore

$$(Y, ZX) \equiv (Y, X)(Y, Z) \equiv (Y, X^{-1}ZX)(Y, X) \pmod{D}$$

or

$$(Y, X^{-1}ZX) \equiv (Y, X)(Y, Z)(Y, X)^{-1} \pmod{D}.$$

But $(Y, X)(Y, Z)(Y, X)^{-1} = (Y, Z)(Y, Z; X, Y)$; and, since L is self-conjugate, (X, Y), like X, belongs to L; and hence

$$(Y, Z; X, Y) = (Y, Z, X'),$$

say, belongs to $(M, N, L) = B \leqslant D$. This gives finally

$$(Y, X^{-1}ZX) \equiv (Y, Z) \pmod{D}. \tag{1}$$

Now (Y, Z^{-1}) belongs, like Y, to M, because M is self-conjugate. Hence, by (1),

$$(Y, Z^{-1}, X^{-1}ZX) = (Y, Z^{-1}, Z) \pmod{D}. \tag{2}$$

But, by (2.15),

$$(Y, Z^{-1}X^{-1}ZX) = (Y, X^{-1}ZX)(Y, Z^{-1})(Y, Z^{-1}, X^{-1}ZX).$$

Therefore, by (1) and (2),

$$(Y, Z^{-1}X^{-1}ZX) \equiv (Y, Z)(Y, Z^{-1})(Y, Z^{-1}, Z) \pmod{D}.$$

But, by (2.15),

$$(Y, Z)(Y, Z^{-1})(Y, Z^{-1}, Z) = (Y, Z^{-1}Z) = (Y, E) = E.$$

Hence, finally, $(Y, Z^{-1}X^{-1}ZX) \equiv E \pmod{D}$; *i.e.* as we had to prove, $(Y, Z^{-1}X^{-1}ZX)$ belongs to D.

From Theorem 2.3 a number of similar results may be deduced. For example, $(H, K; L, M)$ is contained in the join of (H, K, L, M) and (H, K, M, L); and it is also contained in the join of (L, M, H, K) and (L, M, K, H).

Another interesting result is: each of the four groups (H, K, L, M), (H, K, M, L), (L, M, H, K), and (L, M, K, H) is contained in the join of the other three.

It is understood here that H, K, L, and M are any four self-conjugate subgroups of the given group G, though not necessarily distinct.

As we shall make no use of these theorems in the present paper, we leave the proof of them to the reader.

2.4. *Central series: elementary results.*

We have been dealing hitherto in §§2.1–2.3 with theorems which apply without restriction to groups of arbitrary generality. We shall now consider the special case of p-groups.

Definition. By a *central series* of the p-group G will be understood any monotonic sequence of self-conjugate subgroups of G,

$$E = K_0 \leqslant K_1 \leqslant K_2 \leqslant \ldots \leqslant K_{m-1} \leqslant K_m = G, \tag{2.41}$$

leading from E to G, and with the property that

$$(K_i, G) \leqslant K_{i-1} \quad (i = 1, 2, \ldots, m). \tag{2.42}$$

That is, the commutator of any element of G with any element of K_i belongs to K_{i-1}. Or, again, K_i/K_{i-1} belongs to the central of G/K_{i-1}.

For convenience we write $K_0 = K_{-1} = K_{-2} = \ldots$; and we agree to call the number m the (formal) *length* of the series (2.41). Also for any series of G (or monotonic sequence of subgroups leading from E to G) we shall call the number of terms in the series including either E or G but not both the length of the series.

That every p-group possesses a central series follows from (1.11). For, if we define Z_0 to be E, Z_1 to be the central of G, and then by recurrence Z_2, Z_3, ... such that $Z_i/Z_{i-1} = Z_1(G/Z_{i-1})$, the central of G/Z_{i-1}, then $Z_0 < Z_1 < Z_2 < \ldots$, so that this series, which is obviously a central series of G, can end only in G itself. This series

$$E = Z_0 < Z_1 < Z_2 < \ldots < Z_{c-1} < Z_c = G \tag{2.43}$$

will be called the *upper central series* of G, for it has the property that, if (2.41) is *any* central series of G, then

$$K_i \leqslant Z_i \quad (i = 0, 1, 2, \ldots, c). \tag{2.44}$$

For $i = 0$, this is obvious. Suppose then that for a certain i we have proved that $K_i \leqslant Z_i$. Then, by (2.42), $(K_{i+1}, G) \leqslant K_i \leqslant Z_i$, and hence every element of K_{i+1} corresponds in the homomorphism $G \to G/Z_i$ to a self-conjugate element of G/Z_i, *i.e.* $K_{i+1} \leqslant Z_{i+1}$. Therefore (2.44) holds for all i, by induction.

The upper central series (2.43) of a p-group is of frequent occurrence in the literature; of much greater importance for our present theory, however, is the less familiar *lower central series*

$$G = H_1 > H_2 > \ldots > H_{c'} > H_{c'+1} = E, \tag{2.45}$$

defined by recurrence by the equation

$$H_i = (H_{i-1}, G),$$

or, what is equivalent, by (2.24),

$$H_i = \underbrace{(G, G, \ldots, G)}_{i}.$$

To see that (2.45) is a central series of G we need only point out that if, for any i, $H_i > E$, then, by (1.12), there exists a self-conjugate subgroup K of G, contained in and of index p in H_i; by (1.13), H_i/K belongs to the central of G/K, and hence $H_{i+1} = (H_i, G) \leqslant K < H_i$; so that the inequalities in (2.45) are justified and the series can only terminate in E.

Corresponding to (2.44) we have for the lower central series the property

$$H_i \leqslant K_{m-i+1} \quad (i = 1, 2, \ldots, c') \tag{2.46}$$

for any central series (2.41) of length m. This follows by induction. For, if $H_i \leqslant K_{m-i+1}$, then $H_{i+1} = (H_i, G) \leqslant (K_{m-i+1}, G) \leqslant K_{m-i}$ by (2.42).

The class of a p-group. The inequality (2.44) shows that no central series can be of length less than c, that of the upper central series; while (2.46) shows that no central series can be of length less than c', that of the lower central series. Therefore

$$c = c',$$

and this number c is called the *class* of G. By (2.45), the class may be defined as the least number c such that every simple commutator in G of weight greater than c is equal to E. For instance, an Abelian group is of class 1 and conversely. And, generally speaking, the class c of a p-group G measures the extent to which G diverges from an Abelian group in its properties.

It is clear from (2.46) that $H_i \leqslant Z_{c-i+1}$. However,

$$H_i > Z_{c-i}, \tag{2.47}$$

for, if $H_i \leqslant Z_{c-i}$, we should have successively

$$H_{i+1} \leqslant Z_{c-i-1}, \quad \ldots, \quad H_c \leqslant Z_0 = E,$$

and the length of the lower central series would be less than c.

The reason for the slight asymmetry of the notations for the upper and lower central series will become apparent in the sequel. Here it will suffice to draw the reader's attention to it.

In addition to the two series H and Z, we may introduce conveniently, at this point, the *derived series* θ of G. The i-th derived group θ_i of G is defined by recurrence by the equations

$$\theta_i = (\theta_{i-1}, \theta_{i-1}), \quad \theta_0 = G. \tag{2.48}$$

We shall employ the symbols H, Z, and θ throughout the sequel constantly in the sense defined in the present section. If it is necessary

to refer to the members of the central or derived series of a group K, we shall make use of the notations

$$H_i(K), \quad Z_j(K), \quad \theta_m(K).$$

If, however, the symbols H, Z, or θ are used without arguments, we shall adopt the convention that it is always the group denoted in the context by G that is referred to; so that, by definition,

$$H_i = H_i(G),$$

etc.

Similar conventions apply to the *arithmetical invariants* c, the class, and d, the number of elements in an M.B. And we shall tacitly adopt it also for any other characteristic subgroups and arithmetical invariants which may be defined later on.

It will also be convenient to have a special notation for the groups derived from the characteristic subgroups of factor groups. This will be made clear by an example. Suppose that K is self-conjugate in G. Then we shall denote the subgroup of G which "corresponds" to the subgroup $Z_\alpha(G/K)$ of G/K by $Z_\alpha(G \div K)$. For example,

$$Z_{\alpha+1}(G) = Z_1\Big(G \div Z_\alpha(G)\Big)$$

is the definition of the Z_α's of G.

Before concluding the present section, we shall prove an important elementary property of the three series Z, H, and θ which have just been defined. It is

THEOREM 2.49. (i) *If K is any self-conjugate subgroup of G not contained in Z_α, then K contains elements of $Z_{\alpha+1} - Z_\alpha$ and its order exceeds p^α.* (ii) *If K is any self-conjugate subgroup of G which does not contain H_α, then neither does $\{K, H_{\alpha+1}\}$ contain H_α and the index of K in G is at least p^α.* (iii) *If K is any subgroup of G which does not contain θ_α, then neither does $\{K, \theta_{\alpha+1}\}$ contain θ_α and the index of K in G exceeds p^α.*

Proof. (i) K and $K_1 = K \wedge Z_\alpha$, being both self-conjugate in G, and $K > K_1$, since K is not contained in Z_α, we can find a subgroup K_2 self-conjugate in G, contained in K, and containing K_1 as a subgroup of index p (1.12). K_2/K_1 belongs to the central of G/K_1 by (1.13). Hence, since K_1 lies in Z_α, K_2 must lie in $Z_{\alpha+1}$. But K_2 does not lie in Z_α. Hence $K_2 - K_1$ lies in $Z_{\alpha+1} - Z_\alpha$; K possesses elements of $Z_{\alpha+1} - Z_\alpha$. By the same argument, K possesses elements of $Z_{\beta+1} - Z_\beta$ for every

$$\beta = 0, 1, \ldots, \alpha.$$

Each of the groups $K \wedge Z_\beta$ for $\beta = 0, 1, ., \alpha$ is a *proper* subgroup of the succeeding one $K \wedge Z_{\beta+1}$. Hence the order of $K \wedge Z_{\alpha+1}$ and *a fortiori* that of K is at least $p^{\alpha+1}$.

(ii) If $K_1 = K \wedge H_\alpha$, then, by hypothesis, $K_1 < H_\alpha$. Also K_1 is self-conjugate in G. Therefore we can find a subgroup K_2, self-conjugate in G, containing K_1, and of index p in H_α (1.12). Then H_α/K_2 belongs to the central of G/K_2 by (1.13). Hence K_2 contains $H_{\alpha+1}$, and so

$$\{K_1, H_{\alpha+1}\} \leqslant \{K_2, H_{\alpha+1}\} = K_2 < H_\alpha.$$

But, by definition, $K_1 = K \wedge H_\alpha$. Therefore $\{K, H_{\alpha+1}\} < H_\alpha$. Hence the subgroups $\{K, H_\beta\}$ for $\beta = 1, 2, \ldots, \alpha$ all contain the next succeeding ones $\{K, H_{\beta+1}\}$ as *proper* subgroups. The index of $\{K, H_{\alpha+1}\}$ in G is at least p^α. *A fortiori*, the index of K in G cannot be less than p^α.

(iii) This result is equivalent to Burnside's minimal basis Theorem 1.2 applied to θ_α*.

All three theorems are (as the reader will observe) elementary in the sense that they depend solely on the results detailed in (1.1).

We must mention, in conclusion, the self-evident

THEOREM 2.491. *If K is any self-conjugate subgroup of G, then the lower central and derived series of G/K are obtained by joining K to the corresponding subgroups of G,*

$$H_i(G \div K) = \{K, H_i\}, \quad \theta_j(G \div K) = \{K, \theta_j\}.$$

More generally, if $f(G)$ is any commutator form, then

$$f(G \div K) = \{K, f(G)\}.$$

On the other hand, there is clearly no similar result for the upper central series either of K or of G/K, nor for the commutator subgroups of K itself.

2.5. We are now in a position to prove the main result of the present paragraph, viz.

THEOREM 2.5. *If*

$$G = K_m \geqslant K_{m-1} \geqslant \ldots \geqslant K_1 \geqslant K_0 = K_{-1} = K_{-2} = \ldots = E$$

* It is given in practically the same form as here in Burnside, *Theory of groups*, 2 ed., 123.

is any central series of the p-group G, then

$$(K_j, H_i) \leqslant K_{j-i} \quad (i = 1, 2, \ldots; j = m, m-1, \ldots),$$

where $H_i = (H_{i-1}, G)$ *is the i-th term of the lower central series of G.* $(H_1 = G.)$

In words, the commutator of any element of K_j with any element of H_i belongs to K_{j-i}.

The proof is by induction over i, the result being true for $i = 1$ by the definition of a central series. Suppose, then, that $(K_j, H_{i-1}) \leqslant K_{j-i+1}$ for all $j = m, m-1, \ldots$, and consider the group

$$(K_j, H_i) = (H_i, K_j) = (H_{i-1}, G, K_j).$$

Applying Theorem 2.3, we see that $(K_j, H_i) \leqslant D$, where D is the join of the two groups (G, K_j, H_{i-1}) and (K_j, H_{i-1}, G). Now $(G, K_j) \leqslant K_{j-1}$, and hence $(G, K_j, H_{i-1}) \leqslant (K_{j-1}, H_{i-1})$, which is less than or equal to $K_{j-1-i+1} = K_{j-i}$ by hypothesis. Again $(K_j, H_{i-1}) \leqslant K_{j-i+1}$ by hypothesis; and hence $(K_j, H_{i-1}, G) \leqslant (K_{j-i+1}, G) \leqslant K_{j-i}$. Since both the groups (G, K_j, H_{i-1}) and (K_j, H_{i-1}, G) are contained in K_{j-1}, it follows that their join D is also contained in K_{j-1}. Hence $(K_j, H_i) \leqslant D \leqslant K_{j-i}$, *i.e.* (K_j, H_i) is also contained in K_{j-i}.

With the upper central series Z_j for the series K_j, we obtain

THEOREM 2.51. $(Z_j, H_i) \leqslant Z_{j-i}$ *and, in particular, every element of* Z_j *is permutable with every element of* H_j.

With the lower central series H_i for the series K_j, we obtain

THEOREM 2.52. $(H_i, H_j) \leqslant H_{i+j}$ *and, in particular, if* $2i > c$, *the class of G, then* H_i *is Abelian.*

For, if $2i > c$, $(H_i, H_i) \leqslant H_{2i} \leqslant H_{c+1} = E$.

More generally, we have

THEOREM 2.53. *If* $f(G)$ *is any complex commutator subgroup of weight w in G, then* $f(G) \leqslant H_w$, *and, in particular, if* $w > c$, *the class of G, then* $f(G) = E$.

The proof is by induction over w; for, if $w = 1$, then $f = G = H_1$. Suppose that the theorem is true for all commutator subgroups of weight less than w in G; and in accordance with (2.24) let $f = (f_1, f_2)$, where $f_1 = f_1(G)$ and $f_2 = f_2(G)$ are of weights w_1 and w_2 respectively. Then $w_1 + w_2 = w$, and w_1 and w_2 are both less than w. By hypothesis, therefore, $f_1 \leqslant H_{w_1}$ and $f_2 \leqslant H_{w_2}$. Hence $f \leqslant (H_{w_1}, H_{w_2}) \leqslant H_{w_1+w_2} = H_w$ by (2.52), *i.e.* the

theorem is true of any commutator subgroup $f(G)$ of weight w in G. Hence it is true in general.

Thus the vanishing ($= E$) of all the *simple* commutators of weight greater than c has, as a consequence, the vanishing of all the *complex* commutators of weight greater than c.

As an immediate corollary of (2.53) we have

THEOREM 2.54. *The i-th derived group θ_i of G is contained in H_{2^i}. Further, every element of θ_i is permutable with every element of Z_{2^i}. If the α-th derived group of G is greater than E, then the class of G is at least 2^α.*

For from $\theta_0 = G$ and $\theta_{i+1} = (\theta_i, \theta_i)$ it follows that θ_i can be expressed as a commutator subgroup $f(G)$ of weight 2^i. The first corollary follows from (2.51). The second is obvious.

The lower limit for the class of G in terms of the length of the derived series which is contained in (2.54) is, as we shall now show, a best possible one. Indeed, we shall construct (for all p) a group of arbitrarily high class in which the relation

$$\theta_i = H_{2^i}$$

is satisfied for each $i = 0, 1, 2, \ldots$.

Let $P_1, P_2, \ldots, P_r$ be r elements of order p permutable in pairs, and generating an Abelian group A_r of order p^r and type $(1, 1, \ldots, 1)$. We consider with Burnside* the group of p-automorphisms of A_r, that is by definition a Sylow subgroup of order a power of p in the group of all automorphisms of A_r. This group of p-automorphisms B_r is of order $p^{\frac{1}{2}r(r-1)}$. And if we denote by $I_{i,j}$ the automorphism of A_r which replaces P_i by $P_i P_j$ and leaves unchanged all the P_k's other than P_i, then we may suppose that B_r is generated by the $\frac{1}{2}r(r-1)$ elements $I_{i,j}$ for which $i < j$. The $I_{i,j}$ are all of order p, and we have

$$(I_{i,j}, I_{\alpha,\beta}) = E$$

if $i \neq \beta$ and $j \neq \alpha$, while

$$(I_{i,j}, I_{j,k}) = I_{i,k}.$$

It is easy to verify from these equations† that the derived group of B_r is generated by those $I_{i,j}$ for which $j \geqslant i+2$. More generally, $H_\alpha(B_r)$ is generated by those $I_{i,j}$ for which $j \geqslant i+\alpha$, while $\theta_\beta(B_r)$ is generated by those $I_{i,j}$ for which $j \geqslant i+2^\beta$. Thus the class of B_r is equal to $r-1$; while $\theta_\beta(B_r) > E$ implies that $2^\beta < r$. Taking $r = 2^\nu + 1$, we have a group of class 2^ν whose ν-th derived group is greater than E. And in all these

* Burnside, *loc. cit.* (1912).

† With the help of Theorems 2.81 and 2.82 this verification becomes trivial.

groups B_r we have $\theta_i = H_{2^i}$ for all i. Indeed, if $f(G)$ is any commutator form of weight w in G, then B_r has the property

$$f(B_r) = H_w(B_r).$$

Again, we have

THEOREM 2.55. *$H_i(H_j) \leqslant H_{ij} = H_{ij}(G)$, and, in particular, if $ij > c$, the class of G, then the class of H_j is less than i.*

Here $H_i(H_j)$ denotes, of course, the i-th term in the lower central series of H_j, *i.e.* $H_i(H_j) = (\underbrace{H_j, H_j, \ldots, H_j}_{i})$. The weight of $H_j = H_j(G)$ in G being equal to j, the weight of $H_i(H_j)$ is ij; whence the result.

From (2.53) and (2.55) combined, there can be obtained a limit for the class of any $f(G)$ of weight w in G say, viz. if $iw > c$, the class of G, then the class of $f(G)$ must be less than i.

THEOREM 2.56. *If the self-conjugate subgroup K of G belongs to H_i, then the factor groups of the upper and lower central series of K are all of order p^i at least, the last terms of each series being possible exceptions. In particular, if K is non-Abelian, the order of the central of K is at least p^i, while (more precisely) the order of $K/H_2(K)$ is greater than p^i.*

In his paper* already cited, Burnside considers the question, what properties a group must have in order that it may be the derived group H_2 of some group G of prime-power order. He proves that, if H_2 is non-Abelian, then its derived group $H_2(H_2) = \theta_2(G)$ is of index at least p^3 in H_2 and that its central cannot be cyclical. Theorem 2.56 may be regarded as a generalization of this, provided that we replace the clause that the central of a non-Abelian H_2 cannot be cyclical by the weaker clause that its order is at least p^2. Without this weakening, the analogue of Burnside's result for $i > 2$ no longer holds true for the most general p-group.

Proof of (2.56). We may assume that K is non-Abelian. Let $K_1 = K \wedge Z_i$. Then, since $K \leqslant H_i$, $K_1 \leqslant Z_i$, it follows from (2.51) that every element of K_1 is permutable with every element of K. Hence K_1 belongs to the central of K, and is therefore less than K. Hence K is not contained in Z_i, and so by (2.49), (i), it follows that the order of K_1 is at least p^i. The central of K is (*a fortiori*) of order at least p^i. Again, if $Z_a(K) < K$, so that $K/Z_{a-1}(K)$ is non-Abelian, it results that the order of $Z_a(K)/Z_{a-1}(K)$ is also at least p^i, by exactly the same argument.

* Burnside, *loc. cit.* (1913).

Thus the factor groups of the upper central series, with the possible exception of the last, are all of order p^i at least.

Next write $K_2 = H_2(K)$ for the derived group of K. By hypothesis, $K_2 > E$. We have to prove that the order of K/K_2 is at least p^{i+1}. To do this, let K_3 be a self-conjugate subgroup of G of index p in the self-conjugate subgroup K_2 (such a group exists by 1.12). Then K/K_3 is non-Abelian and therefore its central is of order p^i at least, by what has already been proved. But the central of K/K_3 cannot be of index less than p^2 in K/K_3. Hence the order of K/K_3 is at least p^{i+2}, that of K/K_2 at least p^{i+1}.

Finally, if for any α we have $H_\alpha(K) > E$, then let K_4 be a self-conjugate subgroup of G of index p in the self-conjugate subgroup $H_\alpha(K)$. Then the central of K/K_4 does not contain $H_{\alpha-1}(K)/K_4$ by (2.47), the class of K/K_4 being just α. But, since K belongs to $H_i = H_i(G)$, which is also $H_i(G \div K_4)$, it follows from (2.51) that every element of $Z_i(G/K_4)$ is permutable with every element of K/K_4. Hence $Z_i(G/K_4)$ cannot contain $H_{\alpha-1}(K)/K_4$. Therefore, by (2.49), (i), the order of $H_{\alpha-1}(K)/K_4$ is at least p^{i+1}, and so the order of $H_{\alpha-1}(K)/H_\alpha(K)$ is at least p^i. Thus the factor groups of the lower central series of K are all of order at least p^i, with the possible exception of the last one.

Applying Theorem 2.56 to the β-th derived group θ_β of G, which, according to (2.54), lies in H_{2^β}, we obtain

THEOREM 2.57. *If $\theta_{\beta+1} > E$, then the order of $\theta_\beta/\theta_{\beta+1}$ is greater than p^{2^β}, while the factor groups of the upper and lower central series of θ_β are all, with the possible exceptions of the last members of the two series, of order p^{2^β} at least.*

If $\theta_\delta > E$, then G is of order $p^{2^\delta+\delta}$ at least.

For $\theta_{\beta+1}$ is the derived group of θ_β. And, further, if $\theta_\delta > E$, the order of θ_δ is at least p, and that of $\theta_\beta/\theta_{\beta+1}$ is at least equal $p^{2^\beta+1}$ for each $\beta = 0, 1, \ldots, \delta-1$. Hence, the order of G being p^n, we must have

$$n \geqslant 1 + \sum_{\beta=0}^{\delta-1} (2^\beta+1) = 2^\delta+\delta.$$

For $\delta = 1, 2, 3, \ldots$ this gives $n \geqslant 3, 6, 11, 20, \ldots$, where p^n is the order of a group whose δ-th derived group differs from E. The first two of these are the correct or best possible values, since there are non-Abelian groups of order p^3 (the quaternion group, for example) and non-metabelian groups of order p^6. For higher values of δ the number $2^\delta+\delta$ is, so far as we know, merely a lower bound for n. It may be pointed out, however, that, if $p = 2$, there exists a group of order $2^{2^{\delta+1}-1}$ whose ν-th derived

group differs from E, viz. the Sylow subgroup of that order of the symmetric group on $2^{\delta+1}$ symbols. Thus for $p=2$ the limit $2^\delta+\delta$ is of the right order of magnitude.

When p is odd, we can also obtain an upper bound, viz. $(p^{\delta+1}-1)/(p-1)$, *i.e.* $1+p+p^2+\ldots+p^\delta$, for the lower limit of n in terms of δ, by considering the Sylow subgroup whose order is a power of p in the symmetric group on $p^{\delta+1}$ symbols. However, a better limit is obtained from the groups of automorphisms B_r mentioned in connection with Theorem 2.54. For, if $r=2^\delta+1$, then B_r has its δ-th derived group of order p, viz. $\{I_{1,r}\}$, in our previous notation. Hence the lower limit of n cannot exceed the number $2^{\delta-1}(2^\delta+1)$. There is still a wide gap, however, between the numbers $2^\delta+\delta$ and $2^{\delta-1}(2^\delta+1)$.

The reader will have observed that there is a considerable analogy between the properties of the groups H and Z, though clearly not so strict a one as to amount to a genuine dualism. It might be expected that there would exist a series of characteristic subgroups corresponding to the θ series in the same way as the Z series corresponds to the H series. Apparently this is not the case. For it is well known from trivial examples (the non-Abelian groups of order p^3) that a p-group may possess *more than one maximum self-conjugate Abelian subgroup*. Thus there is no unique analogue of the derived group θ_1 in the sense suggested. We may, however, assert the following

THEOREM 2.58. *If $2i \leqslant c$, the class of G, then none of the maximum self-conjugate Abelian subgroups of G is contained in Z_i; their orders are all greater than p^i.*

Proof. Let A be such a subgroup of G and suppose that $A \leqslant Z_i$. Then, by (2.51), every element of A is permutable with every element of H_i. Hence, since A is Abelian, it must belong to the central of $\{A, H_i\}$. But, by (2.47), H_{c-i} is not contained in Z_i. And, therefore, since $2i \leqslant c$, *a fortiori* H_i is not contained in Z_i. Hence $\{A, H_i\}$ contains A as a *proper* subgroup. If, then, A' is any subgroup self-conjugate in G, contained in $\{A, H_i\}$, and containing A as a subgroup of index p, then A' is Abelian, and so A is *not* a maximum self-conjugate Abelian subgroup of G. Thus the idea that $A \leqslant Z_i$ leads to a contradiction. Therefore A is not contained in Z_i and so, by (2.49), (i), its order is p^{i+1} at least.

Finally, we may mention

THEOREM 2.59. *If $f(G, H_2, H_3, \ldots)$ is any complex commutator subgroup of weights $w_1, w_2, w_3, \ldots$ in $G, H_2, H_3, \ldots$ respectively, then*

$$f(G, H_2, H_3, \ldots) \leqslant H_w,$$

where $$w = w_1 + 2w_2 + 3w_3 + \ldots,$$

and, in particular, $f = E$ *if* w *is greater than the class* c *of* G.

For the weight of H_i in G is equal to i, and so the weight of f in G, when f is expressed in terms of G as sole component, is precisely w. The result then follows from (2.53).

2.6. The Theorems 2.53-2.59 all follow from the result (2.52). We must now turn to consider the analogous consequences of the more general (2.5). Throughout the present section,

$$G = K_m \geqslant K_{m-1} \geqslant \ldots \geqslant K_0 = K_{-1} = K_{-2} = \ldots = E \tag{2.61}$$

will denote an arbitrary central series of the p-group G.

Then—corresponding to the most general result (2.59) of the preceding set of theorems—we have

THEOREM 2.6. *If* $f = f(K_i, G, H_2, H_3, \ldots)$ *is any complex commutator subgroup of weights* $1, w_1, w_2, w_3, \ldots$ *in* $K_i, G, H_2, H_3, \ldots$ *respectively, then* $f \leqslant K_{i-w}$, *where*

$$w = w_1 + 2w_2 + 3w_3 + \ldots,$$

and, in particular, $f = E$ *whenever* $w \geqslant i$.

The proof is by induction over w, the result being true for $w = 0$, $f = K_i$. Suppose, then, that $w > 0$ and that the result is true (for all i) for all smaller values of the sum $\Sigma j w_j$. Since $w > 0$ and the weight of f in K_i is 1, we must have $f = (f_1, f_2) = (f_2, f_1)$, where we may suppose without loss of generality that $f_1 = f_1(K_i, G, H_2, \ldots)$ and $f_2 = f_2(G, H_2, \ldots)$ are of weights $1, w_1', w_2', \ldots$, and $0, w_1'', w_2'', \ldots$ respectively in the components

$$K_i, G, H_2, \ldots$$

respectively. Then $w_j = w_j' + w_j''$ for each $j = 1, 2, \ldots$; and if we write $w' = \Sigma j w_j'$ and $w'' = \Sigma j w_j''$, then $w'' > 0$ (for otherwise f_2 could have no components), and so $w' = w - w'' < w$. By hypothesis, then, the theorem is true for f_1 and so $f_1 \leqslant K_{i-w'}$. On the other hand, by (2.59), we have $f_2 \leqslant H_{w''}$. Hence $f \leqslant (K_{i-w'}, H_{w''})$. Therefore, by (2.5), f is contained in $K_{i-w'-w''} = K_{i-w}$, as required.

2.8. We shall prove in this section some theorems on the generation of the subgroups H and of certain other important subgroups, these results being required in §4.1.

THEOREM 2.81. *If* $P_1, P_2, \ldots, P_r$ *is any basis of the group* G *and if* $Q_1, Q_2, \ldots, Q_s$ *are the various simple commutators of weight* w *in the*

components P_i, then

$$H_w = \{H_{w+1}, Q_1, Q_2, \ldots, Q_s\}.$$

Proof. $G = H_1 = \{P_1, P_2, \ldots, P_r\}$ by hypothesis. The theorem is true then for $w = 1$ and so we may argue by induction over w. Let $R_1, R_2, \ldots, R_t$ be the various simple commutators of weight $w-1$ in the components P_i, and suppose that it has already been proved that

$$H_{w-1} = \{H_w, R_1, R_2, \ldots, R_t\}.$$

Every element of H_{w-1} can therefore be expressed as a product of R's multiplied by an element of H_w. Now, by definition, $H_w = (H_{w-1}, G)$; and every element of G may be expressed as a product of P's. Hence, if A is any product of P's, B any product of R's, and T any element of H_w, we are required to prove that (A, BT) can be expressed as a product of Q's multiplied by an element of H_{w+1}.

Now, by (2.15),

$$(A, BT) = (A, T)(A, B)(A, B, T),$$

and of these three factors both (A, T) and (A, B, T) belong to $H_{w+1} = (G, H_w)$, since T belongs to H_w. There remains the factor (A, B) with $A = \Pi_1 P$ and $B = \Pi_2 R$. If each of the products Π_1 and Π_2 contains only one factor so that $A = P_\alpha$ and $B = R_\beta$, say, then $(A, B) = (P_\alpha, R_\beta)$ is the inverse of (R_β, P_α), which is a Q. Suppose next that $A = P_\alpha$ while B contains just σ factors R with $\sigma > 1$ and that the result has been proved already when B has less than σ factors. We may write $B = B' R_\beta$, where B' has $\sigma - 1$ factors. Then, by (2.15),

$$(A, B) = (A, R_\beta)(A, B')(A, B', R_\beta),$$

and, by hypothesis, both (A, R_β) and (A, B') belong to the group $\{H_{w+1}, Q_1, Q_2, \ldots, Q_s\}$. Since (A, B', R_β) belongs to

$$(G, H_{w-1}, H_{w-1}) \leqslant H_{2w-1},$$

by (2.53), and this is less than or equal to H_{w+1}, since $w > 1$, it follows that, when B contains σ factors and $A = P_\alpha$, then also (A, B) is contained in $\{H_{w+1}, Q_1, Q_2, \ldots, Q_s\}$.

Now suppose that A is the product of τ factors P, and that the result has been proved for all B and all A having less than τ factors. We may write $A = A' P_\alpha$, where A' has $\tau - 1$ factors. Then, by (2.16),

$$(A, B) = (A', B)(A', B, P_\alpha)(P_\alpha, B).$$

And, by hypothesis, both (A', B) and (P_α, B) belong to the group $\{H_{w+1}, Q_1, Q_2, \ldots, Q_s\}$. But (A', B, P_α) belongs to $(G, H_{w-1}, G) \leqslant H_{w+1}$

by (2.53). Hence (A, B) belongs to $\{H_{w+1}, Q_1, Q_2, \ldots, Q_s\}$ for all $B = \Pi_2 R$ and all A having τ factors P.

The result required now follows by induction over τ.

We may express (2.81) in a slightly different form adapted to p-group theory, viz.

THEOREM 2.82. *If $P_1, P_2, \ldots, P_d$ is any minimal basis of the p-group G, then every element of H_w may be expressed as a product of simple commutators of weight greater than or equal to w in the components P.*

If the class of G is c, then all the commutators of weight greater than c are equal to E and may be ignored. There remain only a finite number of any given weight less than or equal to c in the components P. This makes it possible to find an upper limit (quite a crude one, as will appear later) for the invariant $d(H_w/H_{w+1})$ in terms of w and $d = d(G)$. Namely, for $w > 1$, we have

$$d(H_w/H_{w+1}) \leqslant \tfrac{1}{2}d(d-1)\,d^{w-2}. \tag{2.83}$$

Proof. The number just written is simply equal to the number of (formally) distinct simple commutators $(P_{\alpha_1}, P_{\alpha_2}, \ldots, P_{\alpha_w})$ of weight w in the P's for which $\alpha_1 < \alpha_2$. Since $(P, P) = E$ for all P, and so $(P_{\alpha_1}, P_{\alpha_2}, \ldots, P_{\alpha_w}) = E$ wherever $\alpha_1 = \alpha_2$, it will suffice in order to prove (2.83) to show that

$$(P_{\alpha_1}, P_{\alpha_2}, \ldots, P_{\alpha_w})^{-1} \equiv (P_{\alpha_2}, P_{\alpha_1}, P_{\alpha_3}, \ldots, P_{\alpha_w}) \pmod{H_{w+1}}.$$

For then it will not be necessary to include in the basis of $H_w \pmod{H_{w+1}}$ more than one of the two elements $(P_{\alpha_1}, P_{\alpha_2}, \ldots, P_{\alpha_w})$ and

$$(P_{\alpha_2}, P_{\alpha_1}, P_{\alpha_3}, \ldots, P_{\alpha_w}).$$

Now the congruence is trivial for $w = 2$, for by (2.11) it is then an equality. Assuming it to hold for commutators of weight less than w, we have $\pmod{H_w}$ the congruence

$$(P_{\alpha_1}, P_{\alpha_2}, \ldots, P_{\alpha_{w-1}})^{-1} \equiv (P_{\alpha_2}, P_{\alpha_1}, P_{\alpha_3}, \ldots, P_{\alpha_{w-1}}).$$

To complete the induction argument, it only remains to show that, if A and B are two elements of H_{w-1} which satisfy $A^{-1} \equiv B \pmod{H_w}$, *i.e.* if AB lies in H_w, then, for all P of G, we must have

$$(A, P)^{-1} \equiv (B, P) \pmod{H_{w+1}}.$$

Now if AB lies in H_w, then (AB, P) lies in H_{w+1}. But, by (2.16), $(AB, P) = (A, P)(A, P, B)(B, P)$. And, since A and B lie in H_{w-1},

(A, P, B) lies in $(H_{w-1}, G, H_{w-1}) \leqslant H_{2w-1} \leqslant H_{w+1}$ if $w > 1$. Hence $(A, P)(B, P)$ lies in H_{w+1}, as required.

Next we have the following general principle, which allows more precise conclusions than (2.83) to be drawn if the invariants of G/H_2 are known.

THEOREM 2.84. *Let $f(X, A_1, A_2, \ldots, A_r)$ be any complex commutator of total weight w in the components $X, A_1, \ldots, A_r$ and of weight 1 in X. Then, for given elements $A_1, \ldots, A_r$ of G and variable element X in G, the subgroup of G/H_{w+1} formed by the cosets $f(X, A_1, A_2, \ldots, A_r)H_{w+1}$ is Abelian and homomorphic with G, the correspondence*

$$X \to \phi(X) = f(X, A_1, A_2, \ldots, A_r)H_{w+1}$$

being a homomorphism.

Proof. That the group formed by the cosets $\phi(X)$ of H_{w+1} is Abelian follows from the fact that $f(X, A_1, \ldots, A_r)$ is of weight w and so belongs to H_w; and H_w/H_{w+1} is Abelian. To prove the homomorphism, we must show that, for all X and Y of G,

$$\phi(X)\phi(Y) = \phi(XY).$$

We shall argue by induction over w, since, for $w = 1$, the theorem reduces to the homomorphism between G and G/H_2; because then $f(X, \ldots) = X$ and $\phi(X) = XH_2$. We may assume then that $f(X, \ldots) = (f_1, f_2)$, where $f_1 = f_1(X, A_1, \ldots, A_r)$ and $f_2 = f_2(X, A_1, \ldots, A_r)$ are of weights w_1 and w_2 respectively and $w_1 + w_2 = w$; and also that the theorem is valid for whichever of the two commutators f_1 and f_2 actually contains X (to weight 1) as a component. We may suppose, without loss of generality, that this is f_1. Then f_2 is independent of X; it is a fixed element of G. And if we write $\phi_1(X) = f_1 H_{w+1}$ and $\phi_2 = f_2 H_{w+1}$, so that $\phi(X) = \big(\phi_1(X), \phi_2\big)$, then, from the congruence

$$\phi_1(XY) \equiv \phi_1(X)\phi_1(Y) \pmod{H_{w_1+1}}, \tag{1}$$

which we may assume since the theorem is valid for f_1, we must deduce that

$$\phi(XY) = \phi(X)\phi(Y). \tag{2}$$

Suppose that $\phi_1(XY) = \phi_1(X)\phi_1(Y)\bar{\phi}$ and write ϕ' for $\phi_1(X)\phi_1(Y)$. Then, by (1), $\bar{\phi}$ lies in H_{w_1+1}. Hence $\phi(XY) = \big(\phi_1(XY), \phi_2\big) = (\phi'\bar{\phi}, \phi_2)$ is equal to

$$(\phi', \phi_2)(\phi', \phi_2, \bar{\phi})(\bar{\phi}, \phi_2) \tag{3}$$

by (2.16). But, in (3), $(\bar{\phi}, \phi_2)$ lies in $(H_{w_1+1}, H_{w_2}) \leqslant H_{w_1+w_2+1} = H_{w+1}$ by

(2.52). Similarly $(\phi', \phi_2, \overline{\phi})$ lies in H_{w+1}. Hence, the ϕ's all being cosets of H_{w+1} by definition, we may write

$$\phi(XY) = (\phi', \phi_2). \tag{4}$$

But, since $\phi' = \phi_1(X)\phi_1(Y)$, another application of (2.16) gives for the right-hand side of (4) the expression

$$\big(\phi_1(X), \phi_2\big)\big(\phi_1(X), \phi_2, \phi_1(Y)\big)\big(\phi_1(Y), \phi_2\big).$$

And since the middle factor lies in $(H_{w_1}, H_{w_2}, H_{w_1}) \leqslant H_{w+1}$, and since $\phi(Y) = \big(\phi_1(Y), \phi_2\big)$ and $\phi(X) = \big(\phi_1(X), \phi_2\big)$, this gives simply (2).

As a corollary of this, we have

THEOREM 2.85. *If $f(P_1, P_2, \ldots, P_s)$ is any complex commutator of weight w in the elements $P_1, \ldots, P_s$ of G; then its order* mod H_{w+1} *divides the order* mod H_2 *of each P_i $(i = 1, 2, \ldots, s)$.*

Thus, if f is a *simple* commutator and the P's are a minimal basis of G whose orders relative to H_2 are known, we can obtain an upper limit for the *order* of H_w/H_{w+1} for each w in terms of w and these orders of the P's. We need only apply Theorem 2.85 to the elements Q_j of Theorem 2.81. The details must be left to the reader.

It is more important to mention the fact that in (2.85) the group H_2 cannot be replaced by a higher H. That is, we cannot say that the order of $f(P_1, P_2, \ldots, P_s) \pmod{H_{k+a}}$ with $a > 1$ necessarily divides the order mod H_{a+1} of each P_i. If it was so, then clearly the order of a commutator could never exceed the orders of any of its components, a result contradicted by the group defined by the relations

$$P^{2^{a-1}} = Q^2 = E, \quad Q^{-1}PQ = P^{-1},$$

since (P, Q) of order 2^{a-1} has a higher order than Q when $a > 2$.

It will be one of the main results of §4 to show that for a *regular** p-group the extension of (2.85), to which we have just referred as being illegitimate in general, remains valid. In particular then, it will be shown that, if the order of G is p^n with $n < p$, then the order of a commutator cannot exceed the order of any component (4.25).

§3. *A formula for* $(PQ)^x$.

3.1. A large part of our subsequent work (including the theory of regular groups contained in §4) will depend on the use of a certain

* For the definition of *regular cf.* §4·1.

formula for the powers $(PQ)^x$ of any product PQ of two elements of a group. The proof of this formula is the main object of §3. The actual proof occupies §§3.3–3.4. In the present section we explain the formula in the form in which it is proved, while in §3.2 we shall derive the form in which it will be required for application to p-group theory.

The theorem which will be proved is

THEOREM 3.1. *For any two elements P and Q of any group G let the formally distinct complex commutators of P and Q be arranged in order of increasing weights, the order being otherwise arbitrary, thus:*

$$R_1, R_2, \ldots, R_i, \ldots. \tag{3.11}$$

Then there exists a series of integer-valued polynomials

$$f_1(x), f_2(x), \ldots, f_i(x), \ldots \tag{3.12}$$

[and $f_1(x) = f_2(x) = x$] all vanishing for $x = 0$ and such that the degree of $f_i(x)$ does not exceed the weight w_i of R_i in P and Q, and with the property that, for all P and Q and all positive integers x,

$$(PQ)^x = R_1^{f_1(x)} R_2^{f_2(x)} \ldots R_i^{f_i(x)} \ldots, \tag{3.13}$$

this equation being interpreted to mean that, if R_λ, $\lambda = \lambda(c)$, is the last term of the series (3.11) which is of weight less than c in P and Q, then

$$(PQ)^x \equiv R_1^{f_1(x)} R_2^{f_2(x)} \ldots R_\lambda^{f_\lambda(x)} \pmod{H_c(K)}, \tag{3.14}$$

where $K = \{P, Q\}$; this being true for all $c = 2, 3, \ldots$.

Thus we may take either $R_1 = P$ and $R_2 = Q$ or else $R_2 = P$ and $R_1 = Q$. Then R_3 starts the commutators of weight 2, and may be either (P, Q) or (Q, P) at choice; R_4 is then either (Q, P) or (P, Q) respectively. With R_5 the commutators of weight 3 begin, and so on.

It is to be emphasised that the coefficients of the polynomials $f_i(x)$ depend only on the order (3.11) chosen for the complex commutators R. It does not depend at all on P, Q, G, or x; so that, if the R's are written in terms of the symbols P and Q with multiple stops as explained in (2.1), we may substitute in (3.13) for P and Q any two elements of any group whatever, and for x any positive integer.

If, however, the order in which the R's are arranged is altered (consistently with the condition that $w_i \leqslant w_j$ whenever $i < j$ must continue to hold in the new arrangement), then, in general, the f's will have to be replaced by different integer-valued polynomials.

The proof of Theorem 3.1 which is contained in §§3.3–3.4 will be seen to give at the same time a method for calculating in a finite number

of steps the (rational number) coefficients of any particular $f_i(x)$, once the order (3.11) of the *preceding* R_j's is given.

3.2. It will be observed that (3.1) contains no reference to p-groups; the formula (3.13) is universally valid in any group whatever. For our purpose it is necessary to specialise.

First, we may point out that any integer-valued polynomial $f(x)$ of degree less than or equal to w and with $f(0) = 0$ may be expressed in the form*

$$f(x) = a_1\binom{x}{1} + a_2\binom{x}{2} + \ldots + a_w\binom{x}{w} \tag{3.21}$$

with *rational integer* coefficients $a_1, a_2, \ldots, a_w$, where

$$\binom{x}{r} = \frac{x(x-1)(x-2)\ldots(x-r+1)}{1.2.3.\ldots.r}. \tag{3.22}$$

Further, it is well known that

(3.23) *If* $x = p^\alpha \geqslant p^\beta > w > 0$, p *being a prime, then* $\binom{x}{w}$ *is divisible by* $p^{\alpha-\beta+1}$.

From this it follows that $f(p^\alpha)$ is also divisible by $p^{\alpha-\beta+1}$, if $f(x)$ is of the form (3.21).

Applying these remarks to the exponents $f_i(x)$ of Theorem 3.1, we obtain

THEOREM 3.2. *If* p *is a prime,* α *is a positive integer,* P *and* Q *are any two elements of the group* G, *and*

$$R_1, R_2, \ldots, R_i, \ldots \quad (R_1 = P,\ R_2 = Q)$$

are the various formally distinct complex commutators of P *and* Q *arranged in order of increasing weights, then integers* $n_1, n_2, \ldots, n_i, \ldots$ *can be found* $(n_1 = n_2 = p^\alpha)$ *such that*

$$(PQ)^{p^\alpha} = R_1^{n_1} R_2^{n_2} \ldots R_i^{n_i} \ldots, \tag{3.21}$$

and if the weight w_i *of* R_i *in* P *and* Q *satisfies the inequality* $w_i < p^\beta \leqslant p^\alpha$, *then* n_i *is divisible by* $p^{\alpha-\beta+1}$.

The right-hand side of (3.21), being in general an infinite product, is to be interpreted, as explained in Theorem 3.1, as giving an infinite sequence of ever more exact congruences for the left-hand side $(PQ)^{p^\alpha}$.

---

* *Cf.* Pólya-Szegö, *Aufgaben und Lehrsätze*, 2 (1925), 132, Problem 85.

An arithmetical lemma.

In order not to interrupt the later stages of the argument, it will be convenient to deal at once with a purely arithmetical result on the truth of which everything depends.

Definition 3.24. Let $\lambda_1, \lambda_2, \ldots, \lambda_w$ be indeterminates. We shall say that a certain set of conditions (C), to be fulfilled by the λ's, are *conditions* (Π) if and only if C is the logical sum of a number of alternatives. each of which alternatives is the logical product of a number of *elementary conditions*, each elementary condition being either of the form

$$\lambda_i < \lambda_j$$

or else of the form $\lambda_i = \lambda_j$.

By the distributive law of logic, it follows that the logical product of a number of conditions (Π) is also a condition (Π). The same holds (trivially) with sum for product. Indeed, we might define conditions as being (Π) if and only if they could be derived from a number of elementary conditions of the forms $\lambda_i < \lambda_j$ or $\lambda_i = \lambda_j$ by repeated applications of the operations of (logical-) sum- and product-formation.

The result we shall need may now be formulated as follows:

THEOREM 3.25*. *Let the conditions C required to be satisfied by the w indeterminates $\lambda_1, \lambda_2, \ldots, \lambda_w$ be conditions (Π) in the sense of definition* 3.24. *Let x be a variable positive integer. Then the number of distinct ordered-sets of w positive integers none exceeding x*

$$(\lambda_1, \lambda_2, \ldots, \lambda_w)$$

which satisfy the conditions C is given by an expression of the form

$$a_1\binom{x}{1} + a_2\binom{x}{2} + \ldots + a_w\binom{x}{w},$$

where the a's are non-negative integers which depend only on C but not on x.

Two ordered-sets of w positive integers, say $(\lambda_1, \lambda_2, \ldots, \lambda_w)$ and $(\lambda_1', \lambda_2', \ldots, \lambda_w')$, will be called *equivalent* if, whenever $\lambda_i = \lambda_j$ or $\lambda_i < \lambda_j$, then $\lambda_i' = \lambda_j'$ or $\lambda_i' < \lambda_j'$ respectively, this being true for all pairs of values of the suffixes i and j. The sets $(\lambda_1, \lambda_2, \ldots, \lambda_w)$ then fall into mutually exclusive *classes of equivalent sets*. Evidently the number k of *different* λ's in any set is a function only of the class to which that set belongs. Further,

* My cordial thanks are due to the Referee for supplying me with the very simple proof of this result which is given below.

the total number of distinct sets in a given class which have all their λ's less than or equal to x will be equal to the number of solutions in positive integers less than or equal to x of a set of inequalities of the form

$$\lambda_{\rho_1} < \lambda_{\rho_2} < \ldots < \lambda_{\rho_k},$$

with k (as before) equal to the number of distinct λ's in each set of the class. But this number is $\binom{x}{k}$.

Now if one set of any given class satisfies the given conditions C, then every set of that class will do so, as is to be seen at once by comparing the definition of equivalence with that of conditions (Π). Therefore, if we denote by a_k the number of classes whose sets all (i) satisfy C, and (ii) have just k different λ's, it follows that the number of solutions of C in positive integers none exceeding x is equal to

$$a_1\binom{x}{1}+a_2\binom{x}{2}+\ldots+a_w\binom{x}{w},$$

as required.

3.3. *Sketch of the proof.*

In proving the formula (3.1) we may clearly suppose that $\{P, Q\}$ is the free group with two generators P and Q; for any formula obtained on this supposition will hold true *a fortiori* in any other group.

First of all, it is necessary to arrange the various distinct complex commutators in P and Q in an order

$$R_1, R_2, R_3, \ldots, R_i, \ldots, \tag{3.31}$$

to be chosen once for all. This order is to be one of increasing weight, but is otherwise completely arbitrary. Thus, if w_i denotes the weight of R_i in P and Q, we must have

$$w_1 \leqslant w_2 \leqslant \ldots \leqslant w_i \leqslant w_{i+1} \leqslant \ldots. \tag{3.32}$$

In particular, R_1 and R_2 are P and Q not necessarily respectively*; R_3 and R_4 are (P, Q) and (Q, P); and so on. We omit from the sequence (3.31) only those commutators which are trivially equal to E due to having two initial components the same, *i.e.* only those of the form $(R, R, \ldots)$.

The order of the R's being decided, we shall deduce the required formula as the result of a succession of *steps* or elementary transformations, starting

* To this extent the formula we actually prove is more general than (3.1).

from the expression

$$PQPQP \dots QPQ, \tag{3.33}$$

which is the product of $2x$ factors alternately equal to P and Q and which is obviously the same as $(PQ)^x$.

Each *step* consists in passing from an expression of the form

$$\dots RS \dots$$

to the expression $\dots SR(R, S) \dots$,

what is denoted by dots being unaffected. Thus, each step introduces a new factor which is the commutator of two previously arising factors; in this way, therefore, only the complex commutators in P and Q can be introduced. Moreover, by (2.12), no step can affect the *value* of the expression. At every point then, we shall be manipulating *a product of R's which is equal to* $(PQ)^x$.

We must now describe the rules by which the successive steps are to be made, in order to verify that an expression of the required form will actually be arrived at.

The *collecting process*. We shall divide the steps into a succession of *stages*, the steps of the $(i+1)$-th stage all following those of the i-th stage. These stages may be described by recurrence. Suppose that at the end of the $(i-1)$-th stage we have arrived at an expression of the form

$$R_1^{n_1} R_2^{n_2} \dots R_{i-1}^{n_{i-1}} . S_1 S_2 \dots S_{\nu_i}, \tag{3.34}$$

where each S is an R_j having $j \geqslant i$. We may speak of what is on the left of the stop in (3.34) as the arranged part, and what is on the right of it as the unarranged part. Then the i-th stage consists in removing the R_i's (if any) in the unarranged part all to the left of the unarranged part. This will involve introducing new factors of the form (R_j, R_i) of weights

$$w_j + w_i > w_i,$$

and therefore, by (3.32), equal to R_k's with $k > i$. Hence we shall arrive at an expression of the form

$$R_1^{n_1} R_2^{n_2} \dots R_{i-1}^{n_{i-1}} R_i^{n_i} . T_1 T_2 \dots T_{\nu_{i+1}}, \tag{3.35}$$

where each T is an R_j having $j > i$. For the T's are either newly introduced factors, or else are S's which are not R_i's; in either case, then, the T's are R_j's having $j > i$.

Now (3.33) is clearly of the form (3.34) with $i = 1$. Hence, by induction, we can arrive at an expression of the form (3.35) as the end of the i-th stage.

The number n_i is equal to the number of R_i's occurring in the unarranged part of (3.34). And, in order to be able to calculate it, we must now specify more closely the succession of steps which constitutes the i-th stage of the collecting process.

Let $S_{i_1}, S_{i_2}, \ldots, S_{i_n}$ $(n = n_i)$ be the S's which are equal to R_i and let $i_1 < i_2 < \ldots < i_n$. Then the i-th stage will begin by moving S_{i_1} steadily to the left of the other S's; this obviously takes $i_1 - 1$ steps. Next, S_{i_2} will be moved steadily to the left of the other S's (apart from S_{i_1} which is already "collected") and this takes $i_1 + i_2 - 3$ steps, since the collecting of S_{i_1} has involved introducing just $i_1 - 1$ new factors all on the left of S_{i_2}, past all of which therefore S_{i_2} must be moved. After collecting S_{i_2}, we next collect S_{i_3} by the same procedure; then $S_{i_4}, S_{i_5}, \ldots$, up to S_{i_n}. When S_{i_n} has been collected, the i-th stage is over, and

$$S_1 S_2 \ldots S_{\nu_i}$$

has been transformed into

$$S_{i_1} S_{i_2} \ldots S_{i_n} . T_1 T_2 \ldots T_{\nu_{i+1}},$$

which is the same as

$$R_i^{n_i} . T_1 T_2 \ldots T_{\nu_{i+1}}.$$

The collected part of (3.34) remains unchanged throughout the i-th and all subsequent stages.

It is clear that the steps which we have hitherto described, viz. (i) the choice of an order for the R's and (ii) the collecting process in its successive stages, together suffice to determine the sequence of numbers

$$n_1, n_2, \ldots, n_i, \ldots$$

without ambiguity—the n's being, of course, functions of the positive integer x. In particular, it is clear that $n_1 = n_2 = x$, since there are x P's and Q's in (3.33) and none arise subsequently.

The *labelling process*. The calculation of these exponents n_i, or rather the discussion of the rules by means of which they may be calculated, constitutes the essential part of the proof of (3.1). This calculation may be achieved with the help of the following idea. We shall suppose that each R_i as it arises during the collecting process may be assigned a *label* $\Lambda_i = (\lambda_1, \lambda_2, \ldots, \lambda_w)$, where $w = w_i$ is the weight of R_i, consisting of a sequence of w_i symbols λ_j; and that this may be done in such a way that no two R_i's

are assigned the same label. That this can actually be done, with the λ's positive integers less than or equal to x, may be seen thus. To begin with, we must label the P's and Q's of (3.33); to do this, we can simply write (3.33) as

$$P^{(1)}\,Q^{(1)}\,P^{(2)}\,Q^{(2)}\,P^{(3)}\ldots Q^{(x-1)}\,P^{(x)}\,Q^{(x)}, \tag{3.36}$$

where $P^{(\alpha)}$ stands for "the P whose label is α." Clearly no two P's and no two Q's have the same label. Suppose then that at any moment of the collecting process all the factors of the expression are labelled in the way desired; and let the next step consist in passing from

$$\ldots RS \ldots$$

to

$$\ldots SR(R,\ S)\ldots.$$

Let $(\lambda_1, \lambda_2, \ldots, \lambda_{w_i})$ be the label of $R = R_i$ and let $(\lambda_1', \lambda_2', \ldots, \lambda'_{w_j})$ be the label of $S = R_j$. Now the weight w_k of $(R,\ S) = R_k$ is given by $w_k = w_i + w_j$. Hence we may take for the label of $(R,\ S)$ simply $(\lambda_1, \lambda_2, \ldots, \lambda_{w_i}, \lambda_1', \lambda_2', \ldots, \lambda'_{w_j})$. Now it is clear from the rules of collecting that any other $R_k = (R_i,\ R_j)$ must involve either a different R_i or a different R_j; so that the label assigned to it will differ from that given to $(R,\ S)$ either in its first w_i or in its last w_j symbols. Thus, the new term is labelled as desired; its label differs from that of all the previously arising R_k's. And hence, by induction, the whole labelling process carried out in the way described has the desired property that no two R_i's are assigned the same label.

It is also clear that the symbols λ of the labels so constructed are all positive integers less than or equal to x. If we express any R_i as a complex commutator in P's and Q's, then its label $(\lambda_1, \lambda_2, \ldots, \lambda_{w_i})$ consists simply in the labels of the component P's and Q's taken in the correct order, *i.e.* we need only replace each P and Q in R_i by its label and replace all stops by commas in order to obtain the label of R_i.

The calculation of the n_i. The remainder of the discussion consists in formulating the conditions E_i which the w_i positive integers less than or equal to x, $(\lambda_1, \lambda_2, \ldots, \lambda_{w_i})$, of the label Λ_i of an R_i need to satisfy in order that an R_i shall arise during the collecting process which actually possesses this particular label. Since no two R_i's have the same label, n_i is simply the number of solutions in positive integers less than or equal to x of the conditions E_i. *It will be shown that the conditions E_i are conditions* (Π) *in the sense defined in* §3.2, this being true for each i.

Theorem 3.1 then follows with the help of the arithmetical lemma of §3.2.

3.3. *Details of the proof.*

In order to formulate the conditions E_i (the *existence* conditions of the R_i) defined in the preceding section, we must introduce certain auxiliary conditions P_{ij}—the *precedence* conditions for the R_i and R_j. The P_{ij} will only be needed when at the end of a certain stage of the collecting process both R_i's and R_j's occur in the unarranged part of the "expression". If $(\lambda_1, \lambda_2, \ldots, \lambda_{w_i})$ and $(\lambda_1', \lambda_2', \ldots, \lambda_{w_j}')$ are the labels of $R = R_i$ and $S = R_j$ respectively, then P_{ij} is defined to be the conditions which the w_i+w_j positive integers $\lambda_1, \lambda_2, \ldots, \lambda_{w_i}, \lambda_1', \lambda_2', \ldots, \lambda_{w_j}'$, all less than or equal to x, must satisfy in addition to the conditions E_i on the λ's and E_j on the λ''s in order that at the end of some stage of the collecting process R and S shall both occur in the unarranged part, *R preceding or to the left of S*. Since the order of R and S is not disturbed until one of them is collected, this definition is not ambiguous.

To prove the desired result that the conditions E_i are conditions (Π) it will be convenient to show at the same time that the conditions P_{ij} are also conditions (Π). And this will be done by an induction argument over the stages of the collecting process.

Consider the initial expression, (3.33). The conditions E_1 and E_2—the existence conditions of the P and Q—are clearly null, for any integer λ satisfying $0 < \lambda \leqslant x$ can occur as the label of a P and of a Q. E_1 and E_2 therefore certainly satisfy (3.2), *i.e.* are conditions (Π). Next we observe that $P^{(\lambda)}$ precedes $Q^{(\mu)}$ if and only if $\lambda \leqslant \mu$; $P^{(\lambda)}$ precedes $P^{(\mu)}$, or $Q^{(\lambda)}$ precedes $Q^{(\mu)}$, or $Q^{(\lambda)}$ precedes $P^{(\mu)}$ if and only if $\lambda < \mu$. The four sets of conditions P_{11}, P_{12}, P_{21}, and P_{22} therefore are all conditions (Π).

We shall now assume that, in the unarranged part of the "expression" as it stands at the end of the $(i-1)$-th stage, viz. in

$$S_1 S_2 \ldots S_{\nu_i}, \tag{3.41}$$

all the *relevant* conditions E_j and P_{jk} are conditions (Π)—E_j and P_{jk} being "relevant" if both R_j's and R_k's occur in (3.41). The induction argument will then be completed if we can prove from this assumption that this state of affairs will also hold good at the end of the i-th stage, *i.e.* that all relevant conditions for the product

$$T_1 T_2 \ldots T_{\nu_{i+1}} \tag{3.42}$$

are conditions (Π). For we have just seen that this is actually true initially, *i.e.* for $i = 0$.

Let R_j, $j > i$, be a typical S which is *not* collected during the i-th stage, so that R_j is also a T. We write $R_j = R_{j_0}$, $(R_j, R_i) = R_{j_1}$, $(R_j, R_i, R_i) = R_{j_2}$, ...,

and, in general,

$$(R_{j_\alpha}, R_i) = R_{j_{\alpha+1}}. \tag{3.43}$$

Then the only R's which can arise during the i-th stage are clearly of one of the forms $R_{j_1}, R_{j_2}, \ldots, R_{j_n}$ $(n = n_i)$. Hence, if R_k, $k > i$, is any other typical S not collected during the i-th stage, and if we define in the same way $R_{k_0} = R_k$, $R_{k_1} = (R_k, R_i)$, ..., and, generally, $R_{k_{\alpha+1}} = (R_{k_\alpha}, R_i)$, ..., then the only existence and precedence conditions which we need to discuss are

I. The E_{j_α} $(\alpha = 1, 2, \ldots)$—the existence conditions on the $w_j + \alpha w_i$ label-symbols of the R_{j_α},

$$T = (S_{\gamma_0}, S_{\gamma_1}, \ldots, S_{\gamma_\alpha}),$$

where $S_{\gamma_0} = R_j$, $S_{\gamma_1} = S_{\gamma_2} = \ldots = S_{\gamma_\alpha} = R_i$.

II. The $P_{j_\alpha k_\beta}$ $(\alpha, \beta = 0, 1, 2, \ldots)$—the conditions that T shall precede the R_{k_β},

$$\bar{T} = (S_{\delta_0}, S_{\delta_1}, \ldots, S_{\delta_\beta}),$$

where $S_{\delta_0} = R_k$, $S_{\delta_1} = S_{\delta_2} = \ldots = S_{\delta_\beta} = R_i$. And, finally,

III. The $P_{j_\alpha j_\beta}$ $(\alpha, \beta = 0, 1, 2, \ldots)$—the conditions that T shall precede $\bar{T}$, where $S_{\delta_0} = R_j$, the rest being as in II.

We shall show that E_{j_α} is equivalent to

$$\gamma_0 < \gamma_1 < \gamma_2 < \ldots < \gamma_\alpha; \tag{3.44}$$

that $P_{j_\alpha k_\beta}$ is equivalent to

$$\gamma_0 < \delta_0; \tag{3.45}$$

and, finally, that $P_{j_\alpha j_\beta}$ is equivalent to the logical sum of the following $\min(\alpha, \beta) + 1$ or 2 alternatives

$$\left\{\begin{array}{l} \qquad (a) \quad \gamma_0 < \delta_0, \\ \qquad (b_\eta) \quad \gamma_0 = \delta_0, \gamma_1 = \delta_1, \ldots, \gamma_{\eta-1} = \delta_{\eta-1}, \gamma_\eta > \delta_\eta, \\ \text{with } \eta = 1, 2, \ldots, \min(\alpha, \beta), \text{ together with a final alternative} \\ \qquad (c) \quad \gamma_0 = \delta_0, \gamma_1 = \delta_1, \ldots, \gamma_\alpha = \delta_\alpha \\ \text{—}(c) \text{ being omitted except when } \beta > \alpha. \end{array}\right. \tag{3.46}$$

These equivalences are the direct consequence of the rules of the collecting process.

Suppose, for instance, that (3.44) is true. Then the

$$S_{\gamma_\eta} \quad (\eta = 1, 2, \ldots, \alpha)$$

are collected in the order of increasing η. And, since S_{γ_0} lies on the left of S_{γ_1} initially, it has to be passed by S_{γ_1}, and this step gives rise to $(S_{\gamma_0}, S_{\gamma_1})$. The latter arises to the left of S_{γ_1}, and so *a fortiori* to the left of S_{γ_2}, and before the latter is collected. Hence S_{γ_2} has to pass $(S_{\gamma_0}, S_{\gamma_1})$, and this step gives rise to $(S_{\gamma_0}, S_{\gamma_1}, S_{\gamma_2})$. And so on. Finally, $(S_{\gamma_0}, S_{\gamma_1}, \ldots, S_{\gamma_{\alpha-1}})$ arises during the collecting of $S_{\gamma_{\alpha-1}}$, hence to the left of S_{γ_α}, *a fortiori* to the left of S_{γ_α}; so that S_{γ_α} must pass it, giving rise to

$$(S_{\gamma_0}, S_{\gamma_1}, \ldots, S_{\gamma_\alpha}) = T.$$

Conversely, T can arise only if S_{γ_α} is collected after $(S_{\gamma_0}, S_{\gamma_1}, \ldots, S_{\gamma_{\alpha-1}})$ has already arisen. S_{γ_α} must be collected after $S_{\gamma_{\alpha-1}}$, therefore, *i.e.* $\gamma_{\alpha-1} < \gamma_\alpha$. Applying this to the components

$$(S_{\gamma_0}, S_{\gamma_1}, \ldots, S_{\gamma_\eta}) \quad (\eta = 1, 2, \ldots, \alpha-1)$$

of T in turn, we get (3.44).

Next it is clear that, if S_{γ_0} and S_{δ_0} are two different S's which remain unmoved during the i-th stage, then the order in which any two T's having respectively S_{γ_0} and S_{δ_0} as first component arise is the same as that of S_{γ_0} and S_{δ_0}. If, for instance, S_{γ_0} precedes S_{δ_0}, then those T's which have S_{γ_0} as first component all lie between S_{γ_0} and S_{δ_0}, those having S_{δ_0} as first component all lie to the right of S_{δ_0}. Hence the equivalence of (3.45) and $P_{j_\alpha k_\beta}$.

Finally, if, in case III, $\gamma_0 = \delta_0, \gamma_1 = \delta_1, \ldots, \gamma_{\eta-1} = \delta_{\eta-1}$, but $\gamma_\eta \neq \delta_\eta$, for some $\eta \leqslant \min(\alpha, \beta)$, then we have $T = (T_0, S_{\gamma_\eta}, S_{\gamma_{\eta+1}}, \ldots, S_{\gamma_\alpha})$ and $\bar{T} = (T_0, S_{\delta_\eta}, \ldots, S_{\delta_\beta})$, where $T_0 = (S_{\gamma_0}, S_{\gamma_1}, \ldots, S_{\gamma_{\eta-1}})$. And hence, by the preceding argument, T will precede $\bar{T}$ if and only if (T_0, S_{γ_η}) precedes (T_0, S_{δ_η}); and this will be so if and only if S_{δ_η} is collected *before* S_{γ_η}, *i.e.* $\delta_\eta < \gamma_\eta$ since then (T_0, S_{δ_η}) will arise first and to the right of T_0, so that S_{γ_η} has to pass (T_0, S_{δ_η}) before passing T_0 and giving rise to (T_0, S_{γ_η}). Secondly, if $\gamma_0 \neq \delta_0$, then, as before, T precedes $\bar{T}$ if and only if S_{γ_0} precedes S_{δ_0} or $\gamma_0 < \delta_0$. Thirdly, if $\gamma_0 = \delta_0, \gamma_1 = \delta_1, \ldots, \gamma_\eta = \delta_\eta, \ldots$ for $\eta = 0, 1, 2, \ldots, \min(\alpha, \beta)$, then $T = \bar{T}$ if $\alpha = \beta$ and if $\alpha \neq \beta$, T precedes $\bar{T}$ or not according as $\alpha < \beta$ or not. Therefore, (3.46) is equivalent to T precedes $\bar{T}$, *i.e.* to $P_{j_\alpha k_\beta}$.

We may now complete the proof that the conditions E_{j_α}, $P_{j_\alpha k_\beta}$, and $P_{j_\alpha j_\beta}$ are conditions (Π).

For, by (3.44), the conditions E_{j_α} for the existence of T are equivalent to one condition E_j, α conditions E_i, one condition P_{ji}, and $\alpha-1$

conditions P_{ii}, holding respectively between various subsets of the $w_j+\alpha w_i$ label-symbols of T. But, by the induction-hypothesis, each of these $2\alpha+1$ conditions are conditions (Π)—since the R_i's and R_j's are already present in (3.41). Hence so is their logical product E_{j_α}.

Similarly $P_{j_\alpha k_\beta}$ is the same as P_{jk} on a certain subset (the labels of S_{γ_0} and S_{δ_0}) of the $w_j+w_k+(\alpha+\beta)w_i$ labels of T and $\bar{T}$. Since P_{jk} is (Π), so is $P_{j_\alpha k_\beta}$.

Finally, $P_{j_\alpha j_\beta}$ is the logical sum of a number of alternatives (a), (b_η), and (c). Of these (c) consists entirely of equalities and so is (Π). (a) and the (b_η) consist partly of equalities, supplemented by a single condition either P_{jj} [for (a)] or P_{ii} [for the (b_η)], either of which is, by hypothesis, a condition (Π). Hence, the alternatives being all (Π), so is $P_{j_\alpha j_\beta}$.

This concludes the proof. The induction argument has been shown to work. All the precedence and existence conditions are conditions (Π). The number n_i of solutions of E_i in positive integers less than or equal to x is, therefore, equal to $f_i(x)$, where f_i is an integer-valued polynomial in x, vanishing for $x=0$, whose degree cannot exceed the number w_i of the symbols conditioned by E_i. This is precisely what is asserted in Theorem 3.1.

4. *Regular p-groups.*

4.1. *Definition* 4.11. The p-group G will be called *regular* if, given any positive integer α and any pair of elements P and Q of G, it is always possible to find elements S_3, S_4, ..., S_r all belonging to the derived group $H_2(K)$ of $K=\{P, Q\}$ and satisfying the equation

$$(PQ)^{p^\alpha} = P^{p^\alpha} Q^{p^\alpha} S_3^{p^\alpha} S_4^{p^\alpha} \dots S_r^{p^\alpha}. \tag{4.12}$$

The elements S and the number r will depend in general on α, P, and Q. The preceding § 3 yields as its main result

THEOREM 4.1. *If, in the p-group G, every subgroup of the form $\{P, Q\}$—i.e. every subgroup which can be generated by two elements P and Q of G—is of class less than p, then G is regular.*

COROLLARY 4.13. *Every p-group of class less than p is regular.*

COROLLARY 4.14. *Every p-group of order less than or equal to p^p is regular.*

Proof of (4.1). If $\{P, Q\}$ is of class less than p, then, by (2.53), every complex commutator of P and Q of weight greater than or equal to p must be equal to E. Hence, if in Theorem 3.2 R_r is the last term of the

series $R_1 = P$, $R_2 = Q$, R_3, ... which has a weight less than p, we may write

$$(PQ)^{p^\alpha} = P^{p^\alpha} Q^{p^\alpha} R_3^{n_3} R_4^{n_4} \dots R_r^{n_r}.$$

But, since, for $\lambda = 3, 4, \dots, r$, the weight w_λ of R_λ is less than p, therefore, by (3.2), $n_\lambda = m_\lambda p^\alpha$, say, is divisible by p^α. Writing $S_\lambda = R_\lambda^{m_\lambda}$ and noting that R_λ being for $\lambda = 3, 4, \dots, r$ of weight greater than or equal to 2 in P and Q must lie in $H_2(K)$, where $K = \{P, Q\}$, the result follows at once from definition (4.11.).

It will be seen from this theorem that the regular p-groups form an important and extensive class. Indeed, from one point of view, they are entitled to be considered the *general* case, irregular groups as exceptional. For, if we consider the groups of order p^n for fixed n and arbitrary p, it is clear that only a finite number can be irregular, namely certain of those for which $p < n$.

It may be mentioned here that, although all groups of order p^n having $n \leqslant p$ are regular, there exists for every p *an irregular group of order* p^{p+1}, viz. the Sylow subgroup of that order in the symmetric group of order p^2! This Sylow subgroup is of class p. It can be generated by two elements of order p, but, nevertheless, contains elements of order p^2; and such a state of affairs will be shown impossible in a regular group.

Immediate consequences of (4.11) are: (1) every subgroup or factor-group of a regular group is regular; and (2) every p-group whose elements other than E are all of order p is regular; and (3) G is regular if and only if all its subgroups of the form $\{P, Q\}$ are regular.

For applications we shall need by the side of (4.12) the analogous result

THEOREM 4.14. *If α is any positive integer and P and Q are any two elements of the regular p-group G, then it is always possible to find elements $S_3, S_4, \dots, S_s$ belonging to the derived group $H_2(L)$ of $L = \{P, R\}$, where $R = (P, Q)$, and satisfying the equation*

$$(P^{p^\alpha}, Q) = R^{p^\alpha} S_3^{p^\alpha} S_4^{p^\alpha} \dots S_s^{p^\alpha}. \tag{4.15}$$

The S's and also the number s will depend in general on α, P, and Q.

(4.15) is obtained by writing R for Q in (4.12). This gives, on the left,

$$(PR)^{p^\alpha} = (Q^{-1} PQ)^{p^\alpha} = Q^{-1} P^{p^\alpha} Q,$$

and on the right, say,

$$P^{p^\alpha} R^{p^\alpha} S_3^{p^\alpha} S_4^{p^\alpha} \dots S_s^{p^\alpha},$$

where $S_3, S_4, \ldots, S_s$ are certain elements of $H_2(L)$. But

$$(P^{p^\alpha}, Q) = P^{-p^\alpha} Q^{-1} P^{p^\alpha} Q,$$

and substituting in this the value found for $Q^{-1} P^{p^\alpha} Q$, we obtain (4.15), as stated.

4.2. *The orders of products and commutators.*

The present section is devoted to deriving the simple elementary properties of regular groups on which the more precise results of (4.4) and (4.5) all depend.

THEOREM 4.21. *If G is a regular p-group, then the product of the p^α-th powers of two or more elements of G is itself the p^α-th power of some element of G.*

Proof. It will suffice to show that, if P and Q are any two elements of G, then there exists an element R of G satisfying the equation

$$P^{p^\alpha} Q^{p^\alpha} = R^{p^\alpha}.$$

For then the result for the product of more than two p^α-th powers will follow by repeated applications.

Now, if G is Abelian, we may satisfy the equation with $R = PQ$. We may suppose then that G is non-Abelian and (arguing by induction over the order of G) that the theorem holds for every proper subgroup of G. Since G is regular, we may apply (4.12) and this gives

$$P^{p^\alpha} Q^{p^\alpha} = R_r^{p^\alpha} \ldots R_2^{p^\alpha} R_1^{p^\alpha} (PQ)^{p^\alpha}, \tag{1}$$

with $R_1, R_2, \ldots, R_r$ all in $H_2(K)$, $K = \{P, Q\}$, and so *a fortiori* all in $H_2 = H_2(G)$. Hence $\{PQ, R_1, R_2, \ldots, R_r\} = L$ say, is contained in $\{PQ, H_2\}$; and, since G is not Abelian, $\{PQ, H_2\}$ is a proper subgroup of G. Therefore L is a proper subgroup of G, and so the theorem is true of L by hypothesis. Hence the right-hand side of (1), being the product of the p^α-th powers of a number of elements of L, is itself the p^α-th power of some element R of L, *i.e.* $P^{p^\alpha} Q^{p^\alpha} = R^{p^\alpha}$.

THEOREM 4.22. *If P and Q are any two elements of the regular group G and if P^{p^α} is the least power of P to be permutable with Q, then the order of (P, Q) is p^α.*

Since P^{p^α} is clearly the least power of P to lie in the central of $\{P, Q\}$, we may state (4.22) in the alternative form: *the order of (P, Q) is equal to the order of P relative to the central of $\{P, Q\}$.*

COROLLARY 4.23. *If P^{p^s} is the least power of P to be permutable with Q, then Q^{p^a} is the least power of Q to be permutable with P.*

COROLLARY 4.24. *The order of (P, Q) cannot exceed the orders of either P or Q relative to the central of $\{P, Q\}$.*

More generally, by repeated applications of (4.24), we have

THEOREM 4.25. *In a regular p-group G, the order of any complex commutator of weight greater than 1 and containing the element P as a component can never exceed the order of P relative to the central of G.*

It will be convenient to prove (4.22) and its corollaries simultaneously with the important

THEOREM 4.26. *If P and Q are any two elements of the regular group G, then the order of PQ cannot exceed the orders of both P and Q. More generally, the order of the product of any number of elements of G cannot exceed the orders of all the factors.*

Proof of (4.22)–(4.26). This batch of theorems are all trivially true when G is Abelian. We may, therefore, argue by induction over the order of G and assume that G is non-Abelian and that (4.22)–(4.26) are all true of every proper subgroup of G.

Since G is regular, we may use formula (4.15), giving

$$(P^{p^a}, Q) = R^{p^a} S_3^{p^a} S_4^{p^a} \dots S_s^{p^a}, \tag{1}$$

where $R = (P, Q)$ and $S_3, S_4, \dots, S_s$ all belong to $H_2(L)$, where $L = \{P, R\}$. Now, since G is non-Abelian, L is a proper subgroup of G. By hypothesis, then, (4.25) is true of L, and, since the generators of $H_2(L)$ according to (2.82) all involve R as component, it follows that none of these generators can exceed in order the order of R. Now the S_λ's ($\lambda = 3, 4, \dots, s$) are all in $H_2(L)$ and so are products of these generators of $H_2(L)$. And, since by hypothesis (4.26) is true of L, it follows that the order of any S_λ is certainly no greater than the order of R. If then $R^{p^a} = E$, we have $S_\lambda^{p^a} = E$ for $\lambda = 3, 4, \dots, s$, and so, by (1), $(P^{p^a}, Q) = E$, *i.e.* P^{p^a} is permutable with Q.

Conversely, suppose that $(P^{p^a}, Q) = E$. Then P^{p^a} lies in the central of $K = \{P, Q\}$ and *a fortiori* in the central of $L = \{P, R\}$. Since the generators of $H_2(L)$ according to (2.82) all involve P as component, and since (4.25) is true of L, it follows that none of these generators can have an order exceeding that of P relative to the central of L, certainly then

none can be of order greater than p^α. As before, therefore, $S_\lambda^{p^\lambda} = E$ for $\lambda = 3, 4, \ldots, s$, and so (1) yields $R^{p^\alpha} = E$.

Thus each of the two equations $(P^{p^\alpha}, Q) = E$ and $(P, Q)^{p^\alpha} = E$ implies the other; and this is precisely (4.22). With (4.22), we have also proved its corollaries. There remains (4.26). Here we may use the formula (4.12), viz.

$$(PQ)^{p^\lambda} = P^{p^\lambda} Q^{p^\lambda} S_3^{p^\lambda} S_4^{p^\lambda} \ldots S_r^{p^\lambda}, \tag{2}$$

where now $S_3, S_4, \ldots, S_r$ all belong to $H_2(K)$, where $K = \{P, Q\}$. Suppose that both P^{p^α} and Q^{p^α} are equal to E. Then, as we have just proved (4.25) for G, it follows that none of the generators of $H_2(K)$ according to (2.82) can have an order exceeding p^α. And, since $H_2(K)$ is a proper subgroup of G, and so by hypothesis (4.26) is true in $H_2(K)$, it follows that none of the elements of $H_2(K)$ can have an order exceeding p^α. Therefore $S_\lambda^{p^\lambda} = E$ for $\lambda = 3, 4, \ldots, r$; and (2) then gives $(PQ)^{p^\alpha} = E$ as required. The second clause of (4.26) then follows by repeated applications of the first clause, and all is proved.

THEOREM 4.27. *If P and Q are any two elements of the regular p-group G, and if the order of (P, Q) is p^α, then, for $\alpha_1 + \alpha_2 \geqslant \alpha$, $P^{p^{\alpha_1}}$ is permutable with $Q^{p^{\alpha_2}}$, while, for $\alpha_1 + \alpha_2 < \alpha$, $(P^{p^{\alpha_1}}, Q^{p^{\alpha_2}})$ is of order $p^{\alpha - \alpha_1 - \alpha_2}$.*

Proof. If $\alpha_1 \geqslant \alpha$, then, by (4.22), $P^{p^{\alpha_1}}$ is permutable with Q, and therefore with all $Q^{p^{\alpha_2}}$; if, however, $\alpha_1 < \alpha$, then, by (4.22), $(P^{p^{\alpha_1}})^{p^{\alpha - \alpha_1}} = P^{p^\lambda}$ is the first power of $P^{p^{\alpha_1}}$ to be permutable with Q, and therefore, again by (4.22), the order of $(P^{p^{\alpha_1}}, Q)$ is equal to $p^{\alpha - \alpha_1}$. Hence, if $\alpha_2 \geqslant \alpha - \alpha_1$, then, by (4.22), $Q^{p^{\alpha_2}}$ is permutable with $P^{p^{\alpha_1}}$. While if $\alpha_2 < \alpha - \alpha_1$ then, by (4.22), $Q^{p^{\alpha - \alpha_1}}$ being the first power of Q to be permutable with $P^{p^{\alpha_1}}$, $(Q^{p^{\alpha_2}})^{p^{\alpha - \alpha_1 - \alpha_2}}$ is the first power of $Q^{p^{\alpha_2}}$ to be permutable with $P^{p^{\alpha_1}}$, so that, by (4.22), the order of $(Q^{p^{\alpha_2}}, P^{p^{\alpha_1}})$ or of $(P^{p^{\alpha_1}}, Q^{p^{\alpha_2}})$ must be just $p^{\alpha - \alpha_1 - \alpha_2}$.

Conversely, of course, if $(P^{p^{\alpha_1}}, Q^{p^{\alpha_2}})$ is of order p^β, then, provided that $\beta > 0$, (P, Q) must be of order $p^{\alpha_1 + \alpha_2 + \beta}$.

THEOREM 4.28. *If P and Q are any two elements of the regular p-group G and if $P^{p^\alpha} = Q^{p^\alpha}$, then $(PQ^{-1})^{p^\alpha} = E$, and conversely.*

Proof. (i) Let $P^{p^\alpha} = Q^{p^\alpha}$. Then, by (4.12), we have

$$(PQ^{-1})^{p^\lambda} = S_3^{p^\alpha} S_4^{p^\alpha} \ldots S_r^{p^\alpha},$$

where the S's are all elements of the derived group of

$$K = \{P, Q^{-1}\} = \{P, Q\}.$$

But, since Q^{p^α} is a power of both P and Q, it belongs to the central of K. Hence, by (4.25), no complex commutator in P and Q^{-1} of weight greater than 1 can be of order greater than p^α. But $H_2(K)$ is generated by such commutators by (2.82). Therefore, by (4.26), no element of $H_2(K)$ can have an order greater than p^α. $S_i^{p^\alpha} = E$ for $i = 3, 4, \ldots, r$. $(PQ^{-1})^{p^\alpha} = E$.

(ii) Let $(PQ^{-1})^{p^\alpha} = E$. Writing $S = PQ^{-1}$, we have

$$P^{p^\alpha} = (SQ)^{p^\alpha} = Q^{p^\alpha} T_3^{p^\alpha} \ldots T_s^{p^\alpha}$$

by (4.12), where $T_1, T_2, \ldots, T_s$ are elements of the derived group of $K = \{S, Q\} = \{P, Q\}$. Since $S^{p^\alpha} = E$, none of the complex commutators in K which involve S as a component can have an order greater than p^α. But $H_2(K)$ is generated by such commutators, and therefore, by (4.26), no element of $H_2(K)$ can be of order greater than p^α. $T_i^{p^\alpha} = E$ for $i = 3, 4, \ldots, r$. $P^{p^\alpha} = Q^{p^\alpha}$.

4.3. *The characteristic subgroups* $\mho_\alpha$ *and* Ω_β.

From now on, G will denote any regular p-group as defined in (4.11). In the present section we shall discuss the results (4.21) and (4.26) of the preceding section and the important (4.28), leaving to a subsequent section the consequences of (4.22) and (4.27).

The invariant μ. We shall denote by p^μ the highest order of any element of G. μ is to be placed by the side of the numbers c and d as an important arithmetical invariant of the group. When any ambiguity is possible, we write $\mu = \mu(G)$.

The subgroups $\mho_\alpha$. By (4.21), the p^α-th powers of the elements of G form a group. This group is clearly a characteristic subgroup of G and will be denoted by $\mho_\alpha$ or $\mho_\alpha(G)$.

The subgroups Ω_β. By (4.26), the elements of G whose orders divide p^β form a group. This group is clearly a characteristic subgroup of G and will be denoted by Ω_β or $\Omega_\beta(G)$.

Since G possesses an element of order p^μ but none of order $p^{\mu+1}$ we have

$$(4.31) \qquad \begin{cases} G = \mho_0 > \mho_1 > \ldots > \mho_{\mu-1} > \mho_\mu = E, \\ G = \Omega_\mu > \Omega_{\mu-1} > \ldots > \Omega_1 > \Omega_0 = E. \end{cases}$$

It will sometimes be convenient to write $\mho_\lambda = G$, $\Omega_\lambda = E$ for λ negative, and $\mho_\lambda = E$, $\Omega_\lambda = G$ for $\lambda > \mu$.

Another obvious result is that

$$(4.32) \qquad \begin{cases} \mho_\alpha \leqslant \Omega_{\mu-\alpha} & (\alpha = 0, 1, \ldots, \mu), \\ \mho_{\alpha-1} \not\leqslant \Omega_{\mu-\alpha} & (\alpha = 1, 2, \ldots, \mu), \end{cases}$$

which again expresses that p^μ is the highest order of an element of G.

The factor-groups $\mho_\alpha/\mho_{\alpha+1}$ and $\Omega_{\alpha+1}/\Omega_\alpha$ have, for $\alpha = 0, 1, \ldots, \mu-1$, all their elements other than E of order p, *i.e.*

$$\mu(\mho_\alpha/\mho_{\alpha+1}) = \mu(\Omega_{\alpha+1}/\Omega_\alpha) = 1.$$

More generally, for $\alpha = 0, 1, \ldots, \mu-\beta$, we have

$$\mu(\mho_\alpha/\mho_{\alpha+\beta}) = \mu(\Omega_{\alpha+\beta}/\Omega_\alpha) = \beta.$$

Again, if K is any subgroup of G, then $\Omega_\alpha(K) = K \wedge \Omega_\alpha(G)$. While if K is self-conjugate in G, then $\mho_\alpha(G \div K) = \{K, \mho_\alpha(G)\}$. In general, an exact expression for $\mho_\alpha(K)$ and $\Omega_\alpha(G \div K)$ is not possible. However, we have, as is easily seen,

$$(4.33) \qquad \begin{cases} \mho_\alpha\{\mho_\beta(G)\} = \mho_{\alpha+\beta}(G), \\ \Omega_\alpha(G \div \Omega_\beta) = \Omega_{\alpha+\beta}(G), \end{cases}$$

which are true for $\alpha, \beta \geqslant 0$.

THEOREM 4.3. *The order p^{ω_α} of $\Omega_\alpha/\Omega_{\alpha-1}$ is equal to the order of $\mho_{\alpha-1}/\mho_\alpha$ for each $\alpha = 1, 2, \ldots, \mu$. Further, the invariants ω_α satisfy*

$$(4.34) \qquad \omega_1 \geqslant \omega_2 \geqslant \ldots \geqslant \omega_\mu.$$

Proof. In view of the inequalities (4.31), the first part of the theorem is equivalent to the statement that the order of $\mho_\alpha$ is equal to the order of G/Ω_α for $\alpha = 1, 2, \ldots, \mu$; and we shall prove it in this form.

Now, by (4.28), the elements of any coset of Ω_α in G all have the same p^α-th power; and conversely, if two elements of G have the same p^α-th power, then they belong to the same coset of Ω_α. Thus the number of p^α-th powers is equal to the number of cosets of Ω_α. This is the result required.

As for (4.34), it will be sufficient to prove that $\omega_1 \geqslant \omega_2$ or that the order of Ω_2/Ω_1 does not exceed the order of Ω_1. For then, applying this result to G/Ω_β, we obtain with the help of (4.33) that $\omega_{\beta+1} \geqslant \omega_{\beta+2}$.

Now $\mu(\Omega_2) = 2$. [We assume that $\mu(G) \geqslant 2$, otherwise there is nothing to prove.] Hence, by (4.32), applied to the group Ω_2, we have

$$\mho_1(\Omega_2) \leqslant \Omega_1(\Omega_2) = \Omega_1(G) = \Omega_1.$$

The order of $\Omega_2/\mho_1(\Omega_2)$ cannot, therefore, be less than the order of Ω_2/Ω_1. But, by the first part of (4.3) which has already been proved, the order of $\Omega_2/\mho_1(\Omega_2)$ is equal to the order of $\Omega_1(\Omega_2) = \Omega_1$. Hence the order of Ω_1 cannot be less than that of Ω_2/Ω_1.

The type and the type-invariants.

We shall now introduce for any regular p-group G a system of invariants strictly analogous in every respect to those used in the theory of Abelian groups. These new invariants, which we shall denote by

$$\mu_1, \mu_2, \ldots, \mu_\omega,$$

are completely determined by the invariants

$$\omega_1, \omega_2, \ldots, \omega_\mu,$$

defined in Theorem 4.3. Conversely, the ω_α's are completely determined by the μ_β's. The definition is as follows:

(4.35) *μ_β is equal to the number of ω_α's which are greater than or equal to β, for $\beta = 1, 2, \ldots, \omega = \omega_1$.*

Obviously $$\mu = \mu_1 \geqslant \mu_2 \geqslant \ldots \geqslant \mu_\omega.$$

Somewhat less evident is the fact that, reciprocally,

(4.36) *ω_α is equal to the number of μ_β's which are greater than or equal to α, and the order p^A of G satisfies the equations*

$$A = \omega_1+\omega_2+\ldots+\omega_\mu = \mu_1+\mu_2+\ldots+\mu_\omega.$$

These results and other similar ones may be made obvious by introducing what we shall call the *figure* of the group. This is a matrix F of indefinite extent whose terms are all either 0 or 1, which is defined by the rule: for $\alpha = 1, 2, \ldots$ the α-th row of F consists first of ω_α terms all equal to 1 and then all 0's. (We write $\omega_\nu = 0$ if $\nu > \mu$.) This means that $f_{\alpha\beta} = 1$ for $\beta = 1, 2, \ldots, \omega_\alpha$ and $f_{\alpha\beta} = 0$ for $\beta > \omega_\alpha$.

Now it is clear from (4.35) that the β-th column of F consists first of μ_β terms all equal to 1 and then all 0's. (We write $\mu_\nu = 0$ if $\nu > \omega$.) This means that $f_{\alpha\beta} = 1$ for $\alpha = 1, 2, \ldots, \mu_\beta$ and $f_{\alpha\beta} = 0$ for $\alpha > \mu_\beta$.

The number $A = \omega_1+\omega_2+\ldots+\omega_\mu$ is equal to the number of $f_{\alpha\beta}$ which are equal to 1 and this gives the second part of (4.36), viz.

$$A = \mu_1+\mu_2+\ldots+\mu_\omega.$$

The relation between the ω_α's and the μ_β's is clearly a reciprocal one, and this gives the first part of (4.36).

In accord with the usual practice in Abelian p-group theory, we shall say that G is of

$$type\ (\mu_1, \mu_2, \ldots, \mu_\omega);$$

though for many purposes the ω_α's are more useful than the μ_β's. The μ_β's will be called the *type-invariants* of G.

As in Abelian groups, the number of type-invariants is equal to $\omega = \omega_1$, where p^ω is the order of the subgroup formed by the elements of G of order p (together with E).

THEOREM 4.37. *If K is any subgroup of G, then $\mu_\beta(K) \leqslant \mu_\beta(G)$, $\omega_\alpha(K) \leqslant \omega_\alpha(G)$, and $f_{\alpha\beta}(K) \leqslant f_{\alpha\beta}(G)$, for all $\alpha, \beta = 1, 2, \ldots$.*

We cannot, however, assert that, given invariants $\mu_\beta(K)$ satisfying these inequalities, there will necessarily exist a subgroup K of G fulfilling them.

Proof. The conditions are not, of course, independent. It will suffice to prove $\mu(K) \leqslant \mu(G)$, which is obvious, together with $\omega_\alpha(K) \leqslant \omega_\alpha(G)$ for $\alpha = 1, 2, \ldots, \mu(K)$. For then the other inequalities follow, as is easily seen by comparing the figures of K and G.

Now $\Omega_{\alpha-1}(K) = \Omega_\alpha(K) \wedge \Omega_{\alpha-1}(G)$, and therefore the order of

$$\Omega_\alpha(K)/\Omega_{\alpha-1}(K)$$

is the same as the order of

$$\{\Omega_\alpha(K), \Omega_{\alpha-1}(G)\}/\Omega_{\alpha-1}(G).$$

But $\{\Omega_\alpha(K), \Omega_{\alpha-1}(G)\} \leqslant \Omega_\alpha(G)$. Hence it cannot exceed the order of $\Omega_\alpha(G)/\Omega_{\alpha-1}(G)$, *i.e.* $\omega_\alpha(K) \leqslant \omega_\alpha(G)$.

Analogous to (4.37) is

THEOREM 4.38. *If K is any self-conjugate subgroup of G, then*

$$\mu_\beta(G/K) \leqslant \mu_\beta(G),$$

and $\omega_\alpha(G/K) \leqslant \omega_\alpha(G)$, and $f_{\alpha\beta}(G/K) \leqslant f_{\alpha\beta}(G)$, for all $\alpha, \beta = 1, 2, \ldots$.

Proof. It is obvious that $\mu(G/K) \leqslant \mu(G)$. It will suffice then to show that $\omega_\alpha(G/K) \leqslant \omega_\alpha(G)$ for $\alpha = 1, 2, \ldots, \mu(G/K)$. But $\mathfrak{V}_\alpha(G \div K)$ is equal to the join of $\mathfrak{V}_\alpha(G)$ and K. Hence its index $p^{\omega_\alpha(G/K)}$ in

$$\mathfrak{V}_{\alpha-1}(G \div K) = \{\mathfrak{V}_\alpha(G \div K), \mathfrak{V}_{\alpha-1}(G)\}$$

is equal to the index in $\mathfrak{V}_{\alpha-1}(G)$ of the meet of $\mathfrak{V}_{\alpha-1}(G)$ and $\mathfrak{V}_\alpha(G \div K)$. But this meet contains $\mathfrak{V}_\alpha(G)$, and therefore this index cannot exceed the index $p^{\omega_\alpha(G)}$ of $\mathfrak{V}_\alpha(G)$ in $\mathfrak{V}_{\alpha-1}(G)$. This gives $\omega_\alpha(G/K) \leqslant \omega_\alpha(G)$.

4.4. *Relations of permutability.*

In the present section we shall develop a set of theorems which, unlike those proved hitherto, have only trivial analogues in the theory of Abelian groups. We shall be concerned, roughly speaking, with the relations between the "*power-order*" or "$\mathfrak{V}-\Omega$"-structure of G and its "$H-Z$" or "*commutator*"-structure.

It is to be emphasised that this section is *not* required for the proof of the basis-theorem in (4.5). It is placed first merely because the results are regarded as of greater theoretical importance than (4.5).

THEOREM 4.4. *If L and M are any two self-conjugate subgroups of the regular group G, then*

$$\big(\mathfrak{V}_\alpha(L),\ \mathfrak{V}_\beta(M)\big) = \mathfrak{V}_{\alpha+\beta}\big((L,\ M)\big)$$

for all $\alpha, \beta = 0, 1, 2, \ldots$.

Proof. (i) If $\mathfrak{V}_{\alpha+\beta}\big((L, M)\big) = E$, then, for all P of L and Q of M, we must have $(P, Q)^{p^{\alpha+\beta}} = E$. Hence, by (4.27), P^{p^α} is permutable with Q^{p^β}, *i.e.* $\big(\mathfrak{V}_\alpha(L),\ \mathfrak{V}_\beta(M)\big) = E$. But, (ii) if $\big(\mathfrak{V}_\alpha(L),\ \mathfrak{V}_\beta(M)\big) = E$, then, conversely for all P of L and Q of M, we must have P^{p^α} permutable with Q^{p^β} and hence, by (4.27), the order of (P, Q) cannot exceed $p^{\alpha+\beta}$, *i.e.* $\mathfrak{V}_{\alpha+\beta}\big((L, M)\big) = E$.

Now both $\big(\mathfrak{V}_\alpha(L),\ \mathfrak{V}_\beta(M)\big)$ and $\mathfrak{V}_{\alpha+\beta}\big((L, M)\big)$ are self-conjugate subgroups of G; applying the results just proved to their two factor groups, we have

(i) $$\big(\mathfrak{V}_\alpha(L),\ \mathfrak{V}_\beta(M)\big) \leqslant \mathfrak{V}_{\alpha+\beta}\big((L,\ M)\big),$$

(ii) $$\mathfrak{V}_{\alpha+\beta}\big((L,\ M)\big) \leqslant \big(\mathfrak{V}_\alpha(L),\ \mathfrak{V}_\beta(M)\big),$$

whence the result.

Repeated application of (4.4) yields the general result

THEOREM 4.41. *If $L_1, L_2, \ldots, L_r$ are any self-conjugate subgroups of the regular group G, and if*

$$L = f(L_1, L_2, \ldots, L_r)$$

is any complex commutator subgroup of weight 1 in each of the r components L_i†,

† The subgroups L_i are not necessarily distinct; and the theorem is still true if we allow an L_i to occur *formally* more than once in f, but it then merely yields a poorer result.

and if, for any integers $\alpha_1, \alpha_2, \ldots, \alpha_r$ *all greater than or equal to* 0, *we write*

$$\mathfrak{V}_{\alpha_i}(L_i) = M_i,$$

$$M = f(M_1, M_2, \ldots, M_r),$$

then we have the identity

$$M = \mathfrak{V}_\alpha(L),$$

where $\alpha = \alpha_1 + \alpha_2 + \ldots + \alpha_r$.

The proof is by induction over r, the theorem coinciding with (4.4) when $r = 2$. We may assume without loss of generality that $L = (L', L'')$ and $M = (M', M'')$, where $L' = f'(L_1, L_2, \ldots, L_s)$, $L'' = f''(L_{s+1}, \ldots, L_r)$, $M' = f'(M_1, M_2, \ldots, M_s)$, and $M'' = f''(M_{s+1}, \ldots, M_r)$, for some s satisfying $0 < s < r$; and that the theorem is true for f' and for f'', these being complex commutator forms of weights s and $r-s$, less than the weight r of f. If, then, $\alpha' = \alpha_1 + \alpha_2 + \ldots + \alpha_s$ and $\alpha'' = \alpha_{s+1} + \ldots + \alpha_r$, we have, by hypothesis, $M' = \mathfrak{V}_{\alpha'}(L')$ and $M'' = \mathfrak{V}_{\alpha''}(L'')$. Hence, by (4.4),

$$M = \mathfrak{V}_{\alpha'+\alpha''}(L) = \mathfrak{V}_\alpha(L).$$

As an immediate corollary of (4.41) we may deduce an important system of identities relating the H-series of the $\mathfrak{V}$-subgroups of the (regular) G with the $\mathfrak{V}$-series of the H-subgroups of G; together with a second system relating the derived series of the $\mathfrak{V}$-subgroups with the $\mathfrak{V}$-series of the derived subgroups. These identities are contained in

THEOREM 4.42. *For all integers* $\alpha, \delta = 0, 1, 2, \ldots$ *and* $\beta = 1, 2, \ldots$, *we have*

$$H_\beta(\mathfrak{V}_\alpha) = \mathfrak{V}_{\alpha\beta}(H_\beta), \tag{4.421}$$

and

$$\theta_\delta(\mathfrak{V}_\alpha) = \mathfrak{V}_{\alpha 2^\delta}(\theta_\delta). \tag{4.422}$$

The class of $\mathfrak{V}_\alpha$ *is equal to the greatest number* β *satisfying*

$$\alpha < \frac{1}{\beta}\,\mu(H_\beta). \tag{4.423}$$

The length of the derived series of $\mathfrak{V}_\alpha$ *is equal to the least number* δ *satisfying*

$$\alpha \geqslant \frac{1}{2^\delta}\,\mu(\theta_\delta). \tag{4.424}$$

Proof. For (4.421), take $r = \beta$, $L_1 = L_2 = \ldots = L_r = G$, and

$$L = (L_1, L_2, \ldots, L_r) = H_\beta$$

in Theorem 4.41. Putting $\alpha_1 = \alpha_2 = \ldots = \alpha_r = \alpha$, we obtain $\Sigma \alpha_i = \alpha\beta$, and hence $M = (M_1, M_2, \ldots, M_r) = H_\beta(\mathfrak{V}_\alpha) = \mathfrak{V}_{\alpha\beta}(H_\beta)$.

For (4.422), take $r = 2^\delta$, $L_1 = L_2 = \ldots = L_r = G$, and $L = \theta_\delta$ in Theorem 4.41. Putting $\alpha_1 = \alpha_2 = \ldots = \alpha_r = \alpha$, we obtain $\Sigma \alpha_i = \alpha 2^\delta$, and hence $M = \theta_\delta(\mathfrak{V}_\alpha) = \mathfrak{V}_{\alpha 2^\delta}(\theta_\delta)$.

The results (4.423) and (4.424) are obvious corollaries of (4.421) and (4.422) respectively.

More generally, we have the result: if $f(G)$ is any commutator subgroup of weight w in G, then

$$f(\mathfrak{V}_\alpha) = \mathfrak{V}_{\alpha w}\big(f(G)\big). \tag{4.425}$$

The proof of this, being exactly similar to that of (4.421) and (4.422), may be omitted.

Replacing H_β and θ_δ on the right-hand sides of (4.421) and (4.422) by G, we obtain the results

$$H_\beta(\mathfrak{V}_\alpha) \leqslant \mathfrak{V}_{\alpha\beta},$$

$$\theta_\delta(\mathfrak{V}_\alpha) \leqslant \mathfrak{V}_{\alpha 2^\delta}.$$

Similarly (4.425) gives

$$f(\mathfrak{V}_\alpha) \leqslant \mathfrak{V}_{\alpha w}.$$

Whence

THEOREM 4.426. *The class of $\mathfrak{V}_\alpha/\mathfrak{V}_{\alpha\beta}$ cannot exceed $\beta - 1$; in particular $\mathfrak{V}_\alpha/\mathfrak{V}_{2\alpha}$ is Abelian, and so, if $\alpha \geqslant \frac{1}{2}\mu$, then $\mathfrak{V}_\alpha$ is Abelian. Again, the length of the derived series of $\mathfrak{V}_\alpha/\mathfrak{V}_{\alpha 2^\delta}$ cannot exceed δ.*

This theorem, though a good deal less precise than (4.42), is useful because it may be applied without a knowledge of the H-series of G.

For the class and length of the derived series of G itself, that is, of $\mathfrak{V}_0$, the preceding theorem gives only a trivial result. It is easy to derive a limit for the class of G in terms of the type-invariants. Here we must be content with the result:

(4.427) *The length of the derived series of G exceeds that of $G/\mathfrak{V}_1$ by a number less than or equal to δ, where 2^δ is the least power of 2 to be greater than or equal to μ.*

For if $2^\delta \geqslant \mu$, then $\mathfrak{V}_2 = E$ and by (4.426) the length of the derived series of $\mathfrak{V}_1$ is then less than or equal to δ.

Another corollary of (4.426) is the result:

(4.428) *For $\alpha = 1, 2, \ldots, \mu - 1$, $\mathfrak{V}_{\alpha+1}$ is the principal subgroup of $\mathfrak{V}_\alpha$; and $d(\mathfrak{V}_\alpha) = \omega_{\alpha+1}$.*

For, by (4.426), $\mathfrak{V}_\alpha/\mathfrak{V}_{\alpha+1}$ is Abelian for $\alpha \geqslant 1$; and, by (4.33),

$$\mathfrak{V}_{\alpha+1} = \mathfrak{V}_1(\mathfrak{V}_\alpha).$$

From the first it follows that $D(\mathfrak{V}_\alpha) \leqslant \mathfrak{V}_{\alpha+1}$, and from the second that $D(\mathfrak{V}_\alpha) \geqslant \mathfrak{V}_{\alpha+1}$. Therefore $D(\mathfrak{V}_\alpha) = \mathfrak{V}_{\alpha+1}$; and as a corollary, since the order of $\mathfrak{V}_\alpha/\mathfrak{V}_{\alpha+1}$ is $p^{\omega_{\alpha+1}}$, we have $d(\mathfrak{V}_\alpha) = \omega_{\alpha+1}$.

We may now proceed to consider the relations of permutability connecting *different* members of the $\mathfrak{V}$-series of G. Theorem 4.41 enables us to write down

THEOREM 4.43. *The simple commutator subgroups* $(\mathfrak{V}_{\alpha_1}, \mathfrak{V}_{\alpha_2}, \ldots, \mathfrak{V}_{\alpha_r})$, *having given* r *and given*

$$\alpha = \alpha_1 + \alpha_2 + \ldots + \alpha_r,$$

are all the same, and are identical with $\mathfrak{V}_\alpha(H_r)$. *In particular, for* $r = 2$, *every element of* $\mathfrak{V}_{\alpha_1}$ *is permutable with every element of* $\mathfrak{V}_{\alpha_2}$ *if and only if*

$$\alpha_1 + \alpha_2 \geqslant \mu(H_2).$$

Proof. Take $L_1 = L_2 = \ldots = L_r = G$ and $L = (L_1, L_2, \ldots, L_r)$ in Theorem 4.41. Then $M = (\mathfrak{V}_{\alpha_1}, \mathfrak{V}_{\alpha_2}, \ldots, \mathfrak{V}_{\alpha_r}) = \mathfrak{V}_\alpha(L) = \mathfrak{V}_\alpha(H_r)$. For $r = 2$, we have $(\mathfrak{V}_{\alpha_1}, \mathfrak{V}_{\alpha_2}) = \mathfrak{V}_{\alpha_1+\alpha_2}(H_2), = E$ if and only if $\mu(H_2) \leqslant \alpha_1 + \alpha_2$.

This corollary may also be expressed thus:

(4.431) *If* $\alpha \geqslant \beta$ *and* $\alpha + \beta \geqslant \mu(H_2)$, *then* $\mathfrak{V}_\alpha$ *belongs to the central of* $\mathfrak{V}_\beta$.

It may be remarked that, while naturally we always have

$$\mu(H_2) \leqslant \mu = \mu(G),$$

there are two cases in which a more precise statement is possible. If, namely, G has only one highest μ-invariant, *i.e.* if $\mu_2 < \mu_1$, then G/Ω_{μ_2} is cyclical and so $H_2 \leqslant \Omega_{\mu_2}$ or $\mu(H_2) \leqslant \mu_2$. Again, if G has just two highest μ-invariants, *i.e.* if $\mu = \mu_1 = \mu_2 > \mu_3$, then $G/\Omega_{\mu-1}$ is of order p^2 and so is Abelian; and therefore $H_2 \leqslant \Omega_{\mu-1}$, or $\mu(H_2) \leqslant \mu - 1$.

For later use, it will be convenient to prove at this point

THEOREM 4.432. *If* $H_\beta \leqslant \Omega_\alpha$, *then* $Z_{\beta-1} \geqslant \mathfrak{V}_\alpha$ *and conversely.*

For $H_\beta \leqslant \Omega_\alpha$ is equivalent to $\mathfrak{V}_\alpha(H_\beta) = E$. But, by (4.43),

$$\mathfrak{V}_\alpha(H_\beta) = (\mathfrak{V}_\alpha, \underbrace{G, G, \ldots, G}_{\beta-1}),$$

and the equation $\mathfrak{V}_\alpha(H_\beta) = E$ is therefore equivalent to $\mathfrak{V}_\alpha \leqslant Z_{\beta-1}$.

An interesting result of a somewhat different kind which will also be required subsequently is

THEOREM 4.433. *If* $H_{\beta+1} \leqslant \mathfrak{V}_\alpha$, *then, for all* $k = 0, 1, \ldots$, *we have* $H_{k\beta+1} \leqslant \mathfrak{V}_{k\alpha}$.

The proof is by induction over k. Suppose that we have already proved that $H_{k\beta+1} \leqslant \mathfrak{V}_{k\alpha}$. Then

$$H_{(k+1)\beta+1} = (H_{k\beta+1}, \underbrace{G, G, \ldots, G}_{\beta})$$

$$\leqslant (\mathfrak{V}_{k\alpha}, \underbrace{G, G, \ldots, G}_{\beta}) = \mathfrak{V}_{k\alpha}(H_{\beta+1})$$

by (4.43).

But, since (by hypothesis) $H_{\beta+1} \leqslant \mathfrak{V}_\alpha$, this gives

$$H_{(k+1)\beta+1} \leqslant \mathfrak{V}_{k\alpha}(\mathfrak{V}_\alpha) = \mathfrak{V}_{(k+1)\alpha}.$$

The theorem then follows generally.

It is clear from (4.433) that, from any inequality of the form $H_{\beta+1} \leqslant \mathfrak{V}_\alpha$ with both α and β positive, a limit for the class of G in terms of $\mu(G)$, α, and β can be written down. For instance, with $\alpha = \beta = 1$, it follows that, if $H_2 \leqslant \mathfrak{V}_1$, or, what is clearly equivalent, if $\mathfrak{V}_1$ is the principal subgroup of G, then $H_{k+1} \leqslant \mathfrak{V}_k$ for all k, and $c \leqslant \mu$.

The theorems which we have dealt with hitherto may be described as $(\mathfrak{V}, \mathfrak{V})$-theorems, since they all derive directly from the fundamental result (4.4). We must now consider a class of theorems of a less precise kind, which may be characterised as $(\mathfrak{V}, \Omega)$- and (Ω, Ω)-theorems. These all derive from

THEOREM 4.44. *If* L *and* M *are any two self-conjugate subgroups of the regular group* G, *and if* $N = (L, M)$, *then, for any integers* $\alpha, \beta = 0, 1, 2, \ldots$,

$$\left(\Omega_\alpha(G \div L), \Omega_\beta(G \div M)\right) \leqslant \Omega_{\alpha+\beta}(G \div N), \tag{4.441}$$

while, if $\beta \geqslant \alpha$, *then*

$$\left(\mathfrak{V}_\alpha(L), \Omega_\beta(G \div M)\right) \leqslant \Omega_{\beta-\alpha}(G \div N), \tag{4.442}$$

and, if $\alpha \geqslant \beta$, *then*

$$\left(\mathfrak{V}_\alpha(L), \Omega_\beta(G \div M)\right) \leqslant \mathfrak{V}_{\alpha-\beta}(N). \tag{4.443}$$

It may be well to recall that $\Omega_a(G \div L)$, *e.g.*, denotes the subgroup of all elements of G whose p^a-th powers lie in L; for it is the subgroup of G which "corresponds" to the Ω_a of G/L.

It may also be noted that the inequality signs of (4.44) cannot, in general, be replaced by equalities; for it is by no means necessary that, *e.g.*, $\Omega_{\alpha+\beta}(G \div N)$ should even belong to the derived group of G.

The result (4.44) may be proved either directly or with the help of (4.4). Since it is not subsequently needed, we omit the proof.

Repeated application of (4.44) yields the general result

THEOREM 4.45. *If $L_1, L_2, \ldots, L_r$ are any self-conjugate subgroups of the regular group G, and if*

$$L = f(L_1, L_2, \ldots, L_r)$$

is any complex commutator subgroup of weight 1 in each of the r components L_i, and if, for any integers $\alpha_1, \alpha_2, \ldots, \alpha_r$, we write

$$M_i = \mho_{\alpha_i}(L_i) \quad \text{if} \quad \alpha_i \geqslant 0,$$

and

$$M_i = \Omega_{-\alpha_i}(G \div L_i) \quad \text{if} \quad \alpha_i \leqslant 0,$$

and

$$M = f(M_1, M_2, \ldots, M_r),$$

then we have either

$$M \leqslant \mho_\alpha(L)$$

or

$$M \leqslant \Omega_{-\alpha}(G \div L),$$

according as the number

$$\alpha = \alpha_1 + \alpha_2 + \ldots + \alpha_r$$

is greater than or equal to 0 or less than or equal to 0.

The proof is by induction over r, the theorem following from (4.4) and (4.44) when $r = 2$. We may assume without loss of generality that $L = (L', L'')$ and $M = (M', M'')$, where $L' = f'(L_1, L_2, \ldots, L_s)$, $L'' = f''(L_{s+1}, \ldots, L_r)$, $M' = f'(M_1, M_2, \ldots, M_s)$, $M'' = f''(M_{s+1}, \ldots, M_r)$, for some s satisfying $0 < s < r$; and that the theorem is true both for f' and for f'', these being complex commutator forms of weights s and $r-s$, less than the weight r of f. If, then, $\alpha' = \alpha_1 + \alpha_2 + \ldots + \alpha_s$ and $\alpha'' = \alpha_{s+1} + \ldots + \alpha_r$, we have, by hypothesis, that $M' \leqslant \mho_{\alpha'}(L')$, where, should α' be negative, we take this to mean $M \leqslant \Omega_{-\alpha'}(G \div L')$. Similarly, with the same convention, $M'' \leqslant \mho_{\alpha''}(L'')$. Hence $M = (M', M'')$ belongs to $\left(\mho_{\alpha'}(L'), \mho_{\alpha''}(L'')\right)$ and, by (4.4) and (4.44), this latter group belongs to $\mho_{\alpha'+\alpha''}\left((L', L'')\right) = \mho_\alpha(L)$ whatever the signs of α', α'', and α.

Theorem 4.45 is of wide application. We prefer, however, for the limited deductions which are at present in view to make use of the less general but more precise

THEOREM 4.46. *If L is any self-conjugate subgroup of the regular group G, and if M is the centraliser* of L in G, then $\Omega_a(G \div M)$ is the centraliser of $\mho_a(L)$ in G.*

Proof. By (4.23), if P and Q are any two elements of the regular p-group G, then the two equations $(P^{p^a}, Q) = E$ and $(P, Q^{p^a}) = E$ are equivalent.

Suppose then that P is any element of $\Omega_a(G \div M)$ so that P^{p^a} lies in M. Then $(P^{p^a}, Q) = E$ for all Q of L. Hence, by the remark just made, $(P, Q^{p^a}) = E$ for all Q of L, *i.e.* P belongs to the centraliser of $\mho_a(L)$. Conversely, if P belongs to the centraliser of $\mho_a(L)$ so that $(P, Q^{p^a}) = E$ for all Q of L, then $(P^{p^a}, Q) = E$ for all Q of L, and so P^{p^a} belongs to the centraliser M of L and P belongs to $\Omega_a(G \div M)$.

With $L = G$ and therefore $M = Z_1$ the central of G, we obtain the result that the centraliser of $\mho_a$ is $\Omega_a(G \div Z_1)$. Since the latter clearly contains Ω_a, we have as a corollary

THEOREM 4.461. *Every element of $\mho_a$ is permutable with every element of Ω_a. For $\beta \geqslant a$, $(\mho_a, \Omega_\beta) \leqslant \Omega_{\beta-a} \wedge \mho_a$.*

More generally, using Theorem (2.51), that $(H_\beta, Z_\beta) = E$, we have

$$(4.462) \qquad \begin{cases} \left(\mho_a(H_\beta),\ \Omega_a(G \div Z_\beta)\right) = E, \\ \left(\mho_a(Z_\beta),\ \Omega_a(G \div H_\beta)\right) = E, \end{cases}$$

a result which is, of course, a particular case of (4.44).

As a corollary of this, we have [in analogy with (4.432)]

$$(4.463) \qquad \begin{aligned} &\text{if} \quad H_\beta \geqslant \mho_a, \quad \text{then} \quad Z_\beta \leqslant \Omega_a(G \div Z_1); \\ &\text{if} \quad Z_\beta \geqslant \mho_a, \quad \text{then} \quad H_\beta \leqslant \Omega_a(G \div Z_1). \end{aligned}$$

Perhaps of more interest than these inequalities is

THEOREM 4.464. *If L is any self-conjugate subgroup of the regular group G which contains the central Z_1 of G, and if M is the centraliser of L*

* If L is any subgroup of a group G, the *centraliser* of L in G consists by definition of all those elements of G which are permutable with every element of L. The central of G is simply the centraliser of G itself in G. If L is self-conjugate in G, so is its centraliser in G. The centraliser of L may be defined alternatively as the greatest subgroup M of G satisfying $L, M) = E$.

in G, then $$\mu(G/M) = \mu(L/Z_1).$$

In particular, if L is any greatest self-conjugate Abelian subgroup of G, then

$$\mu(G/L) = \mu(L/Z_1).$$

Proof. $\mu(G/M)$ is the least number μ_1 for which $\Omega_{\mu_1}(G \div M) = G$. But, by (4.46), $\Omega_a(G \div M)$ is the centraliser of $\mathfrak{O}_a(L)$. Hence μ_1 is the least number for which $\mathfrak{O}_{\mu_1}(L) \leqslant Z_1$. It is therefore equal to $\mu(L/Z_1)$.

The corollary results from the well-known fact that, if L is a maximum self-conjugate Abelian subgroup of G, it coincides with its centraliser in G, *i.e.* $L = M$.

The condition that L shall contain Z_1 is clearly of minor importance. If it does not, we may replace Z_1 in the theorem by $L \wedge Z_1$. It may be observed, however, that $L = M$ implies $L \geqslant Z_1$.

A result somewhat similar to (4.464) may be worth mention. It is

THEOREM 4.465. *If L, K are any two self-conjugate subgroups of the regular group G, and if M is the centraliser of L in K, then*

$$\mu((L, K)) = \mu(K/M).$$

With G for K, this gives $\mu((L, G)) = \mu(L/Z_1)$ by comparison with (4.464).

Proof. $\mu((L, K))$ is the greatest number μ_1 for which p^{μ_1} is the order of an element of the form (P, Q) with P in L and Q in K. While $\mu(K/M)$ is the greatest number μ_2 for which p^{μ_2} is the order of an element Q of K relative to M, *i.e.* it is equal to the greatest number μ_2 for which there exists a P of L and a Q of K with the property that $Q^{p^{\mu_2}}$ is the first power of Q to be permutable with P. By (4.22), therefore, P^{μ_2} is the greatest order of an element of the form (P, Q) with P in L and Q in K. Whence $\mu_1 = \mu_2$.

Finally,

THEOREM 4.466. *If H, K, L, M are self-conjugate subgroups of the regular group G satisfying the relations $H \geqslant L$, $K \geqslant M$, $(L, M) = E$, then*

$$\mu((H, K)) \leqslant \mu(H/L) + \mu(K/M).$$

In particular, if L is a greatest self-conjugate Abelian subgroup of G, then $\mu(G/L) \geqslant \frac{1}{2}\mu(H_2)$.

The corollary is obtained by taking $H = K = G$, $L = M$.

(4.5) *Definition.* An ordered set of elements $P_1, P_2, \ldots, P_r$ of a group G, of order $\pi_1, \pi_2, \ldots, \pi_r$ respectively (the numbers π_i being all greater than 1, *i.e.* the elements $P_1, P_2, \ldots, P_r$ being all distinct from the identity E), will be called a *uniqueness basis* of G—more concisely a U.B. of G—if every element P of G can be expressed in one and only one way in the form

$$P = P_1^{x_1} P_2^{x_2} \ldots P_r^{x_r}$$

with $$0 \leqslant x_i < \pi_i \quad (i = 1, 2, \ldots, r).$$

The fundamental theorem of Abelian p-group theory—the so-called "basis theorem"—is that for such a group a U.B. can always be found. It is then comparatively easy to deduce that the number of terms in any two U.B.'s is the same, and the orders of the basis elements also is the same apart from the sequence in which they are written.

It is the object of the present section to extend these theorems of *Abelian* p-group theory to cover all *regular* p-groups. It will be shown further that any U.B. of any regular G possesses ω terms and that the orders of these terms are $p^{\mu_1}, p^{\mu_2}, \ldots, p^{\mu_\omega}$—where ω and the μ's are the invariants of G defined in §4.3.

THEOREM 4.51. *If $P_1, P_2, \ldots, P_r$ is a U.B. of the regular p-group G, then $r = \omega = \omega(G)$ and the orders of the elements P_i are given in descending order of magnitude by $p^{\mu_1}, p^{\mu_2}, \ldots, p^{\mu_\omega}$, where $\mu_1, \mu_2, \ldots, \mu_\omega$ are the type-invariants of G.*

Proof. Let G' denote the Abelian group having as U.B. the set $P_1', P_2', \ldots, P_r'$, where P_i' is of the same order as P_i. This rule determines G' to within an isomorphism. Moreover, by definition of U.B.,

$$P = P_1^{x_1} P_2^{x_2} \ldots P_r^{x_r} \longleftrightarrow P_1'^{x_1} P_2'^{x_2} \ldots P_r'^{x_r} = P'$$

is a 1–1 correspondence (naturally not an isomorphism unless G is Abelian) between the elements of G and those of G'.

Suppose that P' belongs to $\Omega_a(G')$. Then $P_i'^{x_i}$ must also do so, for each i. But the order of $P_i^{x_i}$ is the same as that of $P_i'^{x_i}$. Hence $P_i^{x_i}$ belongs to $\Omega_a(G)$ for all i, and hence P belongs to $\Omega_a(G)$. It follows that the order of $\Omega_a(G)$ is at least equal to that of $\Omega_a(G')$.

Next suppose that P' belongs to $\mho_a(G')$. Then $P_i'^{x_i}$ must also do so, for each i; that is, x_i must be divisible by p^a. Hence $P_i^{x_i}$ must belong to $\mho_a(G)$ for each i, and therefore P also must belong to $\mho_a(G)$. It follows that the order of $G/\mho_a(G)$ is at most equal to that of $G'/\mho_a(G')$.

We have proved then that

$$\text{ord } \Omega_a(G) \geqslant \text{ord } \Omega_a(G'),$$

$$\text{ord } G/\mho_a(G) \leqslant \text{ord } G'/\mho_a(G').$$

But the two left-hand terms in these two inequalities are equal by (4.3), as are also the two right-hand terms. Hence all four orders are equal. The ω-invariants, and therefore also the μ-invariants, of G and G' are the same.

The enunciation now follows at once, from the corresponding result for Abelian groups. And, incidentally, we have proved the following more precise result, that $P_1, P_2, \ldots, P_r$ being a U.B. of the regular group G, the element $P = P_1^{x_1} P_2^{x_2} \ldots P_r^{x_r}$ of G belongs to $\Omega_a(G)$ if *and only if* each of the factors $P_i^{x_i}$ does so; further, that P belongs to $\mho_a(G)$ if *and only if* each of the factors $P_i^{x_i}$ does so.

We now proceed to the proof of the basis theorem, viz. that every regular G possesses at least one U.B.

Definition 4.52. Let

$$G = L_0 > L_1 > L_2 > \ldots > L_\omega = \Omega_1$$

be any series of self-conjugate subgroups of G descending from G to $\mho_1$, with the property (which is implied by the above inequalities, since $G/\mho_1$ is of order p_ω) that L_i/L_{i+1} is of order p $(i = 0, 1, \ldots, \omega-1)$. Such a series will be called an *L-series* of G.

Then we define: the sequence of ω elements of G,

$$Q_1, Q_2, \ldots, Q_\omega,$$

will be called a *canonical basis* or C.B. of G if it satisfies the following two conditions:

(a) There exists an L-series of G with the property that each of the ω sets $L_{i-1} - L_i$ $(i = 1, 2, \ldots, \omega)$ contains just one of the ω elements Q_j.

(b) The product p^A of the orders of the ω elements Q_j is as small as possible subject to condition (a).

Since none of the sets $L_{i-1} - L_i$ is null, the existence of C.B.'s is immediately obvious; and it is easy to prove that, when G is Abelian, our definition coincides with the usual one. Our proof of the basis-theorem will consist in showing that

(4.53) *every C.B. is also a U.B.*

Whether conversely every U.B. is also a C.B. is a question that we shall leave open.

We prove, first, three properties of a C.B., viz. [with the notation of Definition (4.52)],

(*c*) the number p^A is equal to the order of G;

(*d*) two distinct Q_j's can have no power in common other than the identity; and

(*e*) there is at least one Q_i which is of order p^μ and such that the element $S = Q_i^{p^{a-1}}$ is self-conjugate in G.

Here $Q_1, Q_2, \ldots, Q_\omega$ is any given C.B. of G. Hence there exists an L-series

$$G = L_0 > L_1 > \ldots > L_\omega = \mathfrak{O}_1$$

such that one of the Q_j's lies in each of the sets $L_{i-1} - L_i$. Since (by definition) any rearrangement of the Q_j's is also a C.B., and since the arrangement of the Q_j's is irrelevant for each of the three properties (*c*), (*d*), (*e*) to be proved, there will be no loss of generality if we assume that, for each i, Q_i belongs precisely to the set $L_{i-1} - L_i$.

The proof of (*d*) will consist in showing that, if it is not true for the given C.B., then we can construct a set of ω elements $Q_1', Q_1', \ldots, Q_\omega'$ satisfying condition (*a*) and with the product of the orders of the Q_i''s smaller than p^A, contrary to the condition (*b*) satisfied by the Q_i's.

Suppose then that Q_i and Q_j with $i \neq j$ have a power $\neq E$ in common. We may write

$$Q_i^{p^\alpha} = Q_j^{\xi p^\beta} \neq E, \tag{1}$$

where ξ is prime to p and α and β are integers greater than or equal to 0. There will evidently be no loss of generality if we assume in addition that either (i) $\alpha = \beta$ and $i < j$, or (ii) $\alpha < \beta$. Then write $Q_i' = Q_i Q_j^{-\xi p^{\beta-\alpha}}$, and $Q_k' = Q_k$ for all $k \neq i$. Now, by (1), Q_i and $Q_j^{\xi p^{\beta-\alpha}}$ have the same p^α-th power. Hence, by (4.28), the order of Q_i' cannot exceed p^α. But, by (1) again, the order of Q_i is greater than p^α. Hence the product of the orders of all the Q_k''s is less than p^A. It remains only to point out that Q_i' lies in the set $L_{i-1} - L_i$. For in case (i), $i < j$ gives that Q_j belongs to L_i. While in case (ii), $\beta > \alpha$ gives that $Q_j^{p^{\beta-\alpha}}$ belongs to $\mathfrak{O}_1$ and hence *a fortiori* to L_i. In either case, therefore, Q_i' belongs to the same coset of L_i as Q_i, and the proof of (*d*) is complete.

Proof of (*e*). If all the elements Q_i were of order $p^{\mu-1}$ at most, we should have $G = \{\mathfrak{O}_1, Q_1, Q_2, \ldots, Q_\omega\} \leqslant \Omega_{\mu-1}$, which is absurd. Hence

at least one of the Q_i's is of order p^μ. Denote by Q_i the *last* term in the series $Q_1, Q_2, \ldots, Q_\omega$ which is of order p^μ. Then

$$L_i = \{\mathfrak{V}_1, Q_{i+1}, \ldots, Q_\omega\} \leqslant \Omega_{\mu-1};$$

while, since Q_i belongs to L_{i-1}, we have $\mu(L_{i-1}) = \mu$. Hence

$$\omega_\mu(L_{i-1}) = \omega(L_{i-1}/L_i),$$

since $L_i = \Omega_{\mu-1}(L_{i-1})$, and this is 1, since L_{i-1}/L_i is of order p. Hence $\mathfrak{V}_{\mu-1}(L_{i-1})$, equal to K, say, is of order p. Hence $K = \{S\}$, where $S = Q_i^{p^{\mu-1}}$; and since L_{i-1} and therefore K are self-conjugate in G, so is S.

Proof of (*c*). We define

$$W_\alpha = \{\mathfrak{V}_1, \Omega_\alpha\} \quad (\alpha = 0, 1, \ldots, \mu),$$

so that evidently

$$\mathfrak{V}_1 = W_0 \leqslant W_1 \leqslant \ldots \leqslant W_{\mu-1} < W_\mu = G.$$

Hence there exists at least one L-series of G having each of the W_α's among its members. If $L_0, L_1, \ldots, L_\omega$ is any L-series with this property, then the position of any given W_α in this series is clearly determined by the index p^{λ_α} of W_α in G. From the definition of W_α it follows that

$$\lambda_\alpha = \omega(G/W_\alpha) = \omega(G/\Omega_\alpha) = \omega(\Omega_{\alpha+1}/\Omega_\alpha) = \omega_{\alpha+1}.$$

Hence we have

$$W_\alpha = L_{\omega_{\alpha+1}} \quad (\alpha = 0, 1, \ldots, \mu-1). \qquad (1)$$

Choose now for Q_i an element of $L_{i-1} - L_i$ having as small an order as possible, which we denote for the moment by p^{m_i}. To find m_i, let $\omega_{\alpha+1}$ be the first term of the series $\omega_1, \omega_2, \ldots, \omega_\mu, \omega_{\mu+1} = 0$ which is less than i. Then $\omega_{\alpha+1} < i \leqslant \omega_\alpha$. Hence, by (1), $W_\alpha > L_i \geqslant W_{\alpha-1} \geqslant \Omega_{\alpha-1}$. $L_i \geqslant \Omega_{\alpha-1}$ gives $m_i \geqslant \alpha$. Also $W_\alpha > L_i$ implies (with our special type of L-series) that $W_\alpha \geqslant L_{i-1}$ or $L_{i-1} \leqslant \{\Omega_\alpha, \mathfrak{V}_1\}$, so that every coset of $\mathfrak{V}_1$ in L_{i-1} contains elements of order p^α at most and so $m_i \leqslant \alpha$. Combining, we have $m_i = \alpha$. In other words, m_i is equal to the number of ω-invariants of G which are greater than or equal to i. By (4.35) this is μ_i. Hence Q_i is of order p^{μ_i} and the product of the orders of the Q_i's is therefore p^B, where

$$B = \mu_1 + \mu_2 + \ldots + \mu_\omega.$$

By (4.36), p^B is equal to the order of G.

For our special choice of $Q_1, Q_2, \ldots, Q_\omega$ satisfying condition (*a*) the product of orders is equal to the order of G. And therefore (*c*) will be

completely proved if we can show that, for *any* set of elements Q_1', Q_2', ..., Q_ω' satisfying (a), the product of orders is never less than for our special set.

Let L_0', L_1', ..., L_ω' be the L-series (which exists by hypothesis) and for which Q_i' belongs to $L_{i-1}' - L_i'$. Then, if Q_i' belongs to W_α, we must have $W_\alpha \wedge L_{i-1}' > W_\alpha \wedge L_i'$, since the former contains Q_i' and the latter does not. It follows that the number of Q_i''s which belong to W_α cannot exceed $\omega - \omega_{\alpha+1}$ [for, by (1), $p^{\omega-\omega_{\alpha+1}}$ is precisely the order of

$$W_\alpha / L_\omega' = W_\alpha / \mathfrak{V}_1].$$

Since $W_\alpha \geqslant \Omega_\alpha$, the Q_i''s which are not in W_α must all be of order greater than p^α, and so it follows that at least $\omega_{\alpha+1}$ of the Q_i''s are of order greater than p^α; while exactly $\omega_{\alpha+1}$ of the Q_i's were of order greater than p^α. This being true for all α, the product of the orders of the Q_i''s cannot be less than the product of the orders of the Q_i's, which was the result required.

We may now conclude the proof of (4.53) as follows. Let

$$Q_1, Q_2, \ldots, Q_\omega$$

be any C.B. of G (so that we no longer necessarily have Q_i in the set $L_{i-1} - L_i$). Let Q_i be the element whose existence is asserted in (e) and let $K = \{S\}$, where $S = Q_i^{p^{\mu-1}}$; so that K is self-conjugate in G and of order p.

First we shall show that $Q_1 K, Q_2 K, \ldots, Q_\omega K$ is a C.B. of the group G/K; where, in case $\mu = 1$ and so $S = Q_i$, the term $Q_i K = K$ is to be omitted.

For, by property (d), the first power of Q_k $(k \neq i)$ to lie in K is the identity so that Q_k and $Q_k K$ have the same order, while the orders of Q_i and $Q_i K$ are p^μ and $p^{\mu-1}$ respectively. Hence the product of the orders of the elements $Q_k K$ of G/K is equal to p^{A-1}, where p^A is the product of the orders of the Q_k's, equals the order of G, by property (c). Therefore p^{A-1} is the order of G/K, and the elements $Q_k K$ of G/K satisfy condition (b), again by property (c). It remains to verify (a). This is self-evident when $\mu \neq 1$, for then $K \leqslant \mathfrak{V}_1$ and as L-series of G/K we may simply take L_0/K, L_1/K, ..., L_ω/K. But, if $\mu = 1$, then for Q_i we may clearly take that Q_k which lies in $L_{\omega-1}$; so that here $K = L_{\omega-1}$, and for L-series of G/K we may take L_0/K, L_1/K, ..., $L_{\omega-1}/K$. In either case, just one term of our prospective C.B. lies between two adjacent terms of the L-series of G/K, because the same is true of the Q_k's and the L-series of G.

The argument now proceeds by induction. Theorem 4.53 is trivial for groups of order p. We shall assume, therefore, that it is true for all regular groups whose order is less than that of G. In particular, we may assume it for G/K. Hence we may assume that

$$Q_1 K, Q_2 K, \ldots, Q_\omega K$$

is a U.B. for G/K. This means that every element P of G can be expressed in the form

$$P = Q_1^{x_1} Q_2^{x_2} \dots Q_\omega^{x_\omega} S^\xi.$$

But $S = Q_i^{p^{\mu-1}}$ and is permutable with every element of G, by property (e). Hence

$$P = Q_1^{y_1} Q_2^{y_2} \dots Q_\omega^{y_\omega}, \tag{2}$$

where $y_k = x_k$ for $k \neq i$ and $y_i = x_i + \xi p^{\mu-1}$. That the form (2) for P is unique, in the sense that, if P is also given by

$$P = Q_1^{z_1} Q_2^{z_2} \dots Q_\omega^{z_\omega},$$

then for each k we must have $Q_k^{y_k} = Q_k^{z_k}$, follows from the two facts already proved: (i) that every P can be expressed in the form (2), and (ii) that the product of the orders of the Q_k's is equal to the order of G.

This completes the proof of (4.53), and therefore of the basis theorem, in all details.

CONICS SATISFYING EIGHT CONDITIONS

By W. E. BAKER.

[Received 29 March, 1932.—Read 28 April, 1932.]

I. Conics in threefold space may be made to satisfy eight assigned conditions.

I shall use the following symbols:

P denotes that the conics pass through a given point,

l denotes that the conics meet a given straight line once,

u denotes that the planes of the conics pass through a given point,

t denotes that the conics touch a given plane,

T denotes that the conics touch a given plane at a given point,

k denotes that the conics touch a given straight line,

K denotes that the conics touch a given straight line at a given point,

r denotes that the conics meet two given straight lines in corresponding points of projective ranges,

R denotes that the conics meet three given straight lines in corresponding points of projective ranges.

Of these, l, t, and u are each equivalent to one condition,

P is equivalent to two conditions,

k, r, and T are each equivalent to three conditions,

K is equivalent to four conditions,

R is equivalent to five conditions.

These conditions are connected by the following relations:—

$$ul \equiv P + 2(uu). \tag{i}$$

$$T \equiv \tfrac{1}{2}(Pt). \tag{ii}$$

$$k \equiv uut - 2(uuu). \tag{iii}$$

$$K \equiv PP. \tag{iv}$$

$$r \equiv 2(Pl). \tag{v}$$

$$R \equiv 3(PPl). \tag{vi}$$

The relations (i), (ii), and (iii) are due to Schubert (1). It is easy to see that the condition "to touch a given straight line in a given point" is the same as "to pass through two given points". Hence (iv) is true.

Let C represent any five conditions, and let the solution of the problem PlC be N. Let the points of two straight lines x and y be connected by a projectivity in which a point X on x corresponds to a point Y on y. We may set up a correspondence between points Y and Y' on y in the following manner: to each point Y on y there corresponds one point X on x by the projectivity, and through X we can draw N conics meeting y in Y' and satisfying the conditions C, thus defining N points Y'. Thus one point Y leads to N points Y', and in the reverse correspondence, one point Y' leads to N points Y. The correspondence between Y and Y' is (N, N) and there are $2N$ self-corresponding points. Hence rC is twice PlC, which is (v).

Let D represent any three conditions, and let the solution of the problem $PPlD$ be M. Let the points of three straight lines x, y, and z be connected by a projectivity in which a point X on x corresponds to a point Y on y and a point Z on z. We may set up a correspondence between points Z and Z' on z in the following manner: to each point Z on z there corresponds one pair of points X on x and Y on y by the projectivity, and through X and Y we can draw M conics meeting z in Z' and satisfying D. Hence one point Z leads to M points Z'. In the reverse correspondence, through each point Z' we can draw $2M$ conics meeting x and y in a pair of corresponding points X and Y which correspond to $2M$ points Z by the projectivity. This is making use of the relation (v). Hence the correspondence between Z and Z' is $(M, 2M)$, and there are $3M$ self-corresponding points. Hence RD is three times $PPlD$, which is (vi).

If the projective ranges are on two rational curves of degrees x and y or on three rational curves of degrees x, y, and z, the relations (v) and (vi)

are modified:—

$$r \equiv (x+y)(Pl). \qquad \text{(v)}'$$

$$R \equiv (x+y+z)(PPl). \qquad \text{(vi)}'$$

In virtue of the relations (i) to (vi), it is only necessary to find the number of conics which satisfy eight conditions made up of various combinations

TABLE A.

l^8	92 L.	ul^7	34 L.	$u^2 l^6$	8	$u^3 l^5$	1 C.
$l^7 t$	116 H.	$ul^6 t$	52 H.	$u^2 l^5 t$	14 H.	$u^3 l^4 t$	2 C.
$l^6 t^2$	128 H.	$ul^5 t^2$	76 H.	$u^2 l^4 t^2$	24 U.	$u^3 l^3 t^2$	4 C.
$l^5 t^3$	104 H.	$ul^4 t^3$	72	$u^2 l^3 t^3$	24 U.	$u^3 l^2 t^3$	4 C.
$l^4 t^4$	64	$ul^3 t^4$	48	$u^2 l^2 t^4$	16	$u^3 lt^4$	2 C.
$l^3 t^5$	32	$ul^2 t^5$	24 H.	$u^2 lt^5$	8 H.	$u^3 t^5$	1 C.
$l^2 t^6$	16 H.	ult^6	12 H.	$u^2 t^6$	4 H.		
lt^7	8 H.	ut^7	6 H.				
t^8	4						

TABLE B.

Pl^6	18 L.	uPl^5	6	$u^2 Pl^4$	1 C.
$Pl^5 t$	24 H.	$uPl^4 t$	10 H.	$u^2 Pl^3 t$	2 C.
$Pl^4 t^2$	28	$uPl^3 t^2$	16	$u^2 Pl^2 t^3$	4 C.
$Pl^3 t^3$	24	$uPl^2 t^3$	16	$u^2 Plt^3$	4 C.
$Pl^2 t^4$	16	$uPlt^4$	12	$u^2 Pt^4$	2 C.
Plt^5	8 H.	uPt^5	6		
Pt^6	4 H.				

TABLE C.

$P^2 l^4$	4 L.	$uP^2 l^3$	1 C.
$P^2 l^3 t$	6 H.	$uP^2 l^2 t$	2 C.
$P^2 l^2 t^2$	8	$uP^2 lt^2$	4 C.
$P^2 lt^3$	8	$uP^2 t^3$	4 C.
$P^2 t^4$	8		

TABLE D.

$P^3 l^2$	1 C.
$P^3 lt$	2 C.
$P^3 t^2$	4 C.

of l, u, and t. These numbers are given in Table A. Those marked L. are due to Lüroth (**2**), those marked H. are due to Hierholzer (**3**) who proves their duals, those marked C. are due to Cayley (**4**), and those marked U. are due to Ursell (**7**). Two conditions uu make the planes of the conics belong to an axial pencil, three conditions uuu compel the conics to lie in a fixed plane, and if the number of conditions u is greater than three, the number of conics is zero.

Since some of the proofs depend upon the solution of simpler problems involving the condition P, I add tables giving the number of conics for all possible combinations of the four conditions, P, l, u, and t.

For convenience like conditions are collected, so that $u^2l^3t^3$ is to be interpreted *uullttt*, and so on.

Reference should here be made to a paper by Todd (**8**) on the number of space cubics satisfying twelve conditions.

II. The conics may meet a given curve 2, 3, ..., 8 times. The number of such conics is a function of the degree n and the number of apparent double points h, provided that the curve has no actual double points. The solution of a problem which involves the condition that a given curve is met 2, 3, ..., 8 times can be obtained from a functional equation if the solution is known for a sufficient number of particular values of n and h.

For example, let (vi) represent any six conditions, and let

$$A = ll\ (\text{vi}); \quad B = uu\ (\text{vi}); \quad C = P\ (\text{vi});$$

then the solution of 2 (vi) is a function f of n and h such that

$$f(2n,\ 2h+n^2)-2f(n,\ h) = An^2;$$

obtained by considering a compound curve consisting of two curves with the same n and h in general position.

The solution is

$$f = \tfrac{1}{2}(A-C)\,n(n-1)+Bn+Ch.$$

Similarly, the functional equation for 3 (v) depends upon the solution of $l2$ (v);

the functional equation for 4 (iv) depends upon the solution of $l3$ (iv) and 22 (iv);

the functional equation for 5 (iii) depends upon the solution of $l4$ (iii) and 23 (iii);

the functional equation for 6 (ii) depends upon the solution of $l5$ (ii), 24 (ii), and 33 (ii);

the functional equation for 7 (i) depends upon the solution of $l6$ (i), 25 (i), and 34 (i);

and the functional equation for 8 depends upon the solution of $l7$, 26, 35, and 44.

In what follows I use the letters n and h for degree and number of apparent double points respectively when only one curve is in question; m and n with h and k, when there are two curves; m, n, and o with g, h, and k, when there are three curves; and m, n, o, and p with f, g, h, and k, when there are four curves. If the conics meet curves of degrees m, n, ... each once, the number of such conics will contain the factors m, n, ... It is convenient to omit these factors in the formulae.

I give the solution of all problems which involve the conditions 2, 3, 4, 5, 6, 7, and 8 combined with P, l, u, and t.

Those marked B. are due to Berzolari (**5**), whose results I have verified. Severi (**6**) has obtained formulae reducing these problems to dependence upon the three simple conditions l, u, and t; but he gives his results in terms of the degree n and the rank r.

(1) $P^3 2 = \frac{1}{2}n(n-1).$

(2) $P^2 ul2 = \frac{1}{2}n(n-1).$

(3) $P^2 ut2 = n(n-1).$

(4) $P^2 u3 = \frac{1}{6}n(n-1)(n-2).$

(5) $Pu^2 l^2 2 = \frac{1}{2}n(n-1).$

(6) $Pu^2 lt2 = n(n-1).$

(7) $Pu^2 t^2 2 = 2n(n-1).$

(8) $Pu^2 l3 = \frac{1}{6}n(n-1)(n-2).$

(9) $Pu^2 t3 = \frac{1}{3}n(n-1)(n-2).$

(10) $Pu^2 22 = \frac{1}{4}mn(m-1)(n-1).$

(11) $Pu^2 4 = \frac{1}{24}n(n-1)(n-2)(n-3).$

(12) $u^3 l^3 2 = \frac{1}{2}n(n-1).$

(13) $u^3 l^2 t2 = n(n-1).$

(14) $u^3 lt^2 2 = 2n(n-1).$

(15) $u^3 t^3 2 = 2n(n-1).$

(16) $u^3 l^2 3 = \frac{1}{6}n(n-1)(n-2).$

(17) $u^3 lt3 = \frac{1}{3}n(n-1)(n-2).$

(18) $u^3 t^2 3 = \frac{2}{3}n(n-1)(n-2).$

(19) $u^3 l22 = \frac{1}{4}mn(m-1)(n-1).$

(20) $u^3 t22 = \frac{1}{2}mn(m-1)(n-1).$

(21) $u^3 l4 = \frac{1}{24}n(n-1)(n-2)(n-3).$

(22) $u^3 t4 = \frac{1}{12}n(n-1)(n-2)(n-3).$

(23) $u^3 23 = \frac{1}{12}mn(m-1)(n-1)(n-2).$

(24) $u^3 5 = \frac{1}{120}n(n-1)(n-2)(n-3)(n-4).$

B. (25) $P^2 l^2 2 = \frac{3}{2}n(n-1)+h.$

B. (26) $P^2 lt2 = 2n(n-1)+2h.$

B. (27) $P^2 t^2 2 = 2n(n-1)+4h.$

B. (28) $P^2 l3 = \frac{1}{6}n(n-1)(n-2)+(n-2)h.$

B. (29) $P^2 t3 = 2(n-2)h.$

B. (30) $P^2 22 = \frac{1}{2}mn(m-1)(n-1)+\frac{1}{2}n(n-1)h+\frac{1}{2}m(m-1)k.$

B. (31) $P^2 4 = \frac{1}{2}(n-2)(n-3)h-\frac{1}{12}n(n-1)(n-2)(n-3).$

(32) $Pul^3 2 = \frac{5}{2}n(n-1)+h.$

(33) $Pul^2 t2 = 4n(n-1)+2h.$

(34) $Pult^2 2 = 6n(n-1)+4h.$

(35) $Put^3 2 = 6n(n-1)+4h.$

(36) $Pul^2 3 = \frac{1}{2}n(n-1)(n-2)+(n-2)h.$

(37) $Pult3 = \frac{2}{3}n(n-1)(n-2)+2(n-2)h.$

(38) $Put^2 3 = \frac{2}{3}n(n-1)(n-2)+4(n-2)h.$

(39) $Pul22 = mn(m-1)(n-1)+\frac{1}{2}n(n-1)h+\frac{1}{2}m(m-1)k.$

(40) $Put22 = \frac{3}{2}mn(m-1)(n-1)+n(n-1)h+m(m-1)k.$

(41) $Pul4 = \frac{1}{2}(n-2)(n-3)h.$

(42) $Put4 = (n-2)(n-3)h-\frac{1}{12}n(n-1)(n-2)(n-3).$

(43) $Pu23 = \frac{1}{6}mn(m-1)(n-1)(n-2)+\frac{1}{6}n(n-1)(n-2)h$
$+\frac{1}{2}m(m-1)(n-2)k.$

(44) $Pu5 = \frac{1}{6}(n-2)(n-3)(n-4)h-\frac{1}{30}n(n-1)(n-2)(n-3)(n-4).$

(45) $u^2l^4 2 = \frac{7}{2}n(n-1)+h.$

(46) $u^2l^3t2 = 6n(n-1)+2h.$

(47) $u^2l^2t^2 2 = 10n(n-1)+4h.$

(48) $u^2lt^3 2 = 10n(n-1)+4h.$

(49) $u^2t^4 2 = 7n(n-1)+2h.$

(50) $u^2l^3 3 = \frac{5}{6}n(n-1)(n-2)+(n-2)h.$

(51) $u^2l^2t3 = \frac{4}{3}n(n-1)(n-2)+2(n-2)h.$

(52) $u^2lt^2 3 = 2n(n-1)(n-2)+4(n-2)h.$

(53) $u^2t^3 3 = 2n(n-1)(n-2)+4(n-2)h.$

(54) $u^2l^2 22 = \frac{3}{2}mn(m-1)(n-1)+\frac{1}{2}n(n-1)h+\frac{1}{2}m(m-1)k.$

(55) $u^2lt22 = \frac{5}{2}mn(m-1)(n-1)+n(n-1)h+m(m-1)k.$

(56) $u^2t^2 22 = 4mn(m-1)(n-1)+2n(n-1)h+2m(m-1)k.$

(57) $u^2l^2 4 = \frac{1}{12}n(n-1)(n-2)(n-3)+\frac{1}{2}(n-2)(n-3)h.$

(58) $u^2lt4 = \frac{1}{12}n(n-1)(n-2)(n-3)+(n-2)(n-3)h.$

(59) $u^2t^2 4 = 2(n-2)(n-3)h.$

(60) $u^2l23 = \frac{1}{3}mn(m-1)(n-1)(n-2)+\frac{1}{6}n(n-1)(n-2)h$
$+\frac{1}{2}m(m-1)(n-2)k.$

(61) $u^2t23 = \frac{1}{2}mn(m-1)(n-1)(n-2)+\frac{1}{3}n(n-1)(n-2)h$
$+m(m-1)(n-2)k.$

(62) $u^2 l5 = \frac{1}{6}(n-2)(n-3)(n-4)h - \frac{1}{60}n(n-1)(n-2)(n-3)(n-4).$

(63) $u^2 t5 = \frac{1}{3}(n-2)(n-3)(n-4)h - \frac{1}{20}n(n-1)(n-2)(n-3)(n-4).$

(64) $u^2 222 = \frac{5}{8}mno(m-1)(n-1)(o-1) + \frac{1}{4}no(n-1)(o-1)g$
$+\frac{1}{4}om(o-1)(m-1)h + \frac{1}{4}mn(m-1)(n-1)k.$

(65) $u^2 33 = \frac{1}{18}mn(m-1)(n-1)(m-2)(n-2)$
$+\frac{1}{6}n(n-1)(n-2)(m-2)h$
$+\frac{1}{6}m(m-1)(m-2)(n-2)k.$

(66) $u^2 24 = \frac{1}{48}mn(m-1)(n-1)(n-2)(n-3)$
$+\frac{1}{24}n(n-1)(n-2)(n-3)h + \frac{1}{4}m(m-1)(n-2)(n-3)k.$

(67) $u^2 6 = \frac{1}{24}(n-2)(n-3)(n-4)(n-5)h$
$-\frac{7}{720}n(n-1)(n-2)(n-3)(n-4)(n-5).$

B. (68) $Pl^4 2 = 7n^2 - 6n + 4h.$

B. (69) $Pl^3 t2 = 9n^2 - 7n + 6h.$

B. (70) $Pl^2 t^2 2 = 10n^2 - 6n + 8h.$

B. (71) $Plt^3 2 = 8n^2 - 4n + 8h.$

B. (72) $Pt^4 2 = 4n^2 - 2n + 8h.$

B. (73) $Pl^3 3 = n(n-1)(n-2) + 2(2n-3)h.$

B. (74) $Pl^2 t3 = n(n-1)(n-2) + 2(3n-4)h.$

B. (75) $Plt^2 3 = \frac{2}{3}n(n-1)(n-2) + 8(n-1)h.$

B. (76) $Pt^3 3 = 8(n-1)h.$

B. (77) $Pl^2 22 = \frac{1}{4}mn(11mn - 9m - 9n + 7) + \frac{3}{2}n(n-1)h$
$+\frac{3}{2}m(m-1)k + hk.$

B. (78) $Plt22 = \frac{1}{2}mn(7mn - 5m - 5n + 3) + 2n(n-1)h$
$+2m(m-1)k + 2hk.$

B. (79) $Pt^2 22 = 2mn(2mn - m - n) + 2n(n-1)h + 2m(m-1)k + 4hk.$

B. (80) $Pl^2 4 = \frac{1}{2}h^2 + \frac{1}{2}(3n^2 - 15n + 17)h - \frac{1}{8}n(n-1)(n-2)(n-3).$

B. (81) $Plt4 = h^2+(2n^2-10n+11)h-\frac{1}{4}n(n-1)(n-2)(n-3).$

B. (82) $Pt^2 4 = 2h^2+2(n^2-5n+5)h-\frac{1}{3}n(n-1)(n-2)(n-3).$

B. (83) $Pl23 = \frac{1}{12}mn(n-1)(n-2)(5m-3)+\frac{1}{6}n(n-1)(n-2)h$
$+\frac{1}{2}m(m-1)(3n-4)k+(n-2)hk.$

B. (84) $Pt23 = \frac{1}{6}mn(n-1)(n-2)(3m-1)+2m(m-1)(n-1)k$
$+2(n-2)hk.$

B. (85) $Pl5 = \frac{1}{2}(n-4)h^2+\frac{1}{6}(n-4)(n^2-11n+21)h$
$-\frac{1}{120}n(n-2)(n-3)(n-4)(7n-27).$

B. (86) $Pt5 = (n-4)h^2-(n-4)(2n-5)h$
$-\frac{1}{60}n(n-2)(n-3)(n-4)(3n-23).$

B. (87) $P222 = \frac{1}{8}mno\,[9mno-7(mn+no+om)+5(m+n+o)-3]$
$+\frac{1}{2}no(n-1)(o-1)g+\frac{1}{2}om(o-1)(m-1)h$
$+\frac{1}{2}mn(m-1)(n-1)k+\frac{1}{2}m(m-1)hk$
$+\frac{1}{2}n(n-1)kg+\frac{1}{2}o(o-1)gh.$

B. (88) $P33 = \frac{1}{12}mn(m-1)(n-1)(m-2)(n-2)$
$+\frac{1}{6}mn(n-1)(n-2)h+\frac{1}{6}mn(m-1)(m-2)k$
$+(m-2)(n-2)hk.$

B. (89) $P24 = \frac{1}{4}m(m-1)k^2+\frac{1}{2}(n-2)(n-3)hk$
$+\frac{1}{4}m(m-1)(2n^2-10n+11)k$
$-\frac{1}{12}n(n-1)(n-2)(n-3)h$
$-\frac{1}{48}mn(n-1)(n-2)(n-3)(m-3).$

B. (90) $P6 = \frac{1}{4}(n-4)(n-5)h^2-\frac{1}{12}(n-4)(n-5)(n^2+3n-15)h$
$+\frac{1}{240}n(n-2)(n-3)(n-4)(n-5)(n+29).$

(91) $ul^5 2 = 14n^2-13n+6h.$

(92) $ul^4 t2 = 21n^2-19n+10h.$

(93) $ul^3 t^2 2 = 30n^2-26n+16h.$

(94) $ul^2 t^3 2 = 28n^2-24n+16h.$

(95) $ult^4\,2 = 18n^2-16n+12h.$

(96) $ut^5\,2 = 9n^2-8n+6h.$

(97) $ul^4\,3 = \frac{8}{3}n(n-1)(n-2)+2(3n-5)h.$

(98) $ul^3t3 = \frac{11}{3}n(n-1)(n-2)+2(5n-8)h.$

(99) $ul^2t^2\,3 = \frac{14}{3}n(n-1)(n-2)+8(2n-3)h.$

(100) $ult^3\,3 = 4n(n-1)(n-2)+8(2n-3)h.$

(101) $ut^4\,3 = 2n(n-1)(n-2)+4(3n-5)h.$

(102) $ul^3\,22 = \frac{1}{4}mn(23mn-21m-21n+19)$
$+\frac{5}{2}n(n-1)h+\frac{5}{2}m(m-1)k+hk.$

(103) $ul^2t22 = \frac{1}{2}mn(17mn-15m-15n+13)$
$+4n(n-1)h+4m(m-1)k+2hk.$

(104) $ult^2\,22 = 2mn(6mn-5m-5n+4)$
$+6n(n-1)h+6m(m-1)k+4hk.$

(105) $ut^3\,22 = mn(11mn-9m-9n+7)$
$+6n(n-1)h+6m(m-1)k+4hk.$

(106) $ul^3\,4 = \frac{1}{24}n(n-1)(n-2)(n-3)+\frac{1}{2}(5n^2-25n+29)h+\frac{1}{2}h^2.$

(107) $ul^2t4 = h^2+(4n^2-20n+23)h-\frac{1}{12}n(n-1)(n-2)(n-3).$

(108) $ult^2\,4 = 2h^2+2(3n^2-15n+17)h-\frac{1}{3}n(n-1)(n-2)(n-3).$

(109) $ut^3\,4 = 2h^2+2(3n^2-15n+17)h-\frac{1}{2}n(n-1)(n-2)(n-3).$

(110) $ul^2\,23 = \frac{1}{12}mn(n-1)(n-2)(13m-11)+\frac{1}{2}n(n-1)(n-2)h$
$+\frac{1}{2}m(m-1)(5n-8)k+(n-2)hk.$

(111) $ult23 = \frac{1}{6}mn(n-1)(n-2)(9m-7)+\frac{2}{3}n(n-1)(n-2)h$
$+2m(m-1)(2n-3)k+2(n-2)hk.$

(112) $ut^2\,23 = \frac{2}{3}mn(n-1)(n-2)(3m-2)+\frac{2}{3}n(n-1)(n-2)h$
$+2m(m-1)(3n-4)k+4(n-2)hk.$

(113) $ul^2\,5 = \frac{1}{2}(n-4)h^2+\frac{1}{2}(n-4)(n^2-7n+11)h$
$-\frac{1}{120}n(n-2)(n-3)(n-4)(11n-31).$

$$(114)\quad ult5 = (n-4)h^2+\tfrac{1}{3}(n-4)(2n^2-16n+27)h - \tfrac{1}{60}n(n-2)(n-3)(n-4)(9n-29).$$

$$(115)\quad ut^2 5 = 2(n-4)h^2+\tfrac{2}{3}(n-4)(n^2-11n+21)h - \tfrac{1}{15}n(n-2)(n-3)(n-4)(3n-13).$$

$$(116)\quad ul222 = \tfrac{1}{8}mno\,[19mno-17(mn+no+om)+15(m+n+o)-13] + no(n-1)(o-1)g+om(o-1)(m-1)h + mn(m-1)(n-1)k+\tfrac{1}{2}m(m-1)hk + \tfrac{1}{2}n(n-1)kg+\tfrac{1}{2}o(o-1)gh.$$

$$(117)\quad ut222 = \tfrac{1}{2}mno\,[7mno-6(mn+no+om)+5(m+n+o)-4] + \tfrac{3}{2}no(n-1)(o-1)g+\tfrac{3}{2}om(o-1)(m-1)h + \tfrac{3}{2}mn(m-1)(n-1)k+m(m-1)hk + n(n-1)kg+o(o-1)gh.$$

$$(118)\quad ul33 = \tfrac{7}{36}mn(m-1)(n-1)(m-2)(n-2) + \tfrac{1}{6}n(n-1)(n-2)(3m-4)h + \tfrac{1}{6}m(m-1)(m-2)(3n-4)k+(m-2)(n-2)hk.$$

$$(119)\quad ut33 = \tfrac{5}{18}mn(m-1)(n-1)(m-2)(n-2) + \tfrac{2}{3}n(n-1)(n-2)(m-1)h + \tfrac{3}{8}m(m-1)(m-2)(n-1)k + 2(mn-2m-2n+4)hk.$$

$$(120)\quad ul24 = \tfrac{1}{4}m(m-1)k^2+\tfrac{1}{2}(n-2)(n-3)hk + \tfrac{1}{4}m(m-1)(4n^2-20n+23)k + \tfrac{1}{48}mn(n-1)(n-2)(n-3)(m+1).$$

$$(121)\quad ut24 = \tfrac{1}{2}m(m-1)k^2+(n-2)(n-3)hk - \tfrac{1}{12}n(n-1)(n-2)(n-3)h + \tfrac{1}{2}m(m-1)(3n^2-15n+17)k + \tfrac{1}{12}mn(n-1)(n-2)(n-3).$$

$$(122)\quad ul6 = \tfrac{1}{4}(n-4)(n-5)h^2-\tfrac{1}{12}(n-4)(n-5)(8n-21)h - \tfrac{1}{720}n(n-2)(n-3)(n-4)(n-5)(11n-101).$$

$$(123)\quad ut6 = \tfrac{1}{2}(n-4)(n-5)h^2 - \tfrac{1}{12}(n-4)(n-5)(n^2+11n-36)h - \tfrac{1}{180}n(n-2)(n-3)(n-4)(n-5)(2n-47).$$

$$(124)\quad u223 = \tfrac{1}{24}mno(o-1)(o-2)(11mn-9m-9n+7) + \tfrac{1}{6}no(n-1)(o-1)(o-2)g + \tfrac{1}{6}mo(m-1)(o-1)(o-2)h + \tfrac{1}{2}mn(m-1)(n-1)(2o-3)k + \tfrac{1}{6}o(o-1)(o-2)gh + \tfrac{1}{2}n(n-1)(o-2)gk + \tfrac{1}{2}m(m-1)(o-2)hk.$$

$$(125)\quad u25 = \tfrac{1}{4}m(m-1)(n-4)k^2 + \tfrac{1}{6}(n-2)(n-3)(n-4)hk - \tfrac{1}{30}n(n-1)(n-2)(n-3)(n-4)h + \tfrac{1}{12}m(m-1)(n-4)(2n^2-16n+27)k - \tfrac{1}{240}mn(n-2)(n-3)(n-4)(7mn-27m-9n+29).$$

$$(126)\quad u34 = \tfrac{1}{144}mn(m-1)(m-2)(n-1)(n-2)(n-3) + \tfrac{1}{12}n(n-1)(n-2)(n-3)h + \tfrac{1}{12}m(m-1)(m-2)(2n^2-10n+11)k + \tfrac{1}{2}(m-2)(n-2)(n-3)hk + \tfrac{1}{12}m(m-1)(m-2)k^2.$$

$$(127)\quad u7 = \tfrac{1}{5040}n(n-2)(n-3)(n-4)(n-5)(n-6)(13n+239) - \tfrac{1}{60}(n-4)(n-5)(n-6)(2n^2+5n-28)h + \tfrac{1}{12}(n-4)(n-5)(n-6)h^2.$$

B. $(128)\quad l^6 2 = 37n^2-29n+18h.$

$(129)\quad l^5 t2 = 46n^2-32n+24h.$

$(130)\quad l^4 t^2 2 = 50n^2-26n+28h.$

$(131)\quad l^3 t^3 2 = 40n^2-16n+24h.$

B. $(132)\quad l^2 t^4 2 = 24n^2-8n+16h.$

B. $(133)\quad lt^5 2 = 12n^2-4n+8h.$

B. $(134)\quad t^6 2 = 6n^2-2n+4h.$

B. $(135)\quad l^5 3 = \tfrac{1}{3}n(n-1)(19n-32)+6(3n-4)h.$

(136) $l^4t3 = \frac{2}{3}n(n-1)(11n-16)+4(6n-7)h.$

(137) $l^3t^23 = \frac{2}{3}n(n-1)(11n-10)+4(7n-6)h.$

(138) $l^2t^33 = \frac{8}{3}n(n-1)(2n-1)+8(3n-2)h.$

B. (139) $lt^43 = \frac{4}{3}n(n-1)(2n-1)+8(2n-1)h.$

B. (140) $t^53 = \frac{2}{3}n(n-1)(2n-1)+4(2n-1)h.$

B. (141) $l^422 = \frac{1}{2}mn(30mn-23m-23n+16)+n(7n-6)h$
$+m(7m-6)k+4hk.$

(142) $l^3t22 = \frac{1}{2}mn(37mn-25m-25n+13)+n(9n-7)h$
$+m(9m-7)k+6hk.$

(143) $l^2t^222 = 10mn(2mn-m-n)+2n(5n-3)h+2m(5m-3)k+8hk.$

(144) $lt^322 = 2mn(8mn-3m-3n-2)+4n(2n-1)h$
$+4m(2m-1)k+8hk.$

B. (145) $t^422 = mn(10mn-3m-3n-4)+2n(2n-1)h$
$+2m(2m-1)k+8hk.$

B. (146) $l^44 = 2h^2+(7n^2-30n+31)h-\frac{1}{6}n(n-1)(n^2-2n+3).$

(147) $l^3t4 = 3h^2+(9n^2-35n+33)h-\frac{1}{12}n(n-1)(5n^2-13n+18).$

(148) $l^2t^24 = 4h^2+10(n-1)(n-2)h-\frac{2}{3}n(n-1)(n^2-2n+3).$

(149) $lt^34 = 4h^2+4(n-2)(2n-1)h-\frac{2}{3}n(n-1)(n^2-2n+3).$

B. (150) $t^44 = 4h^2+2(2n^2-5n+1)h-\frac{1}{3}n(n-1)(n^2-2n+3).$

B. (151) $l^323 = \frac{1}{6}mn(n-1)(16mn-26m-11n+16)$
$+n(n-1)(n-2)h+m(7mn-9m-6n+7)k$
$+2(2n-3)hk.$

(152) $l^2t23 = \frac{1}{6}mn(n-1)(19mn-26m-11n+10)$
$+m(9mn-10m-7n+6)k$
$+n(n-1)(n-2)h+2(3n-4)hk.$

(153) $lt^2\,23 = \frac{2}{3}mn(n-1)(5mn-4m-2n-2)+\frac{2}{3}n(n-1)(n-2)h$
$+2m(5mn-4m-3n)k+8(n-1)hk.$

(154) $t^3\,23 = \frac{2}{3}mn(n-1)(4mn-2m-n-4)$
$+4m(2mn-m-n-1)k+8(n-1)hk.$

B. (155) $l^3\,5 = 2(n-3)h^2+(n^3-12n^2+40n-42)h$
$-\frac{1}{30}n(n-2)(7n^3-66n^2+188n-189).$

(156) $l^2t5 = (3n-8)h^2+(n^3-13n^2+41n-40)h$
$-\frac{1}{60}n(n-2)(17n^3-156n^2+403n-384).$

(157) $lt^2\,5 = 4(n-2)h^2+\frac{2}{3}(n-2)(n^2-13n+6)h$
$-\frac{2}{15}n(n-1)(n-2)(2n^2-14n+9).$

(158) $t^3\,5 = 4(n-2)h^2-4(n-2)(n+1)h$
$-\frac{2}{15}n(n-1)(n-2)(n^2-7n-3).$

B. (159) $l^2\,222 = \frac{1}{8}mno\,[49mno-37(mn+no+om)+25(m+n+o)-13]$
$+\frac{1}{4}no(11no-9n-9o+7)g$
$+\frac{1}{4}om(11om-9o-9m+7)h$
$+\frac{1}{4}mn(11mn-9m-9n+7)k+\frac{3}{2}m(m-1)hk$
$+\frac{3}{2}n(n-1)kg+\frac{3}{2}o(o-1)gh+ghk.$

(160) $lt222 = \frac{5}{2}mno\,[3mno-2(mn+no+om)+(m+n+o)]$
$+\frac{1}{2}no(7no-5n-5o+3)g+\frac{1}{2}om(7om-5o-5m+3)h$
$+\frac{1}{2}mn(7mn-5m-5n+3)k+2m(m-1)hk$
$+2n(n-1)kg+2o(o-1)gh+2ghk.$

(161) $t^2\,222 = 4mno\,[2mno-(mn+no+om)+1]$
$+2no(2no-n-o)g+2om(2om-o-m)h$
$+2mn(2mn-m-n)k+2m(m-1)hk$
$+2n(n-1)kg+2o(o-1)gh+4ghk.$

B. (162) $l^2\,33 = \frac{1}{9}mn(m-1)(n-1)(5mn-7m-7n+8)$
$+n(n-1)(n-2)(m-1)h+m(m-1)(m-2)(n-1)k$
$+2(2mn-3m-3n+4)hk.$

(163) $lt33 = \frac{1}{18}mn(m-1)(n-1)(13mn-14m-14n+4)$
$+\frac{1}{3}n(n-1)(n-2)(3m-2)h$
$+\frac{1}{3}m(m-1)(m-2)(3n-2)k$
$+2(3mn-4m-4n+4)hk.$

(164) $t^2 33 = \frac{4}{9}mn(m-1)(n-1)(2mn-m-n-4)$
$+\frac{2}{3}mn(m-1)(m-2)k+\frac{2}{3}mn(n-1)(n-2)h$
$+8(mn-m-n)hk.$

B. (165) $l^2 24 = \frac{1}{48}mn(n-1)(5n^2-13n+18)$
$-\frac{1}{48}m^2n(n-1)(n^2+7n-6)$
$-\frac{1}{4}m(9n^2-35n+33)k+\frac{1}{4}m^2(11n^2-45n+45)k$
$-\frac{1}{8}n(n-1)(n-2)(n-3)h+\frac{1}{2}(3n^2-15n+17)h$
$+\frac{3}{4}m(m-1)k^2+\frac{1}{2}hk^2.$

(166) $lt24 = hk^2+m(m-1)k^2+(2n^2-10n+11)hk$
$-\frac{1}{4}n(n-1)(n-2)(n-3)h$
$+\frac{1}{2}m(7mn^2-25mn+22m-5n^2+15n-10)k$
$-\frac{1}{12}mn(n-1)(mn^2+mn-2n^2+4n-6).$

(167) $t^2 24 = 2hk^2+m(m-1)k^2+2(n^2-5n+5)hk$
$-\frac{1}{3}n(n-1)(n-2)(n-3)h$
$+m^2(4n^2-10n+5)k-m(2n^2-7)k$
$-\frac{1}{6}mn^2(m-1)(n^2-1).$

B. (168) $l^2 6 = \frac{1}{6}h^3+\frac{1}{4}(3n^2-27n+50)h^2$
$-\frac{1}{24}(3n^4+6n^3-309n^2+1236n-1376)h$
$-\frac{1}{720}n(n-2)(13n^4-409n^3+3347n^2-9911n+9840).$

(169) $lt6 = \frac{1}{3}h^3+(n^2-9n+15)h^2$
$-\frac{1}{12}(3n^4-10n^3-123n^2+538n-536)h$
$-\frac{1}{180}n(n-2)(n^4-88n^3+809n^2-2252n+1890).$

$$(170)\quad t^2 6 = \tfrac{1}{90}n(n-2)(n^4+17n^3-151n^2-77n+930) - \tfrac{1}{3}(n^4-6n^3-4n^2-3n+92)h + (n^2-9n+10)h^2+\tfrac{2}{3}h^3.$$

$$(171)\quad l223 = \tfrac{1}{24}mno(o-1)[(27mn-19m-19n+11)o - (42mn-26m-26n+10)] + \tfrac{1}{12}o(o-1)(o-2)n(5n-3)g + \tfrac{1}{12}o(o-1)(o-2)m(5m-3)h + \tfrac{1}{4}mn[(11mn-9m-9n+7)o - (14mn-10m-10n+6)]k + \tfrac{1}{2}n(n-1)(3o-4)gk+\tfrac{1}{2}m(m-1)(3o-4)hk + \tfrac{1}{6}o(o-1)(o-2)gh+(o-2)ghk.$$

$$(172)\quad t223 = \tfrac{1}{6}mno(o-1)(8mno-10mn-5mo-5no+4m+4n+2o+2) + \tfrac{1}{6}no(o-1)(o-2)(3n-1)g + \tfrac{1}{6}mo(o-1)(o-2)(3m-1)h + \tfrac{1}{2}mn(7mno-8mn-5mo-5no+4m+4n+3o)k + 2n(n-1)(o-1)gk+2m(m-1)(o-1)hk + 2(o-2)ghk.$$

B. $$(173)\quad l34 = \tfrac{1}{144}mn(m-1)(n-1)[(5n^2-37n+42)m+2(n^2+7n-6)] - \tfrac{1}{8}n(n-1)(n-2)(n-3)(m-2)h + \tfrac{1}{12}m(m-1)(m-2)(5n^2-15n+11)k + \tfrac{1}{2}m(3n^2-15n+17)hk-(2n^2-10n+11)hk + \tfrac{1}{12}m(m-1)(m-2)k^2+\tfrac{1}{2}(m-2)hk^2.$$

$$(174)\quad t34 = (m-2)hk^2 + (2mn^2-10mn+11m-2n^2+10n-10)hk - \tfrac{1}{12}n(n-1)(n-2)(n-3)(3m-4)h + \tfrac{1}{6}mn(m-1)(m-2)(3n-5)k + \tfrac{1}{18}mn(m-1)(n-1)(mn^2-8mn+9m+n^2+n).$$

B. (175) $l25 = \frac{1}{2}(n-4)\,hk^2+\frac{1}{4}m(m-1)(3n-8)\,k^2$
$+\frac{1}{6}(n-4)(n^2-11n+21)\,hk$
$-\frac{1}{4}m(n^3-13n^2+41n-40)\,k$
$+\frac{1}{12}m^2(5n^3-57n^2+175n-168)\,k$
$-\frac{1}{120}n(n-2)(n-3)(n-4)(7n-27)\,h$
$+\frac{1}{240}mn(n-2)(17n^3-156n^2+403n-384)$
$-\frac{1}{240}m^2n(n-2)(21n^3-188n^2+479n-432).$

(176) $t25 = (n-4)\,hk^2+m(m-1)(n-2)\,k^2$
$-(n-4)(2n-5)\,hk$
$-\frac{1}{60}n(n-2)(n-3)(n-4)(3n-23)\,h$
$+\frac{1}{6}m(n-2)(3mn^2-27mn+30m-n^2+13n-6)\,k$
$-\frac{1}{60}mn(n-1)(n-2)(7mn^2-49mn+54m-4n^2$
$+28n-18).$

B. (177) $l7 = \frac{1}{6}(n-6)\,h^3+\frac{1}{12}(n-6)(n^2-21n+50)\,h^2$
$-\frac{1}{120}(n-6)(7n^4-78n^3+27n^2+972n-1640)\,h$
$+\frac{1}{5040}n(n-2)(n-3)(n-6)(29n^3+46n^2$
$-3023n+6980).$

(178) $t7 = \frac{1}{6}\frac{1}{30}n(n-6)(4n^5-39n^4-164n^3+1851n^2-4448n+3300)$
$-\frac{1}{60}(n-6)(3n^4-52n^3+93n^2+268n-520)\,h$
$-(n-6)(2n-5)\,h^2+\frac{1}{3}(n-6)\,h^3.$

B. (179) $2222 = \frac{5}{8}mnop(4mnop-3\Sigma mno+2\Sigma mn-\Sigma m)$
$+\frac{1}{8}\Sigma mno(9mno-7\Sigma mn+5\Sigma m-3)\,k$
$+\frac{1}{2}\Sigma mn(m-1)(n-1)\,hk$
$+\frac{1}{2}\Sigma m(m-1)\,ghk,$

the summations being over the range (m, f), (n, g), (o, h), (p, k).

B. (180) $$233 = \tfrac{1}{72}mno(n-1)(o-1)\,[(17no-22n-22o+20)\,m - (13no-14n-14o+4)]$$
$$+\tfrac{1}{12}no(n-1)(o-1)(n-2)(o-2)\,g$$
$$+\tfrac{1}{12}mo(o-1)(o-2)(5mn-6m-3n+2)\,h$$
$$+\tfrac{1}{12}mn(n-1)(n-2)(5mo-6m-3o+2)\,k$$
$$+\tfrac{1}{6}no(o-1)(o-2)\,gh+\tfrac{1}{6}no(n-1)(n-2)\,gk$$
$$+\tfrac{1}{2}m(m-1)(3no-4n-4o+4)\,hk$$
$$+(n-2)(o-2)\,ghk.$$

B. (181) $$224 = \tfrac{1}{4}n(n-1)\,gk^2+\tfrac{1}{4}m(m-1)\,hk^2$$
$$+\tfrac{1}{4}mn(m-1)(n-1)\,k^2+\tfrac{1}{2}(o-2)(o-3)\,ghk$$
$$+\tfrac{1}{4}n(n-1)(2o^2-10o+11)\,gk$$
$$+\tfrac{1}{4}m(m-1)(2o^2-10o+11)\,hk$$
$$+\tfrac{1}{8}m^2n^2(o-2)(9o-17)\,k$$
$$-\tfrac{1}{8}mn(m+n)(o-2)(7o-11)\,k$$
$$+\tfrac{5}{8}mn(o-1)(o-2)\,k$$
$$-\tfrac{1}{12}o(o-1)(o-2)(o-3)\,gh$$
$$-\tfrac{1}{48}n(n-3)\,o(o-1)(o-2)(o-3)\,g$$
$$-\tfrac{1}{48}m(m-3)\,o(o-1)(o-2)(o-3)\,h$$
$$-\tfrac{1}{8}m^2n^2\,o(o-1)^2+\tfrac{1}{48}mn(m+n)\,o^2(o^2-1)$$
$$-\tfrac{1}{24}mno(o-1)(o^2-2o+3).$$

B. (182) $$44 = \tfrac{1}{144}mn(m-1)(n-1)(2m^2n^2-13m^2n-13mn^2+15m^2 + 15n^2+80mn-93m-93n+108)$$
$$-\tfrac{1}{48}n(n-1)(n-2)(n-3)(m^2-15m+22)\,h$$
$$-\tfrac{1}{48}m(m-1)(m-2)(m-3)(n^2-15n+22)\,k$$
$$-\tfrac{1}{24}n(n-1)(n-2)(n-3)\,h^2$$
$$-\tfrac{1}{24}m(m-1)(m-2)(m-3)\,k^2$$
$$+\tfrac{1}{4}(2m^2n^2-10m^2n-10mn^2+11m^2+11n^2+50mn - 55m-55n+60)\,hk$$
$$+\tfrac{1}{4}(n-2)(n-3)\,h^2k+\tfrac{1}{4}(m-2)(m-3)\,hk^2.$$

B. (183) $35 = \frac{1}{2}(m-2)(n-4)hk^2+\frac{1}{12}m(m-1)(m-2)nk^2$
$+\frac{1}{6}m(n-4)(n^2-11n+21)hk+(n-4)(2n-5)hk$
$+\frac{1}{12}mn(m-1)(m-2)(n^2-9n+15)k$
$-\frac{1}{120}mn(n-2)(n-3)(n-4)(7n-27)h$
$+\frac{1}{60}n(n-2)(n-3)(n-4)(3n-23)h$
$-\frac{1}{720}m^2n(m-1)(n-2)(7n^3-36n^2-67n+216)$
$+\frac{1}{360}mn(m-1)(n-2)(13n^3-84n^2+47n+144).$

B. (184) $26 = \frac{1}{72}m(m-1)k^3+\frac{1}{4}(n-4)(n-5)hk^2$
$+\frac{1}{4}m(m-1)(n^2-9n+15)k^2$
$-\frac{1}{12}(n-4)(n-5)(n^2+3n-15)hk$
$-\frac{1}{48}m^2(n^4+18n^3-265n^2+846n-776)k$
$+\frac{1}{48}m(3n^4-10n^3-123n^2+538n-536)k$
$+\frac{1}{240}n(n-2)(n-3)(n-4)(n-5)(n+29)h$
$-\frac{1}{720}m^2n(n-2)(8n^4-179n^3+1222n^2-3001n$
$+2310)$
$+\frac{1}{720}mn(n-2)(n^4-88n^3+809n^2-2252n+1890).$

B. (185) $8 = \frac{1}{12}(n-6)(n-7)h^3$
$-\frac{1}{24}(n-3)(n-6)(n-7)(n+10)h^2$
$+\frac{1}{240}(n-6)(n-7)(n^4+40n^3-167n^2-146n+760)h$
$+\frac{1}{5040}n(n-3)(n-6)(n-7)(n^4-103n^3-21n^2$
$+2323n-3880).$

III. The conics may be made to meet a curve in points, some of which are fixed.

I shall use the following symbols:—

x' denotes that the conics meet the curve in one fixed and x other points ($x = 1, 2, 3, 4, 5, 6$).

x'' denotes that the conics meet the curve in two fixed and x other points ($x = 1, 2, 3, 4$).

x''' denotes that the conics meet the curve in three fixed and x other points ($x = 1, 2$).

The x''' problems are trivial and their solutions are:—

$$l1''' = n-3.$$

$$t1''' = 2(n-3).$$

$$2''' = \tfrac{1}{2}(n-3)(n-4).$$

The x'' problems can be solved by means of functional equations:—

$1''$: $f(2n, 2h+n^2) = f(n, h)+n(PPl)$;

$2''$: $f(2n, 2h+n^2) = f(n, h)+n(l1'')+(PP2)$;

$3''$: $f(2n, 2h+n^2) = f(n, h)+n(l2'')+(21'')+(PP3)$;

$4''$: $f(2n, 2h+n^2) = f(n, h)+n(l3'')+(22'')+(31'')+(PP4)$;

where, in substituting the various expressions on the right-hand sides of the equations, it is necessary to put $m = n$ and $h = k$ if two curves are in question.

Berzolari (**5**) gives the solutions of $22''$, $l3''$, $31''$, and $4''$.

The x' problems can be solved by means of functional equations:—

$1'$: $$f(2n, 2h+n^2) = f(n, h)+n(Pl);$$

which leads to

$$f = A(n-1)+B,$$

where $A = Pl$ (v) and $B = Pu$ (v);

$2'$: $$f(2n, 2h+n^2) = f(n, h)+n(l1')+(P2);$$

which leads to

$$f = (n-1)[\tfrac{1}{2}(A-C)n+(B+D-A)]+Ch,$$

where

$A = Pll$ (iv); $B = Puu$ (iv); $C = PP$ (iv); and $D = Pul$ (iv);

$3'$: $f(2n, 2h+n^2) = f(n, h)+n(l2')+(21')+(P3)$;

$4'$: $f(2n, 2h+n^2) = f(n, h)+n(l3')+(22')+(31')+(P4)$;

$5'$: $f(2n, 2h+n^2) = f(n, h)+n(l4')+(23')+(32')+(41')+(P5)$;

$6'$: $f(2n, 2h+n^2) = f(n, h)+n(l5')+(24')+(33')+(42')+(51')+(P6)$;

where, in substituting the various expressions on the right-hand sides of

the equations, we have to put $m = n$ and $h = k$ if two curves are in question.

Berzolari (**5**) gives the solutions of $P3't$, $P4'$, $l^4 2'$, $l^3 3'$, $l^2 4'$, $l41'$, $l23'$, $l32'$, $l5'$, $51'$, $24'$, $42'$, $33'$, and $6'$.

Severi (**6**) establishes formulae for the number of conics meeting a curve in α points and having contact of order β with another curve, in terms of the degree n and rank r and of the elementary conditions P, l, u, and t.

References.

1. Schübert, *Kalkül der abzählenden Geometrie* (Leipzig, 1879), 95.
2. Lüroth, *Journal für Math.*, 68 (1868), 185–190.
3. Hierholzer, *Math. Annalen*, 2 (1870), 563–586.
4. Cayley, *Phil. Trans. Royal Soc.*, 158 (1868), 75–143, 145–172; *Collected Math. Papers*, 6 (1893), 191–262, 263–291.
5. Berzolari, *Rend. Ist. Lomb.* (2), 33 (1900), 664–674, 809–821.
6. Severi, *Torino Atti*, 35 (1900), 526; 36 (1901), 74–93.
7. Ursell, *Proc. London Math. Soc.* (2), 30 (1930), 322–338.
8. Todd, *Proc. Royal Soc.* (A), 131 (1931), 286–306.

ASYMPTOTIC PARTITION FORMULAE: (II) WEIGHTED PARTITIONS

By E. MAITLAND WRIGHT.

[Received 24 March, 1932.—Read 28 April, 1932.]

1.1. The first paper* of this series was concerned with plane partitions. A different generalization of the ordinary linear partition of an integer is obtained by considering the partitions into parts of a given form, each part being appropriately weighted. The idea of weighted partitions† seems very natural, but I have not found any reference to the subject in the literature of partitions.

Suppose the parts are to be chosen from the positive integers

$$\mu_1, \mu_2, \mu_3, \ldots, \tag{1.11}$$

each being capable of repetition any number of times in any particular partition; let these integers be weighted respectively

$$\lambda_1, \lambda_2, \lambda_3, \ldots, \tag{1.12}$$

so that, for example, the partition

$$n = 2\mu_h + \mu_j + \mu_k$$

will count $\lambda_h^2 \lambda_j \lambda_k$ towards the total number of partitions of n. Then the generating function of this mode of partition is

$$\prod_{l=1}^{\infty} (1-\lambda_l x^{\mu_l})^{-1} = 1 + \sum_{n=1}^{\infty} p_{\lambda,\mu}(n)\, x^n, \tag{1.13}$$

and $p_{\lambda,\mu}(n)$ is the number of partitions of n in this form.

1.2. In this paper I discuss in detail the particular mode of partition in which the parts are chosen from all the positive integers, and the weights are all equal, real, and positive. In this case, $\mu_l = l$, $\lambda_l = a > 0$, and

$$f(x) = \prod_{l=1}^{\infty} (1-ax^l)^{-1} = 1 + \sum_{n=1}^{\infty} p_a(n)\, x^n. \tag{1.21}$$

---

* *Quart. J. of Math.* (Oxford series), 2 (1931), 177–189. The present paper can be read independently.

† The idea was suggested by Prof. Hardy, whom I should like to thank for much valuable advice in the course of the investigation.

This function $f(x)$ was used by MacMahon*, its importance in his work being due to the fact that the coefficient of a^i in $p_a(n)$ is the number of partitions of n into exactly i parts.

My object is to obtain an asymptotic expansion of $p_a(n)$ valid for large n. The case $a = 1$, that of ordinary linear partitions, was considered by Hardy and Ramanujan in the well-known memoir† which first drew attention to this type of problem. They found an asymptotic expansion of $p_1(n)$, and showed that, if we take a suitable number of terms (the number depending on n), the error is only $O(n^{-\frac{1}{4}})$. Nothing is known, however, as to the convergence or divergence of the series.

The case $a > 1$ requires a new and simpler method. It is interesting to note how much we can prove without the use of the theory of analytic functions. A series of an entirely new form appears, and, when this series is absolutely convergent, its sum is $p_a(n)$. The question of convergence is next considered, and we find that, if $a > a_0$, where a_0 is the root of a certain transcendental equation, the series is absolutely convergent, while if $1 < a < a_0$ the series diverges. So far the methods used are purely algebraic. The convergence in the critical case $a = a_0$ requires slightly deeper methods, but may still be proved *ab initio* without resource to function theory. Here, however, I prefer to shorten my work by adopting a known result, whose proof appears to require the use of Cauchy's theorem.

We find then that, for $a \geqslant a_0$, $p_a(n)$ is the sum of a convergent series. For $1 < a < a_0$, we apply Cauchy's theorem to the function $f(x)$ and find that the same series, now divergent, is an asymptotic expansion of $p_a(n)$. The same method proves the result we have already found for $a > a_0$, but fails to prove the full result for $a = a_0$.

In § 5, I consider the case $a < 1$ and find an asymptotic expansion of $p_a(n)$ in two forms:

(i) as a series of Bessel functions,

(ii) as a series in descending powers of $n^{\frac{1}{4}}$.

Both of these are of an entirely different form from that found for $a > 1$. This might be expected; for, if $a < 1$, the lower powers of a in $p_a(n)$ are of major importance, while if $a > 1$ the higher powers are dominant. The method used for the case $a < 1$ is a modification of that used by Hardy and Ramanujan in their work on $p_1(n)$ quoted above. Their exact trans-

* *Combinatory analysis*, 2 (Cambridge, 1916), 2 *et seq.*

† *Proc. London Math. Soc.* (2), 17 (1918), 75–115.

formations have to be replaced by an asymptotic transformation valid near $x = 1$. I have developed my own contribution to this method in the first paper of this series.

At the end of the paper, I discuss briefly other methods of weighting. These show clearly that the case which I have selected for detailed discussion is the most typical and, probably, the most interesting.

2.1. In § 2, I use only elementary methods*. My notation is as follows:

$$a > 1, \quad b = a^{-1} < 1.$$

n, l, t, s, r, m are integers, n, t, r always positive, l, m, s positive, negative, or zero. Also, $0 < r \leqslant t$.

We take $|x| < 1$, and write

$$f(x) = \prod_{l=1}^{\infty} (1-ax^l)^{-1} = 1 + \sum_{n=1}^{\infty} p_a(n)\, x^n,$$

$$\Phi_t = \Phi_t(x) = \frac{x^{\frac{1}{2}t(t-1)}}{\prod_{s=1}^{t-1} (1-x^s) \prod_{s=1}^{\infty} (1-x^s)} = \sum_{s=\frac{1}{2}t(t-1)}^{\infty} c_{t,s}\, x^s.$$

For $s < \frac{1}{2}t(t-1)$, we put $c_{t,s} = 0$. Then we have

$$\Phi_t(x) = \sum_{s=-\infty}^{+\infty} c_{t,s}\, x^s,$$

and this form is sometimes more convenient†.

We write also

$$\lambda = \lambda(a) = \frac{1}{\log a}\left(\frac{\pi^2}{3} - \sum_{m=1}^{\infty} \frac{1}{m^2 a^m}\right) - \tfrac{1}{2}\log a, \qquad e_t(l) = \exp\left(2\pi i \frac{l}{t}\right),$$

$$\alpha_{t,r} = b^{1/t} e_t(r), \qquad \alpha_t = |\alpha_{t,r}| = \alpha_{t,t} = b^{1/t},$$

$$\phi_{t,r} = \Phi_t(\alpha_{t,r}), \qquad \phi_t = \phi_{t,t} = \Phi_t(\alpha_t),$$

$$S_t(n) = \frac{1}{t} \sum_{r=1}^{t} \{\phi_{t,r}\, e_t(-nr)\},$$

and

$$\mathfrak{S} = \mathfrak{S}(n) = \sum_{t=1}^{\infty} (-1)^{t-1} a^{n/t} S_t(n),$$

whenever the infinite series converges.

* I use the idea of an infinite series, but otherwise the methods are those of algebra.

† The value of the empty product in Φ_1 is unity, that is

$$\Phi_1 = \prod_{s=1}^{\infty} \frac{1}{(1-x^s)}.$$

a_0 is defined by the two conditions

$$a_0 > 0, \quad \lambda(a_0) = 0.$$

I shall prove in Lemma V that there is one and only one a_0 which satisfies these conditions.

$p_a(n)$ is obviously a polynomial in a of the n-th degree with no term independent of a; that is,

$$p_a(n) = \overset{*n}{\sum_{l=1}} g_{n,l} a^l.$$

I use the letter K, with or without suffixes, to denote a positive number, whose value varies from one occurrence to another. K without a suffix is a function of a only. When K depends on other parameters, these are indicated explicitly by suffixes, thus

$$K_\epsilon = K(a, \epsilon).$$

2.2. We now prove four lemmas preliminary to our first theorem. For the present, we take x real and positive.

LEMMA I. *If* $0 < x < 1$, *the series*

$$\Omega_l = \sum_{t=1}^{\infty} (-1)^{t-1} x^{lt} \Phi_t \tag{2.21}$$

converges, and we have

$$\Omega_l = \sum_{n=l}^{\infty} d_{l,n} x^n, \tag{2.22}$$

*where** $$d_{l,n} = \sum_{t=1}^{\infty} (-1)^{t-1} c_{t,n-lt}. \tag{2.23}$$

We have

$$\begin{aligned} \Omega_l &= \sum_{t=1}^{\infty} (-1)^{t-1} x^{lt} \Phi_t \\ &= \sum_{t=1}^{\infty} (-1)^{t-1} x^{lt} \sum_{s=-\infty}^{+\infty} c_{t,s} x^s \\ &= \sum_{n=l}^{\infty} d_{l,n} x^n, \end{aligned} \tag{2.24}$$

where $d_{l,n}$ has the value given, provided that the double series is absolutely convergent.

It is easy to see that $c_{t,s} \geqslant 0$; then the double series is absolutely

* (2.23) is always a finite series, for $c_{t,n-lt}$ is zero when $n - lt < \frac{1}{2}t(t-1)$, and this will always be true for $t > t_1 = t_1(n, l)$.

convergent if $$\sum_{t=1}^{\infty}\sum_{s=-\infty}^{+\infty} c_{t,s}x^{lt+s} = \sum_{t=1}^{\infty} x^{lt}\Phi_t$$

is convergent. This is true if

$$\frac{x^{l(t+1)}\Phi_{t+1}}{x^{lt}\Phi_t} \leqslant \gamma = \gamma(x) < 1,$$

for every $t > t_0 = t_0(l, x)$. But, from the definition of Φ_t, we know that

$$\frac{x^{l(t+1)}\Phi_{t+1}}{x^{lt}\Phi_t} = \frac{x^{l+t}}{1-x^t},$$

and it is obvious that we can make this as small as we please by choosing $t_0 = t_0(l, x)$ suitably and taking $t > t_0$. Hence the double series (2.24) is absolutely convergent.

LEMMA II. *If* $0 < x < 1$, *we have*

$$x\Omega_{l-1} = (1-x^l)\,\Omega_l. \tag{2.25}$$

From the definition of Φ_t we see that

$$x^{t-1}\Phi_{t-1} = \Phi_t(1-x^{t-1}),$$

provided that $t \geqslant 2$. We may write this

$$x^{t-1}\Phi_t = \Phi_t - x^{t-1}\Phi_{t-1}.$$

Then we have

$$\begin{aligned}\Omega_l &= \sum_{t=1}^{\infty}(-1)^{t-1}x^{lt}\Phi_t = x^l\Phi_1 + \sum_{t=2}^{\infty}(-1)^{t-1}x^{lt-t+1}(\Phi_t - x^{t-1}\Phi_{t-1})\\ &= x^l\Phi_1 + x\sum_{t=2}^{\infty}(-1)^{t-1}x^{t(l-1)}\Phi_t - x^l\sum_{t=2}^{\infty}(-1)^{t-1}x^{l(t-1)}\Phi_{t-1}\\ &= x\sum_{t=1}^{\infty}(-1)^{t-1}x^{t(l-1)}\Phi_t + x^l\sum_{t=1}^{\infty}(-1)^{t-1}x^{lt}\Phi_t\\ &= x\Omega_{l-1} + x^l\Omega_l.\end{aligned}$$

This proves the lemma.

LEMMA III. *If* $0 < x < 1$, *and*

(i) *if* $l > 0$, *then* $$\Omega_l = \frac{x^l}{\prod_{s=1}^{l}(1-x^s)}; \tag{2.26}$$

(ii) *if* $l = 0$, *then* $$\Omega_l = 1; \tag{2.27}$$

(iii) *if* $l < 0$, *then* $$\Omega_l = 0. \tag{2.28}$$

If the second of these results is true, the first and third follow from Lemma II. We have, therefore, only to prove that

$$\Omega_0 = 1;$$

that is, that

$$\sum_{t=1}^{\infty} \frac{(-1)^{t-1} x^{\frac{1}{2}t(t-1)}}{\prod_{s=1}^{t-1} (1-x^s)} = \prod_{s=1}^{\infty} (1-x^s). \tag{2.29}$$

This identity is easily proved by the theory of partitions. It is obvious that the right-hand side is the generating function enumerating the number of partitions of n into an even number of unequal parts less the number of partitions of n into an odd number of unequal parts. MacMahon* has shown that

$$\frac{x^{\frac{1}{2}t(t-1)}}{\prod_{s=1}^{t-1} (1-x^s)}$$

enumerates the partitions of n into exactly $t-1$ unequal parts; so that the left-hand side of (2.29) is the generating function of the same function of n as is the right. The term independent of x on each side being unity, the truth of the identity follows at once.

LEMMA IV. *If* $l \leqslant 0$, *then* $d_{l,n} = 0$. *If* $0 < l \leqslant n$, *then* $d_{l,n} = g_{n,l}$. *If* $l > n$, *then* $d_{l,n} = 0$.

If $a > 1$, and $0 \leqslant ax < 1$, we have†

$$f(x) = 1 + \sum_{l=1}^{\infty} \frac{a^l x^l}{\prod_{s=1}^{l} (1-x^s)} = 1 + \sum_{l=1}^{\infty} a^l \Omega_l$$

$$= 1 + \sum_{l=1}^{\infty} \sum_{n=l}^{\infty} d_{l,n} a^l x^n$$

$$= 1 + \sum_{n=1}^{\infty} \sum_{l=1}^{n} d_{l,n} a^l x^n.$$

But we know that

$$f(x) = 1 + \sum_{n=1}^{\infty} p_a(n) x^n = 1 + \sum_{n=1}^{\infty} \sum_{l=1}^{n} g_{n,l} a^l x^n.$$

Hence, if $0 < l \leqslant n$, then $d_{l,n} = g_{n,l}$.

* *Loc. cit.*, 2, 19.

† MacMahon, *loc. cit.*, 2, 2. For $0 \leqslant ax < 1$, the series are all absolutely convergent and the inversion of the order of summation is justified.

The first part of the lemma follows at once from (2.27) and (2.28), and the fact that n is positive. If $l > n$, then $n - lt < 0$ for $t \geqslant 1$; hence, by (2.24),

$$d_{l,n} = \sum_{t=1}^{\infty} (-1)^{t-1} c_{t,n-lt} = 0.$$

2.3. THEOREM I. *If $a > 1$, and if the series $\mathfrak{S}(n)$ is absolutely convergent, then we have*

$$\mathfrak{S}(n) = p_a(n). \tag{2.31}$$

We know that

$$\begin{aligned} a^{n/t} S_t(n) &= \frac{a^{n/t}}{t} \sum_{r=1}^{t} \{\phi_{t,r}\, e_t(-nr)\} \\ &= \frac{1}{t} \sum_{r=1}^{t} \phi_{t,r}\, a_{t,r}^{-n} \\ &= \frac{1}{t} \sum_{r=1}^{t} \sum_{s=-\infty}^{+\infty} c_{t,s}\, a_{t,r}^{s-n} \\ &= \sum_{s=-\infty}^{+\infty} c_{t,s}\, a_t^{s-n} \left(\frac{1}{t} \sum_{r=1}^{t} e_t\{(s-n)\, r\} \right). \end{aligned}$$

But, if $s - n \not\equiv 0 \pmod{t}$,

$$\frac{1}{t} \sum_{r=1}^{t} e_t\{(s-n)\, r\} = 0;$$

while, if $s - n = mt \equiv 0 \pmod{t}$,

$$\frac{1}{t} \sum_{r=1}^{t} e_t\{(s-n)\, r\} = 1.$$

Then
$$a^{n/t} S_t(n) = \sum_{m=-\infty}^{+\infty} c_{t,n+mt}\, b^m.$$

Since the series for $\mathfrak{S}(n)$ is absolutely convergent, the series of positive terms

$$\sum_{t=1}^{\infty} a^{n/t} S_t(n) = \sum_{t=1}^{\infty} \sum_{m=-\infty}^{+\infty} c_{t,n+mt}\, b^m$$

are convergent. Then we have

$$\begin{aligned} \mathfrak{S}(n) &= \sum_{t=1}^{\infty} (-1)^{t-1} a^{n/t} S_t(n) \\ &= \sum_{t=1}^{\infty} (-1)^{t-1} \sum_{m=-\infty}^{+\infty} c_{t,n+mt}\, b^m, \end{aligned}$$

and the last series is absolutely convergent. We may then reverse the order of summation and we have

$$\begin{aligned}\mathfrak{S}(n) &= \sum_{m=-\infty}^{\infty} b^m \Big(\sum_{t=1}^{\infty} (-1)^{t-1} c_{t,\, n+mt} \Big) \\ &= \sum_{m=-\infty}^{+\infty} d_{-m,n}\, b^m = \sum_{l=-\infty}^{+\infty} d_{l,n}\, a^l \\ &= \sum_{l=1}^{n} g_{n,l}\, a^l = p_a(n)\end{aligned}$$

by Lemma IV.

2.4. We now give elementary proofs of three further lemmas.

LEMMA V. *There is one and only one a_0 real and greater than unity such that $\lambda(a_0) = 0$. Also, for $1 < a < a_0$, $\lambda(a) > 0$; for $a > a_0$, $\lambda(a) < 0$.*

Let us write

$$\mu(a) = \lambda(a) \log a = \frac{\pi^2}{3} - \sum_{m=1}^{\infty} \frac{1}{m^2 a^m} - \tfrac{1}{2}(\log a)^2.$$

For $a > 1$, we may differentiate the series term by term. Then

$$\begin{aligned}\frac{d\mu(a)}{da} &= \sum_{m=1}^{\infty} \frac{1}{m a^{m+1}} - \frac{\log a}{a} \\ &= -\frac{1}{a}\left\{ \log\left(1 - \frac{1}{a}\right) + \log a \right\} = -\frac{\log(a-1)}{a}.\end{aligned}$$

So $\dfrac{d\mu(a)}{da} > 0$ for $1 < a < 2$; and $\dfrac{d\mu(a)}{da} < 0$ for $a > 2$.

When $a = 1$, we have

$$\mu(1) = \frac{\pi^2}{3} - \sum_{m=1}^{\infty} \frac{1}{m^2} = \frac{\pi^2}{6}.$$

Hence $\mu(a)$, which is continuous for $a \geqslant 1$, is positive at $a = 1$, increases with a until $a = 2$, and then decreases steadily to $-\infty$ as a tends to $+\infty$. Then there is one and only one $a_0 > 1$ such that

$$\mu(a_0) = 0.$$

Also, if $1 < a < a_0$, we have $\mu(a) > 0$, and, if $a > a_0$,

$$\mu(a) < 0.$$

But for $a > 1$, $\log a > 0$; and so, for this range of values of a, $\lambda(a)$ and $\mu(a)$ have the same sign and vanish together. The lemma is therefore proved.

LEMMA VI. *If $z>0$, $y=e^{-z}$, and $0<b<1$, we have*

$$\log \prod_{s=0}^{\infty}(1-by^s)<-\frac{1}{z}\sum_{m=1}^{\infty}\frac{b^m}{m^2},$$

and

$$-\log \prod_{s=1}^{\infty}(1-y^s)<\frac{\pi^2}{6z}.$$

We have

$$\frac{y}{1-y}=\frac{1}{e^z-1}<\frac{1}{z},$$

and

$$1-y=1-e^{-z}<z.$$

Also, since $0<y<1$,

$$my^{m-1}(1-y)<1-y^m<m(1-y);$$

and so

$$\frac{y^m}{m(1-y^m)}<\frac{y}{m^2(1-y)}<\frac{1}{m^2z},$$

and

$$\frac{1}{m(1-y^m)}>\frac{1}{m^2(1-y)}>\frac{1}{m^2z}.$$

Hence we have

$$-\log \prod_{s=1}^{\infty}(1-y^s)=\sum_{s=1}^{\infty}\sum_{m=1}^{\infty}\frac{y^{ms}}{m}=\sum_{m=1}^{\infty}\frac{y^m}{m(1-y^m)}<\frac{1}{z}\sum_{m=1}^{\infty}\frac{1}{m^2}=\frac{\pi^2}{6z}.$$

Similarly

$$-\log \prod_{s=0}^{\infty}(1-by^s)=\sum_{s=0}^{\infty}\sum_{m=1}^{\infty}\frac{b^m y^{ms}}{m}=\sum_{m=1}^{\infty}\frac{b^m}{m(1-y^m)}>\frac{1}{z}\sum_{m=1}^{\infty}\frac{b^m}{m^2},$$

and so

$$\log \prod_{s=0}^{\infty}(1-by^s)<-\frac{1}{z}\sum_{m=1}^{\infty}\frac{b^m}{m^2}.$$

LEMMA VII. *If $a>1$, then $|\phi_{t,r}|\leqslant\phi_t<Ke^{\lambda t}$.*

Since $0<|\alpha_{t,r}|=\alpha_t<1$, we have

$$|1-\alpha_{t,r}^s|\geqslant 1-\alpha_t^s>0.$$

Then

$$|\phi_{t,r}|=\frac{|\alpha_{t,r}|^{\frac{1}{2}t(t-1)}}{\prod_{s=1}^{t-1}|1-\alpha_{t,r}^s|\prod_{s=1}^{\infty}|1-\alpha_{t,r}^s|}$$

$$\leqslant\frac{\alpha_t^{\frac{1}{2}t(t-1)}}{\prod_{s=1}^{t-1}(1-\alpha_t^s)\prod_{s=1}^{\infty}(1-\alpha_t^s)}=\phi_t.$$

This proves the first part of the lemma. Then we have

$$\phi_t = \frac{a_t^{\frac{1}{2}t(t-1)} \prod_{s=t}^{\infty} (1-a_t^s)}{\left\{\prod_{s=1}^{\infty} (1-a_t^s)\right\}^2} = \frac{b^{\frac{1}{2}(t-1)} \prod_{s=0}^{\infty} (1-ba_t^s)}{\left\{\prod_{s=1}^{\infty} (1-a_t^s)\right\}^2}.$$

Hence, by Lemma VI, since

$$\log a_t = -\frac{\log a}{t},$$

we have

$$\log \phi_t = -\tfrac{1}{2}(t-1)\log a + \log\left\{\prod_{s=0}^{\infty}(1-ba_t^s)\right\} - 2\log\left\{\prod_{s=1}^{\infty}(1-a_t^s)\right\}$$

$$< K + \frac{t}{\log a}\left(\frac{\pi^2}{3} - \sum_{m=1}^{\infty}\frac{b^m}{m^2} - \tfrac{1}{2}(\log a)^2\right) = K + \lambda t,$$

and so

$$\phi_t < Ke^{\lambda t}.$$

2.5. THEOREM II. *For $a > a_0$, and $n \geqslant 1$, we have*

$$p_a(n) = \mathfrak{S}(n).$$

By Lemma V, if $a > a_0$, $\lambda(a) > 0$. Let us put $\lambda = -\rho$. Then we have $\rho = \rho(a) > 0$, and

$$|S_t(n)| \leqslant \frac{1}{t}\sum_{s=1}^{t}|\phi_{t,r}| \leqslant \phi_t < Ke^{-\rho t},$$

by Lemma VII. Also

$$a^{n/t} \leqslant a^n;$$

and so $\mathfrak{S}(n)$ is absolutely convergent for $a > a_0$. Then Theorem II follows at once from Theorem I.

3.1. We now consider the case $a = a_0$. All that is necessary is to prove the absolute convergence of the series $\mathfrak{S}$ for this value of a, and it is possible to do this without using analytic function theory. The work would, however, be longer and more tedious than in the case $a > a_0$. I therefore do not now restrict myself to elementary methods, and, in particular, I quote a non-elementary result (in Lemma VIII) from the paper by Hardy and Ramanujan.

From the results we have so far found we know only that, for $a = a_0$,

$$S_t(n) \leqslant \phi_t < K.$$

This is not enough to prove the absolute convergence of $\mathfrak{S}$, and our first step is to improve this result.

LEMMA VIII. *If* $z > 0$, *and*

$$g(z) = \prod_{s=1}^{\infty} (1-e^{-sz})^{-1},$$

then

$$K < \frac{g(z)}{z^{\frac{1}{2}} \exp(\pi^2/6z)} < K.$$

It is obvious that $g(z)$ decreases steadily as z increases and tends to 1 as z tends to infinity. Then, if $z \geqslant 2\pi$, we have

$$1 < g(z) \leqslant g(2\pi) = K.$$

Also, for this range of values of z,

$$K < z^{\frac{1}{2}} \exp\left(\frac{\pi^2}{6z}\right) < K.$$

Hence the lemma is proved for $z \geqslant 2\pi$. If $0 < z < 2\pi$, we have the result*

$$g(z) = \left(\frac{z}{2\pi}\right)^{\frac{1}{2}} \exp\left(\frac{\pi^2}{6z} - \frac{z}{24}\right) g\left(\frac{4\pi^2}{z}\right),$$

and so

$$\frac{g(z)}{z^{\frac{1}{2}} \exp(\pi^2/6z)} = \frac{1}{(2\pi)^{\frac{1}{2}}} \exp\left(-\frac{z}{24}\right) g\left(\frac{4\pi^2}{z}\right).$$

Then, as before,

$$1 < g\left(\frac{4\pi^2}{z}\right) < g(2\pi) = K.$$

Also

$$\exp\left(-\frac{\pi}{12}\right) < \exp\left(-\frac{z}{24}\right) < 1.$$

Hence

$$K < \frac{g(z)}{z^{\frac{1}{2}} \exp(\pi^2/6z)} < K,$$

as before.†

* Hardy and Ramanujan, *loc. cit.*, 93. The result is stated in a different notation and in a much more general form. It is deduced from elliptic function transformations. In the next paper of this series I shall give a direct proof by Cauchy's theorem of the general result for the function

$$\prod_{l=1}^{\infty} (1-e^{-zl^k})^{-1},$$

where k is a positive integer. If we put $k = 1$, we see that the Hardy-Ramanujan result I use here can be proved without reference to elliptic function transformations.

† A proof of this lemma by the application of Euler's summation formula to the logarithmic differential of $g(z)$ with respect to z,

$$\sum_{l=1}^{\infty} \frac{l}{e^{lz}-1},$$

is fairly simple, but considerably longer than the proof given here. It has the theoretical advantage that the result is thus proved without recourse to the theory of analytic functions of a complex variable.

LEMMA IX. *If* $a > 1$, *then* $\phi_t < Kt^{-1}e^{\lambda t}$.

As in Lemma VII, we have

$$\phi_t = \frac{b^{\frac{1}{2}(t-1)} \prod_{s=0}^{\infty} (1-ba_t^s)}{\left(\prod_{s=1}^{\infty} (1-a_t^s)\right)^2}$$

$$= b^{\frac{1}{2}(t-1)} \prod_{s=0}^{\infty} (1-ba_t^s) \left\{g\left(\frac{\log a}{t}\right)\right\}^2$$

$$< \frac{Kb^{\frac{1}{2}t}}{t} \exp\left\{\left(\frac{\pi^2}{3} - \sum_{m=1}^{\infty} \frac{b^m}{m^2}\right) \frac{t}{\log a}\right\}$$

$$= Kt^{-1}e^{\lambda t}$$

by Lemma VIII and the first part of Lemma VI.

3.2. LEMMA X. $c_{t,s+1} \geqslant c_{t,s} \geqslant 0$.

If $s < \frac{1}{2}t(t-1)$, we have $c_{t,s} = 0$, while if $s = \frac{1}{2}t(t-1)$, then $c_{t,s} = 1$. Hence, for $s < \frac{1}{2}t(t-1)$, the lemma is proved. We have also, for $|x| < 1$,

$$\sum_{s=\frac{1}{2}t(t-1)}^{\infty} c_{t,s}x^s = \frac{x^{\frac{1}{2}t(t-1)}}{\prod_{l=1}^{t-1}(1-x^l) \prod_{l=1}^{\infty}(1-x^l)}$$

$$= \frac{1}{1-x} x^{\frac{1}{2}t(t-1)} \prod_{l=1}^{t-1} (1+x^l+x^{2l}+\ldots) \prod_{l=2}^{\infty} (1+x^l+\ldots)$$

$$= (1+x+x^2+\ldots) \sum_{m=\frac{1}{2}t(t-1)}^{\infty} q_{t,m}x^m,$$

where $q_{t,m} \geqslant 0$. Then, if $s \geqslant \frac{1}{2}t(t-1)$,

$$c_{t,s} = \sum_{m=\frac{1}{2}t(t-1)}^{s} q_{t,m};$$

hence

$$c_{t,s+1} - c_{t,s} = q_{t,s+1} \geqslant 0,$$

and

$$c_{t,s} \geqslant 0.$$

LEMMA XI. *If* $a > 1$, *then*

$$S_t(n) \leqslant \frac{a\phi_t}{t}.$$

We have, as in paragraph 2.3,

$$S_t(n) = \sum_{\substack{s \geqslant \frac{1}{2}t(t-1) \\ s \equiv n \,(\mathrm{mod}\, t)}} c_{t,s}\, b^{s/t},$$

while

$$\phi_t = \sum_{s=\frac{1}{2}t(t-1)}^{\infty} c_{t,s}\, b^{s/t}.$$

Suppose that

$$n \equiv h (\mathrm{mod}\, t), \quad \tfrac{1}{2}t(t-1) \leqslant h < \tfrac{1}{2}t(t+1);$$

there is always one and only one such h. Then

$$S_t(n) = \sum_{l=0}^{\infty} c_{t,\,h+lt}\, b^{l+(h/t)}.$$

Now, by Lemma X, we have

$$\begin{aligned} t c_{t,h+lt} &\leqslant c_{t,h+lt} + c_{t,h+lt+1} + \ldots + c_{t,h+lt+t-1} \\ &\leqslant c_{t,h+lt}\, a + c_{t,h+lt+1}\, a^{1-1/t} + \ldots + c_{t,h+lt+t-1}\, a^{1/t} \\ &= a(c_{t,h+lt} + c_{t,h+lt+1}\, b^{1/t} + \ldots + c_{t,h+lt+t-1}\, b^{(t-1)/t}). \end{aligned}$$

Hence

$$\begin{aligned} t S_t(n) &= \sum_{l=0}^{\infty} t c_{t,h+lt}\, b^{l+(h/t)} \\ &\leqslant a \sum_{l=0}^{\infty} b^{(h/t)+l}(c_{t,h+lt} + \ldots + c_{t,h+lt+t-1}\, b^{(t-1)/t}) \\ &= a \sum_{s=h}^{\infty} c_{t,s}\, b^{s/t} \leqslant a\phi_t, \end{aligned}$$

which proves the lemma.

3.3. THEOREM III. *If $a = a_0$, then $p_a(n) = \mathfrak{S}$.*

If $a = a_0$, we have $\lambda(a) = 0$. Hence, by Lemmas IX and XI,

$$S_t(n) \leqslant \frac{a\phi_t}{t} < \frac{K}{t^2}.$$

Then $\mathfrak{S}$ is absolutely convergent, for

$$\sum_{t=1}^{T} a^{n/t} S_t(n) \leqslant K a^n \sum_{t=1}^{T} \frac{1}{t^2} < K a^n.$$

Hence Theorem III follows from Theorem I.

4.1. To discuss the case $1 < a < a_0$, different methods are applicable. By arguments similar to those already used, we can prove that

$$S_t(n) \geqslant t^{-1}\phi_t > Kt^{-2}e^{\lambda t},$$

and hence that Σ is divergent for $1 < a < a_0$. We see then that we cannot use Theorem I.

We take C_t as the circle in the x-plane with centre at the origin and radius

$$r_t = b^{2/(2t+1)}.$$

Then $a_t = b^{1/t} < r_t < b^{1/(t+1)} = a_{t+1}$, since

$$\frac{1}{t} > \frac{2}{2t+1} > \frac{1}{t+1}.$$

C_0 is any circle with centre at the origin and radius less than b.

From the definition of $f(x)$ as an infinite product, we see at once that

(i) $f(x)$ has a simple pole at $x = a_{t,r}$ for all $t \geqslant 1$, and all $r \leqslant t$, *i.e.* $f(x)$ has t poles on the circumference of the circle

$$|x| = a_t.$$

(ii) $f(x)$ is regular at all points within the unit circle other than these poles.

(iii) Every point of the circumference of the unit circle is an essential singularity of $f(x)$, for every such point is the limit of a sequence of poles.

(iv) $f(x)$ is undefined outside the unit circle.

Then, within the circle C_T, $f(x)$ has poles at the points

$$x = a_{t,r}, \quad 0 < t \leqslant T, \quad 0 < r \leqslant t;$$

and is analytic at every other point within C_T. There are no poles within C_0.

4.2. LEMMA XII. *If $a > 1$, the residue of $f(x)$ at the pole $x = a_{t,r}$ is $(-1)^t t^{-1} a_{t,r}\phi_{t,r}$.*

We write α for $a_{t,r}$ for convenience, and note that $\alpha^t = b$. Then the residue is given by

$$\lim_{x\to\alpha}(x-\alpha)f(x) = \frac{1}{\prod_{s=1}^{t-1}(1-a\alpha^s)\prod_{s=t+1}^{\infty}(1-a\alpha^s)} \lim_{x\to\alpha}\left(\frac{x-\alpha}{1-ax^t}\right). \quad (4.21)$$

We have first

$$\lim_{x\to\alpha}\left(\frac{x-\alpha}{1-\alpha x^t}\right)=-\alpha\lim_{x\to\alpha}\left(\frac{1-x/\alpha}{1-(x/\alpha)^t}\right)=-\frac{\alpha}{t}.$$

Also

$$\prod_{s=1}^{t-1}(1-a\alpha^s)=\prod_{s=1}^{t-1}(1-\alpha^{s-t})=\alpha^{-\frac{1}{2}t(t-1)}\prod_{s=1}^{t-1}(\alpha^{t-s}-1)$$

$$=(-1)^{t-1}\alpha^{-\frac{1}{2}t(t-1)}\prod_{s=1}^{t-1}(1-\alpha^{t-s})=(-1)^{t-1}\alpha^{-\frac{1}{2}(t-1)}\prod_{s=1}^{t-1}(1-\alpha^s);$$

and

$$\prod_{s=t+1}^{\infty}(1-a\alpha^s)=\prod_{s=t+1}^{\infty}(1-\alpha^{s-t})=\prod_{s=1}^{\infty}(1-\alpha^s).$$

Then

$$\frac{1}{\prod_{s=1}^{t-1}(1-a\alpha^s)\prod_{s=t+1}^{\infty}(1-a\alpha^s)}=\frac{(-1)^{t-1}\alpha^{\frac{1}{2}t(t-1)}}{\prod_{s=1}^{t-1}(1-\alpha^s)\prod_{s=1}^{\infty}(1-\alpha^s)}=(-1)^{t-1}\phi_{t,r},$$

and the lemma follows at once from this and (4.21) above.

4.3. LEMMA XIII. *If $T\geqslant 1$, we have*

$$I_T=\frac{1}{2\pi i}\int_{C_T}\frac{f(x)\,dx}{x^{n+1}}=p_a(n)-\sum_{t=1}^{T}(-1)^{t-1}a^{n/t}S_t(n).$$

We have, by Cauchy's theorem and Lemma XII,

$$I_T=\frac{1}{2\pi i}\int_{C_T}\frac{f(x)\,dx}{x^{n+1}}=p_a(n)+\sum_{t=1}^{T}\sum_{r=1}^{t}(-1)^t t^{-1}\alpha_{t,r}^{-n}\phi_{t,r}$$

$$=p_a(n)-\sum_{t=1}^{T}(-1)^{t-1}a^{n/t}S_t(n).$$

4.4. LEMMA XIV.

$$|I_T|\leqslant Ke^{\lambda T}a^{n/T}\quad(T\geqslant 1).$$

If $|x|=r_t$,

$$|f(x)|=\frac{1}{\prod_{n=1}^{\infty}|1-ax^n|}\leqslant\frac{1}{\prod_{n=1}^{\infty}|1-a|x|^n|}=|f(r_t)|.$$

Let us put $y=r_t^{\frac{1}{2}}=b^{1/(2t+1)}$, so that $a=y^{-(2t+1)}$. If $s\leqslant t$, then $2s<2t+1$, and

$$|1-ay^{2s}|=|1-y^{2s-2t-1}|=y^{2s-2t-1}-1$$

$$=y^{2s-2t-1}(1-y^{2t+1-2s}).$$

Then

$$\prod_{s=1}^{t}|1-ay^{2s}| = y^{-t^2}\prod_{s=1}^{t}(1-y^{2t+1-2s}) = y^{-t^2}\prod_{s=1}^{t}(1-y^{2s-1})$$

$$= \frac{\prod_{s=1}^{\infty}(1-y^{2s-1})}{y^{t^2}\prod_{s=0}^{\infty}(1-by^{2s})}.$$

If $s>t$, then $$|1-ay^{2s}| = 1-y^{2s-2t-1},$$

and $$\prod_{s=t+1}^{\infty}|1-ay^{2s}| = \prod_{s=t+1}^{\infty}(1-y^{2s-2t-1}) = \prod_{s=1}^{\infty}(1-y^{2s-1}).$$

Then $$|f(r_t)| = \frac{1}{\prod_{s=1}^{\infty}|1-ay^{2s}|} = \frac{y^{t^2}\prod_{s=0}^{\infty}(1-by^{2s})}{\prod_{s=1}^{\infty}(1-y^{2s-1})^2}.$$

We have, by Lemma VIII,

$$\frac{1}{\prod_{s=1}^{\infty}(1-y^{2s-1})} = \frac{\prod_{s=1}^{\infty}(1-y^{2s})}{\prod_{s=1}^{\infty}(1-y^{s})} = \frac{g\left(\frac{\log a}{2t+1}\right)}{g\left(\frac{2\log a}{2t+1}\right)}$$

$$< K\exp\left(\frac{\pi^2 t}{6\log a}\right).$$

Also, by the first result of Lemma VI, we have

$$\prod_{s=0}^{\infty}(1-by^{2s}) < \exp\left(-\frac{2t+1}{2\log a}\sum_{m=1}^{\infty}\frac{b^m}{m^2}\right) < \exp\left(-\frac{t}{\log a}\sum_{m=1}^{\infty}\frac{b^m}{m^2}\right).$$

Then we have

$$|f(r_t)| \leqslant K\exp\left\{\frac{t}{\log a}\left(\frac{\pi^2}{3}-\sum_{m=1}^{\infty}\frac{b^m}{m^2}\right)-\frac{t^2\log a}{2t+1}\right\}$$

$$< Ke^{\lambda t},$$

and so

$$|I_T| = \left|\frac{1}{2\pi i}\int_{C_T}\frac{f(x)\,dx}{x^{n+1}}\right| \leqslant Ke^{\lambda T}r_T^{-n} < Ke^{\lambda T}a^{n/T}.$$

4.5. THEOREM IV. *If $a>1$, and q is any positive integer, we have*

$$\left|p_a(n)-\sum_{t=1}^{q}(-1)^{t-1}a^{n/t}S_t(n)\right| < Ke^{\lambda q}a^{n/(q+1)}.$$

We have, by Lemmas VII, XIII, and XIV,

$$\left| p_a(n) - \sum_{t=1}^{q} (-1)^{t-1} a^{n/t} S_t(n) \right|$$

$$\leqslant \left| p_a(n) - \sum_{t=1}^{q+1} (-1)^{t-1} a^{n/t} S_t(n) \right| + |a^{n/(q+1)} S_{q+1}(n)|$$

$$\leqslant |I_{q+1}| + K e^{\lambda q} a^{n/(q+1)} \leqslant K e^{\lambda q} a^{n/(q+1)}.$$

4.6. We see that Theorem IV includes Theorem II, but not Theorem III; for, if $a = a_0$, we have

$$K e^{\lambda q} a^{n/(q+1)} = K a^{n/(q+1)} > K,$$

for all values of q; whereas Theorem III asserts that the error term tends to zero as $q \to \infty$. For $1 < a < a_0$, the result is new and shows that $S(n)$ is an asymptotic series for $p_a(n)$.

We observe that the K of Theorem IV is independent of q. To find the best possible approximation to $p_a(n)$ when $a > a_0$, we choose q so that

$$e^{\lambda q} a^{n/(q+1)}$$

is as small as possible. If q were a continuous variable, the best value of q would be

$$q' = \left(\frac{n \log a}{\lambda} \right)^{\frac{1}{2}} - 1;$$

we must take q as $[q']$ or $[q']+1$.

With either value of q the error is less than

$$K e^{2(\lambda n \log a)^{\frac{1}{2}}}.$$

4.7. The principle of our results for $a > 1$ is now clear. If we construct the partial fraction series for $f(x)$, we have the formal series

$$G(x) = \sum_{t=1}^{\infty} \frac{(-1)^{t-1}}{t} \sum_{r=1}^{t} \frac{\phi_{t,r}}{1 - (x/a_{t,r})}.$$

If $a > a_0$, this series converges for all x such that $|x| < 1$ and $x \neq a_{t,r}$ for any $t \geqslant 1$. In this case, it is easy to see that, for $|x| < a_{1,1} = b$,

$$G(x) = 1 + \sum_{n=1}^{\infty} \mathfrak{S}(n) x^n.$$

We can also show that, for $a > a_0$ and $|x| < 1$, $f(x)$ and $G(x)$ represent the same function. Then we have

$$p_a(n) = \mathfrak{S}(n)$$

for $a > a_0$. We may remark that, when $a > a_0$, the series $G(x)$ converges to zero for $|x| > 1$.

5.1. We now consider the case $a < 1$. Entirely different methods are applicable. The notation used hitherto will be replaced by the following:

$a < 1$.

$f(x)$ and $p_a(n)$ have the same meaning as before.

r is an arbitrary, non-negative integer; K without a suffix is now a positive number, whose value varies from one occurrence to another, depending only on the parameters a and r. When K depends on other parameters, these are indicated by suffixes, thus

$$K_\epsilon = K(a, r, \epsilon).$$

$|u(n)| = O\{v(n)\}$ means that $|u(n)| < Kv(n)$, while $|u(n)| = o\{v(n)\}$ means that, for any $\epsilon > 0$,

$$|u(n)| < \epsilon v(n)$$

for $n > K_\epsilon$.

We write
$$A = \sum_{m=1}^{\infty} \frac{a^m}{m^2}, \quad N = \left(\frac{n}{A}\right)^{\frac{1}{2}};$$

$$z = \log \frac{1}{x} = \log \left|\frac{1}{x}\right| - i\theta,$$

where θ is the principal value of $\arg x$.

C is the circle $|x| = e^{-1/N}$.

$C = C' + C''$, where C' is that part of C on which

$$|\theta| < \frac{1}{N}.$$

B_s is the s-th Bernouillian number.

$$\alpha_s = \frac{(-1)^s 2}{(2\pi)^{2s+2}} \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} \frac{m^{2s} a^m}{l^{2s+2}} = \frac{(-1)^s B_{s+1}}{\Gamma(2s+3)} \sum_{m=1}^{\infty} m^{2s} a^m.$$

β_s $(s \leqslant 2r+2)$ is the coefficient of y^s in the expansion of

$$\exp\left(\sum_{t=0}^{r} \alpha_s y^{2s+1}\right)$$

in ascending powers of y. It is obvious that, so long as $s \leqslant 2r+2$, β_s is

independent of r. Also, for $|y|<K$,

$$\left|\exp\Big(\sum_{t=0}^{r}\alpha_s y^{2s+1}\Big)-\sum_{s=0}^{r+1}\beta_s y^s\right|<K|y|^{r+2}. \tag{5.11}$$

$$\gamma_\nu=A^{\frac{1}{2}\nu}\Gamma(\nu+\tfrac{3}{2})\sum_{m=0}^{\nu}\frac{\beta_{\nu-m}}{(4A)^m\Gamma(m+1)\Gamma(\nu-2m+\frac{3}{2})}.$$

$J_s(u)$ is the Bessel function of order s and argument u.

5.2. The theorems which I shall prove in § 5 are as follows:

THEOREM V. *If $a<1$, and if r is any positive integer, we have*

$$p_a(n)=(1-a)^{\frac{1}{2}}\sum_{s=0}^{r}\beta_s\Big(\frac{A}{n}\Big)^{\frac{1}{2}(s+1)}i^{-(s+1)}J_{s+1}\{2i\sqrt{(An)}\}+O\Big(\frac{|J_{r+2}\{2i\sqrt{(An)}\}|}{n^{\frac{1}{2}(r+2)}}\Big).$$

THEOREM VI. *If $a<1$, and if r is any positive integer, we have*

$$p_a(n)=\frac{(1-a)^{\frac{1}{2}}A^{\frac{1}{4}}}{2\pi^{\frac{1}{2}}n^{\frac{3}{4}}}e^{2\sqrt{(An)}}\Big\{\sum_{\nu=0}^{r}\frac{\gamma_\nu}{n^{\frac{1}{2}\nu}}+O(n^{-\frac{1}{2}(r+1)})\Big\}.$$

For $a<1$, $f(x)$ is an analytic function of x, regular within the unit circle, every point of the circumference being an essential singularity; the function is not defined outside the circle. Since C encloses the origin and lies within the unit circle, we have

$$\begin{aligned}p_a(n)&=\frac{1}{2\pi i}\int_C\frac{f(x)\,dx}{x^{n+1}}\\&=\frac{1}{2\pi i}\int_{C'}\frac{f(x)\,dx}{x^{n+1}}+\frac{1}{2\pi i}\int_{C''}\frac{f(x)\,dx}{x^{n+1}}\\&=I'+I''.\end{aligned}$$

It will be seen later that, when n is large, the dominant terms in $p_a(n)$ are supplied by I'. I first prove two lemmas with regard to the values of $f(x)$ on C', C'' respectively.

5.3. LEMMA XV. *If x lies on C', we have*

$$\left|f(x)-(1-a)^{\frac{1}{2}}\exp\Big(\frac{A}{z}\Big)\Big(\sum_{s=0}^{r+1}\beta_s z^s\Big)\right|<Ke^{AN}N^{-r-2}. \tag{5.31}$$

By Poisson's well-known formula, for x real and less than unity, we have

$$\begin{aligned}\log f(x)&=-\sum_{l=1}^{\infty}\log(1-ax^l)\\&=\tfrac{1}{2}\log(1-a)-\sum_{l=-\infty}^{+\infty}\int_0^\infty\log(1-ax^t)\,e^{2\pi lit}\,dt,\end{aligned}$$

where $\sum_{-\infty}^{+\infty}$ denotes $\lim_{k\to\infty} \sum_{-k}^{k}$. Now, by expanding the logarithm, we have

$$\int_0^\infty \log(1-ax^t)\, e^{2\pi l i t}\, dt = -\sum_{m=1}^{\infty} \frac{a^m}{m} \frac{1}{(mz-2l\pi i)}.$$

Hence

$$\log f(x) = \tfrac{1}{2}\log(1-a) + \frac{A}{z} + 2z \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} \frac{a^m}{m^2 z^2 + 4l^2\pi^2}$$

$$= \tfrac{1}{2}\log(1-a) + \frac{A}{z} + \sum_{s=0}^{r} \alpha_s z^{2s+1} + R_{r+1} z^{2r+3}, \qquad (5.32)$$

where

$$R_{r+1} = (-1)^{r+1} 2 \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} \frac{a^m m^{2r+2}}{4^{r+1} l^{2r+2} \pi^{2r+2} (m^2 z^2 + 4l^2\pi^2)},$$

the latter step being an application of the identity

$$\frac{1}{b+c} = \sum_{s=0}^{r} (-1)^s \frac{b^s}{c^{s+1}} + (-1)^{r+1} \frac{b^{r+1}}{c^{r+1}(b+c)}.$$

But, if $\mathfrak{R}(z^2) \geqslant 0$,

$$\frac{1}{|m^2 z^2 + 4l^2\pi^2|} \leqslant \frac{1}{|\mathfrak{R}(m^2 z^2 + 4l^2\pi^2)|} \leqslant \frac{1}{4l^2\pi^2};$$

and so, for such values of z, the double series occurring in R_{r+1} converges absolutely and uniformly, and

$$|R_{r+1}| \leqslant |\alpha_{r+1}|.$$

Hence, by the principle of analytic continuation, (5.32) is true for every x such that $\mathfrak{R}(z^2) > 0$.

Now, when x lies on C', we have $\mathfrak{R}(z) = 1/N$ and $|\theta| < 1/N$, and so

$$\mathfrak{R}(z^2) = \{\mathfrak{R}(z)\}^2 - \theta^2 > 0.$$

Hence, for all x on C',

$$\log f(x) = \tfrac{1}{2}\log(1-a) + \frac{A}{z} + \sum_{s=0}^{r} \alpha_s z^{2s+1} + R_{r+1} z^{2s+1},$$

where $|R_{r+1}| \leqslant |\alpha_{r+1}| < K$. Using (5.11), we have at once

$$\left| f(x) - (1-a)^{\frac{1}{2}} \exp\left(\frac{A}{z}\right) \left\{ \sum_{s=0}^{r+1} \beta_s z^s \right\} \right| < K \left| \exp\left(\frac{A}{z}\right) \right| |z|^{r+2} \leqslant K e^{AN} N^{-r-2}.$$

5.4. LEMMA XVI. *If x lies on C'', and if*

$$\delta = a\left(1 - \frac{1}{\sqrt{2}}\right) > K,$$

we have $$|f(x)| < K \exp\{(A-\delta)N\},$$

and so $$|I''| < K \exp\{(2A-\delta)N\}.$$

We have, for $|x| < 1$,

$$\log f(x) = -\sum_{l=1}^{\infty} \log(1-ax^l)$$

$$= \sum_{l=1}^{\infty}\sum_{m=1}^{\infty} \frac{a^m x^{lm}}{m} = \sum_{m=1}^{\infty} \frac{a^m x^m}{m(1-x^m)};$$

and so, since $|1-x^m| \geqslant 1-|x|^m$, it follows that

$$|\log f(x)| \leqslant \frac{a|x|}{|1-x|} + \sum_{m=2}^{\infty} \frac{a^m|x^m|}{1-|x^m|}$$

$$= \sum_{m=1}^{\infty} \frac{a^m|x^m|}{1-|x^m|} - a|x|\left(\frac{1}{1-|x|} - \frac{1}{|1-x|}\right).$$

Now, for x on C'',

$$1-|x| = \frac{1}{N} + O\left(\frac{1}{N^2}\right),$$

$$|1-x| \geqslant \frac{\sqrt{2}}{N} + O\left(\frac{1}{N^2}\right),$$

and $$|x| = \exp\left(-\frac{1}{N}\right) = 1 + O\left(\frac{1}{N}\right);$$

so we have

$$a|x|\left(\frac{1}{1-|x|} - \frac{1}{|1-x|}\right) \geqslant aN\left(1-\frac{1}{\sqrt{2}}\right) + O(1)$$

$$= \delta N + O(1).$$

By Lemma XV,

$$\sum_{m=1}^{\infty} \frac{a^m|x^m|}{m(1-|x^m|)} = \log f(|x|) = AN + O(1).$$

Therefore, for x on C'',

$$|\log f(x)| < (A-\delta)N + O(1);$$

and so $$|f(x)| < K \exp\{(A-\delta)N\}.$$

Also, $$|I''| \leqslant \frac{1}{2\pi}\int_{C''} \left|\frac{f(x)}{x^n}\right| \left|\frac{dx}{x}\right| < K \exp\{(2A-\delta)N\},$$

since $|x|^n = e^{-n/N} = e^{-A\sqrt{N}}$.

5.5. LEMMA XVII. *If* $s \leqslant r+1$, *and if*

$$P_s = \frac{1}{2\pi i}\int_{1-i}^{1+i} v^s \exp\left\{AN\left(v+\frac{1}{v}\right)\right\} dv,$$

then
$$|P_s - i^{-(s+1)} J_{s+1}(2iAN)| < Ke^{\frac{3}{2}AN}. \tag{5.51}$$

Let Γ_1 denote the contour in the v-plane formed by the sides of the square whose vertices lie at the points $v = \pm 1 \pm i$. If we put $v = it$, Γ_1 transforms into Γ_2, a contour of the same shape and position in the t-plane. Then

$$\begin{aligned} i^{-s-1} J_{s+1}(2iAN) &= i^{s+1} J_{-s-1}(2iAN) \\ &= \frac{i^{s+1}}{2\pi i}\int_{\Gamma_2} t^s \exp\left\{iAN\left(t-\frac{1}{t}\right)\right\} dt \\ &= \frac{1}{2\pi i}\int_{\Gamma_1} v^s \exp\left\{AN\left(v+\frac{1}{v}\right)\right\} dv \\ &= \frac{1}{2\pi i}\left(\int_{1-i}^{1+i} + \int_{1+i}^{-1+i} + \int_{-1+i}^{-1-i} + \int_{-1-i}^{1-i} \ldots dv\right) \\ &= P_s + Q' + Q'' + Q'''. \end{aligned} \tag{5.52}$$

In Q' we put $v = \xi + i$. Then ξ is real, $-1 \leqslant \xi \leqslant 1$, and

$$\Re\left(v+\frac{1}{v}\right) = \xi + \frac{\xi}{1+\xi^2}.$$

It can be seen at once by differentiation that this is an increasing function of ξ in the interval and therefore has its greatest value when $\xi = 1$; hence, in Q',

$$\Re\left(v+\frac{1}{v}\right) \leqslant \tfrac{3}{2}.$$

Then
$$|Q'| < K \exp(\tfrac{3}{2}AN). \tag{5.53}$$

Similarly,
$$|Q'''| < K \exp(\tfrac{3}{2}AN). \tag{5.54}$$

In Q'' we put $v = -1 + i\eta$. Then η is real, $-1 \leqslant \eta \leqslant 1$, and

$$\Re\left(v+\frac{1}{v}\right) = -1 - \frac{1}{1+\eta^2} < -1;$$

hence
$$|Q''| < Ke^{-AN}. \tag{5.55}$$

(5.51) follows immediately from (5.52)–(5.55).

5.6. LEMMA XVIII. *We have*

$$\left| I' - (1-a)^{\frac{1}{2}} \sum_{s=0}^{r+1} \frac{\beta_s P_s}{N^{s+1}} \right| < K N^{-r-3} e^{2AN}.$$

If we put $v = Nz$ in P_s, we have

$$\frac{P_s}{N^{s+1}} = \frac{1}{2\pi i} \int_{(1-i)/N}^{(1+i)/N} z^s \exp\left(\frac{A}{z} + nz\right) dz$$

$$= \frac{1}{2\pi i} \int_{C'} \frac{z^s e^{A/z}}{x^{n+1}} dx.$$

Then, by Lemma XV,

$$\left| I' - (1-a)^{\frac{1}{2}} \sum_{s=0}^{r+1} \frac{\beta_s P_s}{N^{s+1}} \right| = \frac{1}{2\pi} \left| \int_{C'} \left\{ \left(f(x) - (1-a)^{\frac{1}{2}} e^{A/z} \sum_{s=0}^{r+1} \beta_s z^s \right) \Big/ x^{n+1} \right\} dx \right|$$

$$< K N^{-r-2} e^{2AN} \int_{C'} \left| \frac{dx}{x} \right|$$

$$< K N^{-r-3} e^{2AN}.$$

5.7. LEMMA XIX. *For any $s \leqslant r+1$, we have*

$$i^{-s-1} J_{s+1}(2iAN)$$

$$= \frac{e^{2AN}}{2\pi^{\frac{1}{2}} (AN)^{\frac{1}{2}}} \left[\sum_{m=0}^{r+1-s} \frac{(-1)^m \Gamma(s+m+\frac{3}{2})}{2^{2m} \Gamma(m+1) \Gamma(s-m+\frac{3}{2}) (AN)^m} + O\left(\frac{1}{N^{r+2-s}} \right) \right].$$

This lemma is merely a statement in terms of our notation of the known asymptotic expansion of the Bessel function.*

5.8. *Proof of Theorems V and VI.* This is now a matter of very simple calculation. Using Lemmas XVI, XVII, and XVIII, we have

$$\left| p_a(n) - (1-a)^{\frac{1}{2}} \sum_{s=0}^{r+1} \frac{\beta_s i^{-s-1} J_{s+1}(2iAN)}{N^{s+1}} \right|$$

$$\leqslant \left| I' - (1-a)^{\frac{1}{2}} \sum_{s=0}^{r+1} \frac{\beta_s P_s}{N^{s+1}} \right| + |I''| + (1-a)^{\frac{1}{2}} \sum_{s=0}^{r+1} \left| \frac{\beta_s}{N^{s+1}} \right| |P_s - i^{-(s+1)} J_{s+1}(2iAN)|$$

$$< K(e^{2AN} N^{-r-3} + e^{(2A-\delta)N} + e^{\frac{1}{2}AN})$$

$$< K N^{-r-3} e^{2AN}.$$

* See, for example, Whittaker and Watson, *Modern analysis* (1927), 368.

But, by Lemma XIX,

$$i^{-r-2} J_{r+2}(2iAN) \sim \frac{e^{2AN}}{2\pi^{\frac{1}{2}}(AN)^{\frac{1}{2}}}.$$

Hence

$$\left| p_a(n) - (1-a)^{\frac{1}{2}} \sum_{s=0}^{r} \frac{\beta_s}{N^{s+1}} i^{-s-1} J_{s+1}(2iAN) \right|$$

$$< K\{N^{-r-3} e^{2AN} + N^{-r-2} J_{r+2}(2iAN)\}$$

$$< KN^{-r-2} J_{r+2}(2iAN).$$

If we now substitute $(n/A)^{\frac{1}{2}}$ for N, we have Theorem V. To deduce Theorem VI we have first to substitute the asymptotic expansion of the Bessel function given in Lemma XIX. We then write

$$s = \nu - m,$$

and collect the terms in

$$e^{2AN} N^{-\nu-\frac{3}{2}}$$

for each $\nu \leqslant r$. All other terms are contained in one error term, and, if we again substitute for N, we have Theorem VI.

6.1. Let us now return to the general type of weighted partition described in paragraph 1.1. We take λ_l and μ_l such that λ_l is real and positive, $\sum_{l=1}^{\infty} \lambda_l$ is divergent, μ_l increases steadily with l, and

$$\sum_{l=1}^{\infty} \lambda_l x^{\mu_l}$$

converges for $|x| < 1$. Then $f(x)$ will always have at least one essential singularity on the circumference of the unit circle, but no singularities other than poles inside the circle.

There are two cases to consider:

(i) $\lambda_l \leqslant 1$, *except perhaps for a finite sequence of values of* s. Here $f(x)$ can have only a finite number of poles within the unit circle. Let ϕ_a be the residue at any such pole a, let

$$F(x) = \sum_a \frac{\phi_a}{x-a},$$

and $f(x) = F(x) + h(x)$. Then $h(x)$ is analytic and regular within the unit circle. The terms in $F(x)$ give rise to the dominant terms in the

expansion of $p_{\lambda,\mu}(n)$, and we may be able to carry the asymptotic expansion further by applying methods similar to those of the last section to $h(x)$.

(ii) $\lambda_l > 1$ *for an infinite sequence of values of* s. Here we have an infinite sequence of rings of poles, and if, for the sequence of values of l for which $\lambda_l > 1$, $\mu_l^{-1} \log \lambda_l$ decreases steadily after some l, then these rings are of steadily increasing radius. We can then find an asymptotic expansion of $p_{\lambda,\mu}(n)$ by the method of § 4. The expansion may or may not converge. It is interesting to observe [see example (*c*) of the next paragraph] that the expansion may converge, even if $\lambda_l < 1$ for an infinite sequence of values of l.

6 . 2. Here I indicate briefly the results in certain examples of case (ii).

(*a*) $\mu_l = l^k$, where k is a positive integer, and $\lambda_l = a > 1$. The a_0 phenomenon also appears in this case. The series for $p_{\lambda,\mu}(n)$ is convergent for $a > a_0$, and divergent for $1 < a < a_0$, where $a_0 = a_0(k)$ is a decreasing function of k.

(*b*) $\mu_l = l^k$, where k is a positive integer, and $\lambda_l = l^\alpha$, where α is real and positive. We always have convergence in this case, as we might expect, for $\lambda_l \to \infty$, and so we have an analogy with the case $a > a_0$.

(*c*) $$f(x) = \prod_{l=1}^{\infty} \{(1-ax^l)(1-bx^l)\}^{-1} \quad (ab = 1,\ a > 1).$$

Here $a_0 = \exp(\sqrt{\frac{2}{3}}\pi)$, and we have a convergent series for $p_{\lambda,\mu}(n)$ when $a > a_0$.

CAYLEY'S PROBLEM OF THE IN-AND-CIRCUMSCRIBED TRIANGLE

By W. L. Edge.

[Received 17 May, 1932.—Read 19 May, 1932.]

Introduction.

1. A triangle is said to be in-and-circumscribed to a plane curve when each of its vertices is a point of the curve and each of its sides is a tangent of the curve, the three points of contact and the three vertices being six distinct points. The number of in-and-circumscribed triangles of a plane quartic of genus 3 has been obtained in a recent paper*; the number was obtained geometrically, by considering Hesse's (1, 1) correspondence between the plane quartic and a twisted sextic, and is in agreement with the number of in-and-circumscribed triangles, of a curve of any order, which was obtained by Cayley.

Cayley†, in his paper having a similar title to the present one, solves the problem, for a curve of any order, as follows. He first obtains the number of triangles whose vertices lie on three given curves and whose sides touch three other given curves; he then makes two or more of the six curves coincide, and thus, corresponding to the series of different relations of identity between the curves, a series of fifty-two problems presents itself. The last problem of the series, when all the six curves coincide, is the problem of the in-and-circumscribed triangle. The solution of these problems is made to depend upon two things—the functional method and the principle of correspondence. The use of the functional method causes the solution

* Edge, "Octadic surfaces and plane quartic curves", *Proc. London Math. Soc.* (2), 34 (1932), 492–525.

† Cayley, "On the problem of the in-and-circumscribed triangle", *Phil. Trans. Royal Soc.*, 161 (1871), 369–412, or *Collected Papers*, 8, 212–257.

of certain problems of the series to depend upon the solution of previous ones; for example, the problem of the in-and-circumscribed triangle is made to depend upon the solution of nine of the preceding problems of the series, and, when all these nine problems have been solved, the solution of the problem of the in-and-circumscribed triangle is obtained by solving a functional equation. The principle of correspondence can also be applied to the series of problems; those triangles which satisfy the conditions of a problem have their vertices among the united points of a certain correspondence on a curve. The number of united points can easily be obtained; the whole difficulty lies in the fact that, apart from the vertices of the triangles, these united points also include other points. To use Cayley's expression, these other points may be called *heterotypic* solutions. Before a problem can be solved the number of its heterotypic solutions has to be obtained and this, as Cayley remarks, is a matter of some delicacy.

The problem of the in-and-circumscribed triangle is of course a problem concerned simply with a single curve, and there seems *a priori* no need to consider it as the outcome of, or to approach it through, a series of problems concerned with more than one curve. But, whatever may be said about this line of approach, Cayley did succeed in obtaining the solution of the problem in this way, and the problem was not solved by any other method for a long time. It is easy to obtain a correspondence on the curve whose united points include the vertices of the in-and-circumscribed triangles; the difficulty is to obtain the actual number of other united points of the correspondence, *i.e.* the heterotypic solutions of the problem. And there is a further difficulty: when it has been seen that a certain point of the curve is a heterotypic solution and figures among the united points, it still remains to be determined exactly how many times the point is to be counted as a united point of the correspondence. These difficulties were surmounted by Zeuthen, who gave the solution of the problem in his text-book on enumerative geometry*.

Cayley commences to consider the solution of the problem by this second method at the end of his paper, but does not accomplish it. He obtains an expression for the number of united points of a correspondence among whose united points the vertices of all the in-and-circumscribed triangles must occur, but says he has not considered how to account for the heterotypic solutions. Had he done so he would not have been successful, because the expression which he gives for the number of united points and, con-

* *Lehrbuch der abzählenden Methoden der Geometrie* (Leipzig, 1914), 249–253.

sequently, the number of heterotypic solutions which he says are to be accounted for, are not correct*.

2. A curve cannot have in-and-circumscribed triangles unless its order and class are both at least equal to four. In the present paper the problem is considered by stages, being first solved for a plane quartic of genus 3, then for any plane quartic with ordinary singularities (*i.e.* double points, cusps, bitangents, and inflections), and finally for any plane curve with ordinary singularities. The solution of the problem for a plane quartic of genus 3 is given in the first section of the paper, this particular problem having already been solved in another way†. In the second section of the paper the problem is solved for any plane quartic with ordinary singularities. The problem for a quartic curve is worked out first because it is comparatively easy to account for the heterotypic solutions. Moreover, the problem for a quartic curve has an interest of its own; the four intersections of a side of an in-and-circumscribed triangle with the curve consist of the two vertices on that side and of the point of contact counted twice, so that the three vertices and the three points of contact make up all the intersections of the curve with the sides of the triangle. The three sides of the triangle form a cubic curve having two intersections with the quartic wherever it meets it, and which is therefore a *contact cubic*. It was as particular forms of contact cubics that the in-and-circumscribed triangles of a plane quartic of genus 3 were previously obtained‡. Similarly, *a polygon of n sides which is in-and-circumscribed to a quartic curve is a particular form of contact curve of order n.*

The solution of the problem for any plane curve with ordinary singularities is given in the third section of the paper. Here use has to be made of Zeuthen's rule§; in order not to interrupt the course of the argument too much, the rule is merely stated and applied; the details which justify the applications are postponed to the fourth section of the paper, which is concerned solely with the application of Zeuthen's rule to the present problem. This question of the heterotypic solutions has been outstanding for so long that it seems desirable to clear the matter up finally; con-

* There are two unfortunate slips at the end of the paper. The number 132, in the last line on p. 256, ought to be 162. This mistake has invalidated the work on p. 257. Furthermore, a term $-4x^2$, printed in the first equation on p. 257, has been overlooked in the succeeding calculations.

† Edge, *loc. cit.*

‡ Edge, *loc. cit.*

§ Zeuthen, *Bulletin des sciences mathématiques*, 5 (1873), 186.

sequently it is necessary to consider in some detail this crucial question of the application of Zeuthen's rule, and the fourth section of the paper is made somewhat long thereby. In Zeuthen's text-book he assumes the result of applying the rule, as is done in the third section of this paper, and leaves the verification to the reader.

If a point p of a curve is a united point of a correspondence S, a certain number ν of the points which correspond to p in S coincide with p. Then p usually counts ν times among the united points of S; Zeuthen's rule should, strictly speaking, always be applied to see that this is so. In this paper we make no mention of Zeuthen's rule except in the cases when it is *not* so, *i.e.* except in the cases when the number of times p is to be counted among the united points of S is *not* equal to the number of corresponding points which coincide with p. The reader can verify the other cases for himself*. It is for this reason that no mention of Zeuthen's rule occurs when we are concerned with the solution of the problem for a quartic curve.

A fifth section of the paper deals with the first stages of the problem of the in-and-circumscribed polygons of a plane quartic of genus 3.

3. We make use of the idea of the dependence of correspondences upon one another. Let $T_1, T_2, \ldots, T_k$ be k correspondences on an algebraic curve, and let Y_i denote the set of points which correspond to a point p of the curve in the correspondence T_i. Then, if there exist integers $\lambda_1, \lambda_2, \ldots, \lambda_k$, not all zero, such that the set of points

$$\lambda_1 Y_1 + \lambda_2 Y_2 + \ldots + \lambda_k Y_k$$

varies in a linear series as p varies on the curve, the correspondences T_i are said to be dependent upon one another. Any of the integers λ may be positive or negative, the set of points $\lambda_1 Y_1 + \lambda_2 Y_2 + \ldots + \lambda_k Y_k$ being real or virtual. When the correspondences T_i are so dependent we shall write

$$\lambda_1 T_1 + \lambda_2 T_2 + \ldots + \lambda_k T_k \sim 0.$$

Clearly relations of this kind may be multiplied by positive or negative integers, and added and subtracted from one another.

If we have a relation $T + \lambda I \sim 0$, where I is the identical correspondence and λ is an integer, then the set of points Y which correspond to p in the correspondence T, taken with p itself counted λ times, forms a (real or

* One verification has been given in § 17.

virtual) set of points which varies in a linear series as p varies on the curve. Hence λ is the valency of T.

The problem for a plane quartic curve of genus 3.

4. Take any point p on a plane quartic without double points or cusps; the quartic is of class 12, so that ten tangents can be drawn to the curve from any point of itself. Suppose a tangent from p to the curve touches it in t and meets it again in q; that a tangent from q, other than qp, touches the curve in u and meets it again in r; and that a tangent from r, other than rq, touches the curve in v and meets it again in s. To the point p there correspond ten points t and ten points q; to each of these points q there correspond nine points u and nine points r; to each of these ninety points r there correspond nine points v and nine points s. We denote by T the correspondence in which to p correspond the ten points t, by Q the correspondence in which to p correspond the ten points q, by U the correspondence in which to p correspond the ninety points u, by R the correspondence in which to p correspond the ninety points r, by V the correspondence in which to p correspond the 810 points v, by S the correspondence in which to p correspond the 810 points s. The three correspondences Q, R, S are symmetrical correspondences and S, with which we are more immediately concerned, is an (810, 810) correspondence. These correspondences are dependent on one another and satisfy the relations

$$\left.\begin{aligned} 2I+T &\sim 0,\\ 10I+2T+Q &\sim 0,\\ T+2Q+U &\sim 0,\\ 9Q+2U+R &\sim 0,\\ U+2R+V &\sim 0,\\ 9R+2V+S &\sim 0, \end{aligned}\right\} \qquad (1)$$

where I denotes the identical correspondence. The first of these relations holds because the first polar of p touches the curve at p and meets it again in the ten points t; the second relation holds because there are ten tangents from p to the curve, the complete set of points in which any one of these tangents meets the curve consisting of p, one of the points q, and one of the points t counted twice. And so on. From these relations

there follow the relations

$$\left.\begin{aligned} T+2I &\sim 0, \\ Q+6I &\sim 0, \\ U-14I &\sim 0, \\ R-26I &\sim 0, \\ V+66I &\sim 0, \\ S+102I &\sim 0. \end{aligned}\right\} \qquad (2)$$

Hence the valency of S is 102. The number of united points of S is therefore*

$$810+810+2\,.\,3\,.\,102 = 2232.$$

The vertices of any in-and-circumscribed triangle are clearly included among the united points of S; in fact, they are included twice over. For let e, f, g be the vertices of such a triangle. If p is at e, one of the corresponding points q is at g and one of the corresponding points r is at f; hence, since fe is a tangent, other than fg, from f to the curve, one of the corresponding points s is at e. But we may also proceed round the triangle in the opposite direction, since, when p is at e, one of the corresponding points q is at f and one of the corresponding points r is at g. Thus, when p is at e, *two* of the 810 points s which correspond to it are also at e. Hence e counts twice among the united points of S as, similarly, do f and g. Each in-and-circumscribed triangle gives rise in this way to *six* united points of S.

But S possesses another set of united points. Consider any one of the twenty-eight bitangents of δ, and suppose that t_1 and t_2 are its points of contact. Through t_1 then pass nine tangents of the curve, apart from t_1t_2; let one of these meet the curve again, other than at its point of contact, in h. If p is at h, one of the corresponding points q is at t_1. To obtain the corresponding points r we take the nine tangents, other than t_1h, which can be drawn from t_1 to the curve; one of these tangents is t_1t_2, and this meets the curve again, apart from its point of contact t_2, in t_1 itself. Hence one of the points r is at t_1. The nine tangents, other than t_1t_2, which pass through t_1 meet the curve again in points s; hence one of the points s is at h, which is therefore a united point of S. Thus the nine tangents, other than t_1t_2, which can be drawn from t_1 to the curve meet the curve again in united points of S. Since there are fifty-six points of contact of bitangents the number of united points of S so arising is

* The number of united points of an (α, β) correspondence of valency γ on a curve of genus p is $\alpha+\beta+2p\gamma$.

504. These points, united points of S which are not vertices of in-and-circumscribed triangles, are heterotypic solutions of the problem. They may be regarded as vertices of in-and-circumscribed triangles two of whose vertices coincide, but are not proper solutions.

There are no united points of S other than these 504 points and the vertices of in-and-circumscribed triangles; hence, since

$$2232-504=1728=6\times 288,$$

the number of in-and-circumscribed triangles of a plane quartic without double points or cusps is 288.

The problem for any plane quartic curve.

5. We suppose the only singularities of the curve are ordinary double points and cusps, and that the curve is of class X. Then $X-2$ tangents can be drawn to the curve from any point p of itself. If p is a cusp, these $X-2$ tangents include the tangent at the cusp itself, there being only $X-3$ lines through the cusp which touch the curve at points other than the cusp; if p is a double point, the $X-2$ tangents consist of the two tangents at the double point and of $X-4$ lines passing through the double point and touching the curve elsewhere. We define the correspondences T, Q, U, R, V, S exactly as in § 4, and they now satisfy the relations

$$\left.\begin{aligned} 2I+T &\sim 0,\\ (X-2)I+2T+Q &\sim 0,\\ T+2Q+U &\sim 0,\\ (X-3)Q+2U+R &\sim 0,\\ U+2R+V &\sim 0,\\ (X-3)R+2V+S &\sim 0, \end{aligned}\right\}$$

analogous to the relations (1). From these follow the relations

$$\left.\begin{aligned} T+2I &\sim 0,\\ Q+(X-6)I &\sim 0,\\ U-(2X-10)I &\sim 0,\\ R-(X^2-13X+38)I &\sim 0,\\ V+(2X^2-24X+66)I &\sim 0,\\ S+(X^3-20X^2+125X-246)I &\sim 0, \end{aligned}\right\}$$

analogous to (2), and actually reducing to (2) when $X = 12$. Hence the valency of S, whose indices are now $(X-2)(X-3)^2$, is

$$X^3-20X^2+125X-246.$$

Twice the genus of the curve is $X-6+\kappa$, where κ is the number of its cusps, so that the number of united points of S is

$$2(X-2)(X-3)^2+(X-6+\kappa)(X^3-20X^2+125X-246)$$

$$= X^4-24X^3+229X^2-954X+1440+\kappa(X^3-20X^2+125X-246).$$

These united points include the vertices of the in-and-circumscribed triangles, each counted twice. They also include, as in the case of the non-singular curve, those points of the curve through which pass tangents of the curve which meet it again in points of contact of its bitangents. But there are now united points of S associated with the cusps of the curve.

6. Let k be a cusp of the curve; the tangent at k meets the curve in one further point l. The $X-2$ tangents which can be drawn from l to the curve consist of lk and $X-3$ other tangents; the $X-2$ tangents which can be drawn from k to the curve consist of kl and $X-3$ other tangents. Suppose p is at l. The line lk is one of the $X-2$ tangents from p to the curve; the remaining intersection of this tangent, whose point of contact is k, with the curve is k itself; hence one position of q is at k. To obtain r we take any tangent, other than kl, from k to the curve; we have thus $X-3$ tangents to choose from, and, whichever we take, its remaining intersection with the curve is at k, which is therefore a position of r. We can then take kl as the tangent rvs, so that s is at l. Hence l is a united point of S. But l counts $X-3$ times among the united points of S for, in passing from q, which is at k, to r, which is also at k, we have a choice of $X-3$ tangents; of the $(X-2)(X-3)^2$ positions of s which correspond to p when it is at l, $X-3$ coincide with l.

Further, k itself is a united point of S. Let the $X-3$ tangents which pass through k and touch the curve in points other than k do so in $t^{(1)}, t^{(2)}, \ldots, t^{(X-3)}$. Suppose p is at k; then, taking any one of these $X-3$ tangents, say $kt^{(1)}$, we obtain a position of q also at k. To pass from q to r we must take a tangent other than $kt^{(1)}$ from k to the curve; any one of the $X-4$ tangents $kt^{(2)}, \ldots, kt^{(X-3)}$, say $kt^{(2)}$, gives a position of r also at k. To pass now from r to s we have to take a tangent, other than $kt^{(2)}$, from k to the curve; there is, of course, no reason why we should not choose $kt^{(1)}$,

although it was used in passing from p to q; any one of the $X-4$ tangents $kt^{(1)}$, $kt^{(3)}$, ..., $kt^{(X-3)}$ gives a position of s also at k. Thus, when p is at k, $(X-3)(X-4)^2$ of the $(X-2)(X-3)^2$ corresponding positions of s are also at k; in other words, k counts $(X-3)(X-4)^2$ times among the united points of S.

7. If N is the number of in-and-circumscribed triangles and τ the number of double tangents, we have the following equation:

$$X^4-24X^3+229X^2-954X+1440+\kappa(X^3-20X^2+125X-246)$$
$$=6N+2\tau(X-3)+\kappa(X-3)+\kappa(X-3)(X-4)^2.$$

Substituting for τ from the relation

$$2\tau=X^2-10X+32-3\kappa,$$

we obtain, after reduction,

$$6N=X^4-25X^3+242X^2-1016X+1536-3\kappa(3X^2-29X+68),$$

which gives the number N of in-and-circumscribed triangles of a plane quartic of class X with κ cusps.

We may write this last equation in the form

$$6N=(X-4)(X^3-21X^2+158X-384)-3\kappa(X-4)(3X-17),$$

so that N is zero when $X=4$. Hence *a plane quartic with one node and two cusps has no in-and-circumscribed triangles**.

When a curve is reciprocated, an in-and-circumscribed triangle of the curve becomes an in-and-circumscribed triangle of the reciprocal curve; the number of in-and-circumscribed triangles of a curve is therefore the same as the number of in-and-circumscribed triangles of the reciprocal curve. Hence, since a cubic curve cannot have in-and-circumscribed triangles, a curve of class 3 cannot have in-and-circumscribed triangles. If we put $X=3$ in the above expression for N, we find

$$6N=72-24\kappa,$$

and this is, in fact, zero, because a quartic curve of class 3 has three cusps.

* *Cf.* Zeuthen, *Lehrbuch*, 205 (Example 3).

The number of in-and-circumscribed triangles of a quartic curve with δ nodes and κ cusps, for different values of δ and κ, is as shown in the following table.

δ	κ	X	N
0	0	12	288
1	0	10	96
0	1	9	30
2	0	8	32
1	1	7	12
0	2	6	6
3	0	6	8
2	1	5	2
1	2	4	0
0	3	3	0

The problem for a plane curve of any order.

8. When the curve is of order n, greater than 4, it is more difficult to account for all the heterotypic solutions, and this not only because there are far more of them but also because recourse has to be had to Zeuthen's rule to find exactly how many times certain points of the curve are to be reckoned among the united points of S. When $n > 4$, a double tangent of the curve meets it in points other than its two points of contact; such points of intersection of the curve with its double tangents are united points of S, and their existence also causes the points of contact of the double tangents to be united points of S. Again, when $n > 4$, the tangent at an inflection of the curve meets the curve in more than one other point, and the points of intersection of inflectional tangents with the curve, as well as the inflections themselves, are now found to be united points of S. Further, when $n > 4$, the tangents which can be drawn to the curve from a node or a cusp meet the curve again, and these further intersections of the tangents with the curve are also united points of S.

For the purpose of applying Zeuthen's rule to the present problem it may be stated as follows*. Suppose p_0 is a united point of a correspondence S on the curve. When p is near to p_0 a certain number of corresponding points, say $s_1, s_2, \ldots, s_\nu$, will also be near to p_0. When pp_0 (*i.e.* the length of the arc pp_0) is the principal infinitesimal, then $s_1 p, s_2 p, \ldots, s_\nu p$ will also be infinitesimals of certain orders; the number of times that p_0

* *Cf.* Enriques, *Teoria geometrica delle equazioni*, 1 (Bologna, 1929), 160.

must be reckoned among the united points of S is the sum of the orders of these ν infinitesimals. If the orders of these infinitesimals are all equal, as they will be in all the applications of the rule which we shall have occasion to make, we may say that the number of times p_0 must be reckoned among the united points of S is equal to ν, the number of corresponding points which are at p_0, multiplied by the order of the infinitesimal sp.

9. Suppose then that we have a curve C of order n and class X. If p is any point of the curve, there are $X-2$ tangents of the curve passing through it, each of which meets the curve in $n-3$ further points; we thus have $(X-2)(n-3)$ points q which correspond to p in the correspondence Q. Two points of C are corresponding points in Q when the line joining them is a tangent of C at some other point. We thus define the correspondences T, Q, U, R, V, S exactly as for quartic curves. They satisfy the relations

$$\left.\begin{aligned} 2I+T &\sim 0,\\ (X-2)I+2T+Q &\sim 0,\\ (n-3)T+2Q+U &\sim 0,\\ (X-3)Q+2U+R &\sim 0,\\ (n-3)U+2R+V &\sim 0,\\ (X-3)R+2V+S &\sim 0, \end{aligned}\right\}$$

from which follow the relations

$$\left.\begin{aligned} T+2I &\sim 0,\\ Q+(X-6)I &\sim 0,\\ U-(2X+2n-18)I &\sim 0,\\ R-(X^2-13X-4n+54)I &\sim 0,\\ V+(2X^2+2Xn+2n^2-32X-32n+162)I &\sim 0,\\ S+(X^3-20X^2-8Xn-4n^2+157X+76n-486)I &\sim 0. \end{aligned}\right\}$$

The valency of S is therefore*

$$X^3-20X^2-8Xn-4n^2+157X+76n-486.$$

Hence, since the indices of S are now $(n-3)^3(X-2)(X-3)^2$, the number

* The valency of this correspondence is given incorrectly by Cayley (*loc. cit.*, 257), owing to a mistake already referred to. But see Zeuthen, *Lehrbuch*, 251.

of united points of S is

$$2(n-3)^3(X-2)(X-3)^2$$
$$+(X-2n+2+\kappa)(X^3-20X^2-8Xn-4n^2+157X+76n-486),$$

where κ is the number of cusps of C, since the genus of C is one-half of $X-2n+2+\kappa$. After some reductions this number is found to be

$$X^4+X^3(2n^3-18n^2+52n-72)+X^2(-16n^3+144n^2-400n+549)$$
$$+X(42n^3-366n^2+880n-1306)-28n^3+164n^2+152n$$
$$+\kappa(X^3-20X^2-8Xn-4n^2+157X+76n-486).$$

These united points include the vertices of the in-and-circumscribed triangles, each counted twice. They also include heterotypic solutions, which we must now account for.

10. Let us consider first the heterotypic solutions associated with the bitangents of C. Let a bitangent touch C at t_1 and t_2 and meet it again in $j^{(1)}, j^{(2)}, \ldots, j^{(n-4)}$. Any tangent of C, other than t_1t_2, which passes through t_1 meets the curve further, other than at its point of contact, in $n-3$ points; these points, as in the case of a quartic curve when there is only one of them, are all united points of S. Since there are $X-3$ tangents, other than t_1t_2, from t_1 to the curve there are, associated with the point of contact t_1, $(n-3)(X-3)$ united points of S. Hence, if τ is the number of bitangents of C, there are $2\tau(n-3)(X-3)$ heterotypic solutions arising in this way.

The $X-2$ tangents from any one of the points j to C consist of the line jt_1t_2, counted twice, and of $X-4$ other tangents; the first polar of j passes through both t_1 and t_2. Suppose now that p is at a point of contact of a bitangent, say at t_1. Then, since pt_2 is a tangent from p to the curve meeting it again in the points $j^{(1)}, j^{(2)}, \ldots, j^{(n-4)}$, any one of the points j is a point q corresponding to p; take q to be at $j^{(1)}$. Then to pass from q to r we must take a tangent, other than $j^{(1)}t_2$, from among the $X-2$ tangents from $j^{(1)}$ to the curve; we may therefore take the tangent $j^{(1)}t_1$, and the points $j^{(2)}, j^{(3)}, \ldots, j^{(n-4)}$ are positions of r (and so is t_2, but this is irrelevant to our problem). To pass now from r to s we have to take a tangent, other than $j^{(2)}t_1$, from among the $X-2$ tangents from $j^{(2)}$ to C; we may therefore take $j^{(2)}t_2$ whose point of contact is t_2. Since this touches the curve at t_1 there are two positions of s at t_1, and therefore coinciding with p. Hence, since there was a choice of $n-4$ positions j for q and of $n-5$ positions for r,

there are, of the $(n-3)^3(X-2)(X-3)^2$ points which correspond to t_1 in the correspondence S, $2(n-4)(n-5)$ which coincide with t_1. Thus the point of contact of a bitangent counts $2(n-4)(n-5)$ times among the united points of S, so that the number of heterotypic solutions which arise in this way is $4\tau(n-4)(n-5)$.

Suppose now that p is at one of the $n-4$ points j, say at $j^{(1)}$; then to pass from p to q we take any one of the $X-2$ tangents from p to the curve; we may choose pt_1t_2 either in virtue of its point of contact t_1 or in virtue of its point of contact t_2; let us choose it in virtue of its point of contact t_1. Then all the points $j^{(2)}, \ldots, j^{(n-4)}$ are positions of q; take q to be at $j^{(2)}$. To pass from q to r we take any one of the $X-2$ tangents, other than $j^{(2)}t_1$, from $j^{(2)}$ to C; take then the tangent $j^{(2)}t_2$. We have to choose r among the remaining intersections of this line with the curve, other than $j^{(2)}$. No purpose would be served by taking r to be at t_1, nor by taking it to be at $j^{(1)}$, but we may take it to be any one of the points $j^{(3)}, j^{(4)}, \ldots, j^{(n-4)}$. Thus there is a choice here of $n-6$ positions for r; when r is taken at any one of these points, say $j^{(3)}$, then to pass from r to s we take a tangent, other than $j^{(3)}t_2$, from $j^{(3)}$ to the curve; taking then $j^{(3)}t_1$ we see that $j^{(1)}$ is a possible position of s, which therefore coincides with p. Hence any intersection of a bitangent with C, other than its two points of contact, is a united point of S; moreover, it is to be reckoned $2(n-5)(n-6)$ times among the united points of S. For, when p is at $j^{(1)}$, we have seen that there is, in order that s also should be at $j^{(1)}$, a choice of two points of contact among the $X-2$ points of contact of tangents from $j^{(1)}$, followed by a choice of $n-5$ positions of q and a choice of $n-6$ positions of r; thus, of the $(n-3)^3(X-2)(X-3)^2$ points which correspond to $j^{(1)}$ in the correspondence S, $2(n-5)(n-6)$ are at $j^{(1)}$. Since the number of points in which bitangents meet C is altogether $(n-4)\tau$, the number of heterotypic solutions which arise in this way is $2\tau(n-4)(n-5)(n-6)$.

The number of heterotypic solutions which are associated with the bitangents of C is therefore

$$2\tau(n-3)(X-3)+4\tau(n-4)(n-5)+2\tau(n-4)(n-5)(n-6)$$
$$= 2\tau\{(n-3)(X-3)+(n-4)^2(n-5)\}.$$

11. There are $X-4$ tangents to C from a node, and each of these tangents meets C further in $n-4$ points; these $(n-4)(X-4)$ points are united points of S. For denote by $w^{(1)}, w^{(2)}, \ldots, w^{(X-4)}$ the $X-4$ tangents to C from one of its nodes O, and let p be an intersection of one of these tangents, say $w^{(1)}$, with the curve. To pass from p to q we take any one

of the $X-2$ tangents from p to the curve; let us then take $w^{(1)}$. This passes through the node, so that q may be on *either* branch of the curve at the node. To pass now from q to r we take any tangent, other than $w^{(1)}$, from O to the curve; taking any one of the $X-5$ tangents $w^{(2)}, \ldots, w^{(X-4)}$ we have a position of r at the node on the branch other than that on which q is situated. To pass then from r to s we have to take a tangent from r to the curve other than that one of the $X-5$ tangents $w^{(2)}, \ldots, w^{(X-4)}$ which we chose to obtain r; we may, therefore, choose $w^{(1)}$ and so obtain a position of s at p. Of the points which correspond to p in the correspondence S there are $2(X-5)$ which coincide with p, since there is a choice of two positions for q and then a choice of $X-5$ ways of passing from q to r; the point p must be reckoned $2(X-5)$ times among the united points of S.

Associated with each double point of C there are therefore $2(n-4)(X-4)(X-5)$ united points of S, so that, if C has δ nodes, there are $2\delta(n-4)(X-4)(X-5)$ heterotypic solutions arising in this way.

12. We proceed now to consider the heterotypic solutions associated with the cusps of C.

The tangent to C at a cusp meets it further in $n-3$ points, and each of these points counts $X-3$ times among the united points of S exactly as for the quartic curve; in this way there are $(n-3)(X-3)$ united points of S associated with each cusp of C. Further, the cusps themselves are united points of S, and each of them, as in the case of the quartic curve, counts $(X-3)(X-4)^2$ times. But when the order n of C is greater than 4, each of the $X-3$ tangents to C from one of its cusps meets C further in $n-4$ points, and these points also figure among the united points of S. For let p be an intersection of C with one of its tangents from the cusp k, say with the tangent $kt^{(1)}$. Taking $pt^{(1)}$ as a tangent from p to the curve we obtain *two* positions of q at k; to pass from q to r we have to choose any one of the $X-3$ tangents from k, other than $kt^{(1)}$, to the curve. It is of no use to choose the tangent at k; but, if we choose any of the $X-4$ tangents $kt^{(2)}, \ldots, kt^{(X-3)}$, say $kt^{(2)}$, we obtain a position of r at k. Then, in order to pass from r to s, we have to choose a tangent from k other than $kt^{(2)}$; we may, therefore, choose $kt^{(1)}$ and so obtain a position of s at p. Since two of the $(n-3)(X-2)$ positions of q which correspond to p are at k, and since there is a choice of $X-4$ ways of passing from q to r, the point p coincides with $2(X-4)$ of the points s which correspond to it. But it does not count $2(X-4)$ times among the united points of S; for we shall find,

when Zeuthen's rule is applied, that, in order to obtain the number of times which p counts among the united points of S, the number $2(X-4)$ must be multiplied by $\frac{3}{2}$; hence p counts $3(X-4)$ times among the united points of S. In this way there are $3(n-4)(X-3)(X-4)$ united points of S associated with each cusp of C.

The number of heterotypic solutions which are associated with the cusps of C is therefore

$$\kappa(n-3)(X-3)+\kappa(X-3)(X-4)^2+3\kappa(n-4)(X-3)(X-4)$$
$$=\kappa(X-3)\{n-3+(X-4)^2+3(n-4)(X-4)\}.$$

13. We have now to consider, lastly, heterotypic solutions associated with the inflections of C.

If i is a point of inflection of C the $X-2$ tangents which can be drawn to C from i include the inflectional tangent itself; the first polar of an inflection not only touches the curve there but has three-point contact with it; apart from this inflectional tangent there are $X-3$ other tangents from i to C. If m is a point of intersection of C with the tangent at one of its inflections, then the $X-2$ tangents which can be drawn from m to C include the inflectional tangent twice over; the first polar of m touches C at the inflection; the $X-2$ tangents from m consist of the inflectional tangent, counted twice, and of $X-4$ other tangents. Of the $X-2$ tangents which can be drawn to C from a point *near* m, an intersection of C with a tangent at an inflection i, *two* have their points of contact near i; this observation is of importance when we come to apply Zeuthen's rule.

Suppose then that the tangent at an inflection i of C meets C further in the $n-3$ points $m^{(1)}, m^{(2)}, \ldots, m^{(n-3)}$. Suppose p is at i; then, since the $X-2$ tangents from p to C include the tangent at i, we may take q to be any one of the points m, say the point $m^{(1)}$. To pass from q to r we have to take any one of the $X-2$ tangents from $m^{(1)}$ to the curve, other than $m^{(1)}i$; but, since $m^{(1)}i$ counts for two among these $X-2$ tangents, we may use it to obtain r and so choose r to be any one of the $n-4$ points $m^{(2)}, \ldots, m^{(n-3)}$, say $m^{(2)}$. Similarly, since $m^{(2)}i$ counts for two among the $X-2$ tangents from $m^{(2)}$ to the curve, we may use it to obtain positions of s and so choose s to be at i. Hence s coincides with p, and i is a united point of S. Moreover, since there was a choice of $n-3$ points for q and of $n-4$ points for r, when p is at i $(n-3)(n-4)$ of the corresponding points s coincide with it. But, when Zeuthen's rule is applied, it is found that, in order to obtain the number of times i must be counted among the united points of S, this

number must be multiplied by 2; hence each inflection of C is to be reckoned $2(n-3)(n-4)$ times among the united points of S.

Now suppose that p is at any of the points m, say at $m^{(1)}$. To obtain a position of q we choose any one of the $X-2$ tangents which can be drawn from $m^{(1)}$ to the curve; hence, choosing $m^{(1)}i$, we can take q to be at any of the other points m (no purpose would be served by taking it to be at i), say at $m^{(2)}$; it must be remembered that there is really a choice of two among the $X-2$ tangents from $m^{(1)}$ when $m^{(1)}i$ is selected, since there are only $X-4$ others. Then, in order to pass from q to r, we select one of the $X-2$ tangents, other than $m^{(2)}i$, which can be drawn from $m^{(2)}$ to the curve; but, since two of these $X-2$ tangents coincide with $m^{(2)}i$, we may select $m^{(2)}i$ again and so take r to be at any of the points $m^{(3)}, m^{(4)}, \ldots, m^{(n-3)}$. No useful purpose would be served either by taking r to be at i or by taking it to be at $m^{(1)}$. Then, supposing r to be at $m^{(3)}$, for example, we may again take the tangent at i to obtain s and so find a position for s at $m^{(1)}$. Thus $m^{(1)}$ is a united point of the correspondence S. Moreover, since there was a choice of two out of $X-2$ ways of passing from p to q, followed by a choice of $n-4$ positions for q and then by a choice of $n-5$ positions for r, the number of the $(n-3)^3(X-2)(X-3)^2$ corresponding points which coincide with $m^{(1)}$ is $2(n-4)(n-5)$. But $m^{(1)}$ does not count $2(n-4)(n-5)$ times among the united points of S; Zeuthen's rule requires that this number must be multiplied by $\frac{3}{2}$, so that $m^{(1)}$ counts $3(n-4)(n-5)$ times among the united points. Each intersection of the curve with a tangent at an inflection therefore counts $3(n-4)(n-5)$ times among the united points.

If C has ι inflections, the number of heterotypic solutions associated with them is

$$2\iota(n-3)(n-4)+3\iota(n-3)(n-4)(n-5)=\iota(n-3)(n-4)(3n-13).$$

14. The number of united points of S which are not vertices of in-and-circumscribed triangles can now be written down; it is

$$2\tau\{(n-3)(X-3)+(n-4)^2(n-5)\}+2\delta(n-4)(X-4)(X-5)$$
$$+\kappa(X-3)\{n-3+(X-4)^2+3(n-4)(X-4)\}+\iota(n-3)(n-4)(3n-13).$$

Putting in here the values of τ, δ, ι given by Plücker's equations:

$$2\tau=X^2-10X+8n-3\kappa,$$

$$2\delta=n^2-n-X-3\kappa,$$

$$\iota=3X-3n+\kappa,$$

this number, after some reductions, is found to be

$$X^3+X^2(2n^3-18n^2+56n-77)+X(-10n^3+81n^2-229n+322)$$
$$-n(n^3-18n^2+57n+20)+\kappa(X^3-11X^2+[4n+22]X+5n^2-59n+114).$$

When this number is subtracted from the number of united points of S obtained in § 9, the difference must be six times the number of in-and-circumscribed triangles of the curve. Hence, if N is the number of in-and-circumscribed triangles of a curve of order n and class X with κ cusps,

$$6N = X^4+X^3(2n^3-18n^2+52n-73)+X^2(-18n^3+162n^2-456n+626)$$
$$+X(52n^3-447n^2+1109n-1628)+n^4-46n^3+221n^2+172n$$
$$+\kappa\{-9X^2-X(12n-135)-9n^2+135n-600\}.$$

This reduces to the expression found for $6N$ in § 7 when $n = 4$.

It is not immediately obvious, from the form which we have obtained for $6N$, that the number of in-and-circumscribed triangles is the same for a curve as for its reciprocal curve. But when we put

$$\kappa = \xi-3X,$$

we obtain

$$6N = X^4+X^3(2n^3-18n^2+52n-46)+X^2(-18n^3+162n^2-420n+221)$$
$$+X(52n^3-420n^2+704n+172)+n^4-46n^3+221n^2+172n$$
$$+\xi\{-9X^2-X(12n-135)-9n^2+135n-600\},$$

a form which is symmetrical in n and X and is the form given by Cayley*. It is now immediately obvious that the number of in-and-circumscribed triangles is the same for a curve as for its reciprocal, for the number of triangles for the reciprocal curve is got merely by interchanging n and X, since $\xi = \kappa+3X = \iota+3n$ and so has the same value for the curve as for its reciprocal. Another symmetrical form of the result is obtained by putting $\kappa = \eta+3n$; then

$$6N = X^4+X^3(2n^3-18n^2+52n-73)+X^2(-18n^3+162n^2-483n+626)$$
$$+X(52n^3-483n^2+1514n-1628)+n^4-73n^3+626n^2-1628n$$
$$+\eta\{-9X^2-X(12n-135)-9n^2+135n-600\}.$$

* *Loc. cit.*, 221 and 251.

When the class of a curve is equal to the order we find, putting $X = n$, that

$$6N = 2n(n-2)(n-4)(n^3-12n^2+54n-91)-30\kappa(n-4)(n-5).$$

The application of Zeuthen's rule.

15. Suppose, for example, we wish to obtain the number of times which a point of intersection of C with a tangent from one of its cusps is to be counted among the united points of S. We have to solve the following problem*:

Two tangents from a cusp K touch a curve in T_1 and T_2 and meet it again in P_1 and P_2 respectively. A point p is taken on the curve near P_1; through p there passes a tangent touching the curve near T_1. This tangent meets the curve in two points near K; let q be one of these. Through q there passes a tangent touching the curve near T_2; this tangent meets the curve in a second point r near K, and through r there passes a tangent touching the curve near T_1. This last tangent meets the curve again in a point s near P_1. If pP_1 is the principal infinitesimal, what is the order of the infinitesimal sp?

The tangent at a point of a curve of order n meets the curve in $n-2$ further points; the two tangents of the curve at two points which are very near each other each meet the curve in $n-2$ further points, and these two sets of $n-2$ points form $n-2$ pairs of points, each pair of points consisting of one point of each set, the two being very near each other. The distance between any pair is an infinitesimal of the same order as the distance between the two points of contact, supposing always that the curvature of the curve is neither very large nor infinitesimal at any of the points concerned. Hence the problem which has just been put may be replaced by the following:

Two tangents from a cusp K touch a curve at T_1 and T_2. A point t is taken near T_1 on the curve; the tangent at t meets the curve in two points near K. Let q be either of these points. Through q there passes a tangent touching the curve near T_2; this meets the curve in a second point r near K. Through r there passes a tangent whose point of contact t' is near T_1. If tT_1 is the principal infinitesimal, what is the order of the infinitesimal tt'?

* See the enunciation of Zeuthen's rule in § 8.

These two problems have the same answer.

To find the answer to either of these problems it is sufficient to solve it for some particular curve; the problem is, of its nature, a question concerning a property of a pair of tangents drawn to a curve from a cusp. It is, when solved for one curve, solved for all, although it would, of course, have to be answered differently if the curve were made to have certain peculiarities. Moreover, we can solve the second problem when the curve is of order 4, whereas the first problem cannot arise for quartic curves. But when the second problem is solved for a quartic curve it is solved for any curve, and therefore the first problem is solved also. In either problem the curve must be of class at least 5.

16. Let us then take a plane quartic with one cusp and two nodes; it is of class 5, and there are two tangents which can be drawn to it from its cusp. We will use ordinary rectangular Cartesian coordinates and consider a bicircular quartic with a cusp at the origin. Take then the curve

$$(x^2+y^2)^2-(x^2+y^2)my+a^2x^2=0,$$

for which the cuspidal tangent is $x=0$. The coordinates of a point of the curve can be expressed as rational functions of a parameter λ; for example, in the form

$$x=\frac{am^2\lambda^3}{m^2\lambda^2+a^2(\lambda^2+1)^2}, \qquad y=\frac{a^2m\lambda^2(\lambda^2+1)}{m^2\lambda^2+a^2(\lambda^2+1)^2}.$$

The parameter of the cusp is $\lambda=0$. The two tangents from the cusp are $my+2ax=0$, which touches the curve at the point T_1 whose parameter is -1, and $my-2ax=0$, which touches the curve at the point T_2 whose parameter is $+1$.

It will be found that, if the tangent at the point λ meets the curve again in the point μ, then

$$\lambda\mu^2\{m^2\lambda^2+a^2(\lambda^2+3)\}+a^2(\lambda^2-1)(2\mu+\lambda)=0. \qquad \text{(A)}$$

This relation gives the two parameters μ_1 and μ_2 of the points in which the tangent at λ meets the curve again, or, alternatively, it gives the three parameters λ_1, λ_2, and λ_3 of the points of contact of the tangents which can be drawn to the curve from μ; it is quadratic in μ and cubic in λ. When $\mu=0$ the relation is simply $a^2\lambda(\lambda^2-1)=0$, so that the three tangents which can be drawn to the curve from the cusp are the tangent at the cusp itself and the tangents at the two points $\lambda=+1$ and $\lambda=-1$.

When μ is small, the cubic equation in λ has one root nearly equal to $+1$ and another nearly equal to -1, the amounts by which these two roots differ from $+1$ and -1 respectively being infinitesimals* of the same order as μ^2. The discriminant of (A), regarded as a quadratic in μ, is

$$-a^2(\lambda^2-1)\{m^2\lambda^4+a^2(\lambda^2+1)^2\}.$$

If λ is slightly less than $+1$, the two values of μ are real and have opposite signs; if λ is slightly greater than $+1$, the two values of μ are conjugate imaginaries. If λ is slightly greater than -1, the two values of μ are real and have opposite signs; if λ is slightly less than -1, the two values of μ are conjugate imaginaries.

We obtain the following relations by considering the equation (A), where $k^2=m^2+4a^2$, and only the infinitesimal of lowest order is written down on the right-hand side.

If μ is infinitesimal, then (A) is satisfied by $\lambda=1+\gamma$ and by $\lambda=-1+\delta$, where

$$\gamma=-\mu^2k^2/2a^2, \qquad \delta=\mu^2k^2/2a^2.$$

If $\lambda=-1+\delta$, where δ is infinitesimal, then (A) is satisfied by

$$\mu_1=(2a^2\delta/k^2)^{\frac{1}{2}}$$

and by $$\mu_2=-(2a^2\delta/k^2)^{\frac{1}{2}},$$

with $$\mu_1+\mu_2=-4a^2\delta/k^2.$$

If $\lambda=1+\gamma$, where γ is infinitesimal, then (A) is satisfied by

$$\mu_1=(-2a^2\gamma/k^2)^{\frac{1}{2}}$$

and by $$\mu_2=-(-2a^2\gamma/k^2)^{\frac{1}{2}},$$

with $$\mu_1+\mu_2=-4a^2\gamma/k^2.$$

We can now give the solution of the second problem of § 15 immediately. Let t be the point $-1+\delta$ near T_1, δ being infinitesimal. The tangent at t meets the curve in two points near the cusp; let q be one of these with parameter μ_1. Through q there passes a tangent touching the curve near T_2 at a point whose parameter is $1+\gamma$, and it meets the curve in a second point near the cusp whose parameter is μ_2. Through this point there passes

* When a complex number is spoken of as being an infinitesimal of a certain order, it is to be understood that its modulus is an infinitesimal of that order.

a tangent touching the curve at t' near T_1, the parameter of t' being $-1+\sigma$, say. Then we have, writing down only the infinitesimals of lowest orders on the right,

$$\mu_1 = (2a^2\delta/k^2)^{\frac{1}{2}},$$

$$\gamma = -\tfrac{1}{2}\mu_1^2 k^2/a^2 = -\delta,$$

$$\mu_2 = -4a^2\gamma/k^2 - \mu_1 = -(2a^2\delta/k^2)^{\frac{1}{2}} + 4a^2\delta/k^2,$$

$$\sigma = \tfrac{1}{2}\mu_2^2 k^2/a^2 = \delta - 4(2a^2\delta^3/k^2)^{\frac{1}{2}}.$$

Now the arc tT_1, as is easily shown, is an infinitesimal of the same order as δ, while the arc tt' is of the same order as $\delta-\sigma$. Hence, if tT_1 is the principal infinitesimal, tt' is an infinitesimal of order $\frac{3}{2}$. Both the problems propounded in § 15 have therefore $\frac{3}{2}$ as their answer. This establishes the application of Zeuthen's rule made in § 12.

17. Suppose now that we wish to verify how many times a cusp itself is to be counted among the united points of S. To apply Zeuthen's rule we have to solve the following problem:

From a cusp K on a curve are drawn two tangents touching the curve in T_1 and T_2 respectively. A point p is taken on the curve near K; through p there passes a tangent touching the curve near T_1 and having a second intersection q with the curve near K; through q there passes a tangent touching the curve near T_2 and having a second intersection r with the curve near K; through r there passes a tangent touching the curve near T_1 and having a second intersection s with the curve near K. If the arc pK of the curve is a principal infinitesimal, of what order is the infinitesimal arc sp?

We use the bicircular quartic for this problem also, supposing that $\mu_1, \mu_2, \mu_3, \mu_4$ are the infinitesimal parameters of p, q, r, s. Further, let pq touch the curve at the point $-1+\delta$, qr at the point $1+\gamma$, and rs at the point $-1+\delta'$. Then, writing down only the infinitesimals of lowest order on the right,

$$\delta = \tfrac{1}{2}\mu_1^2 k^2/a^2, \qquad \mu_2 = -\mu_1 - 2\mu_1^2,$$

$$\gamma = -\tfrac{1}{2}\mu_1^2 k^2/a^2, \qquad \mu_3 = \mu_1 + 4\mu_1^2,$$

$$\delta' = \tfrac{1}{2}\mu_1^2 k^2/a^2, \qquad \mu_4 = -\mu_1 - 6\mu_1^2.$$

The arc Kp is an infinitesimal of order μ_1^2; s is on the opposite side of the

cusp to p since μ_4 has the opposite sign to μ_1, and Ks is an infinitesimal of order $\mu_4{}^2$. Hence the arc ps $(=pK+Ks)$ is also an infinitesimal of order $\mu_1{}^2$, and is therefore of the same order as the infinitesimal arc Kp. The answer to the problem is therefore 1.

Hence the number of times that a cusp of C is to be counted among the united points of S is equal to the number of points which correspond to the cusp and which also coincide with it.

18. We now consider the applications of Zeuthen's rule which give the number of heterotypic solutions associated with the inflections of C.

In order to find how many times an inflection is to be reckoned among the united points of S we have to solve the following problem:

The tangent at a point of inflection P on a curve meets it again in two points Q and R. A point p is taken on the curve near P; through p there passes a tangent whose point of contact t_1 is near P and which meets the curve again in a point q near Q. Through q there passes a second tangent whose point of contact, say t, is also near P and which meets the curve again in a point r near R. Through r there passes a second tangent whose point of contact, say t_2, is near P and which meets the curve again in a point s near P. If Pp is the principal infinitesimal, what is the order of the infinitesimal sp?

Again it will be sufficient if this problem is solved for some particular curve; the curve must be of order at least 5 and of class at least 4. It can be shown that the coordinates of a point on the polar reciprocal of

$$(x^2+y^2)^2-(x^2+y^2)my+a^2x^2=0$$

with respect to the circle $x^2+y^2=a^2$ can be expressed parametrically in the form

$$x=\frac{2a\{m^2\lambda^4+a^2(\lambda^2+1)^2\}}{m^2\lambda^3(\lambda^2-1)}, \qquad y=\frac{a^2(\lambda^2+1)(\lambda^2-3)-m^2\lambda^2}{m\lambda^2(\lambda^2-1)}.$$

This is a rational quintic curve with a point of inflection at $\lambda=0$. But, since this point is at infinity, we take instead the curve

$$x=\frac{am^2\lambda^3(\lambda^2-1)}{2\{m^2\lambda^4+a^2(\lambda^2+1)^2\}}, \qquad y=\frac{m\lambda\{a^2(\lambda^2+1)(\lambda^2-3)-m^2\lambda^2\}}{2\{m^2\lambda^4+a^2(\lambda^2+1)^2\}}.$$

This curve has a point of inflection at the origin, for which $\lambda=0$, the inflectional tangent being $x=0$. This tangent meets the curve again

in the point $(0, -m/2a)$, for which $\lambda = 1$, and in the point $(0, m/2a)$, for which $\lambda = -1$.

It is found that the point whose parameter is μ lies on the tangent at the point whose parameter is λ if

$$\lambda^2\mu\{m^2\mu^2+a^2(\mu^2+3)\}+a^2(\mu^2-1)(2\lambda+\mu) = 0. \qquad \text{(B)}$$

This relation (B) is analogous to the relation (A); it gives the two points λ which are points of contact with the two tangents which can be drawn from μ to the curve, or, alternatively, it gives the parameters of the three points μ in which the curve is met by its tangent at λ. If $\lambda = 0$, the relation is simply $a^2\mu(\mu^2-1) = 0$; the three intersections of the curve with the tangent at its inflection consist of the inflection itself and of the two points whose parameters are ± 1. The equation is quadratic in λ and cubic in μ; regarded as a quadratic in λ its discriminant is

$$-a^2(\mu^2-1)\{m^2\mu^4+a^2(\mu^2+1)^2\}.$$

Notice also that, when $\mu = 1$, both roots of the resulting quadratic in λ are $\lambda = 0$; and similarly when $\mu = -1$; the two tangents which can be drawn to the curve from a point where it is met by the tangent at its inflection both coincide with the tangent at the inflection.

When λ is small, the equation (B) has one root μ nearly equal to $+1$ and a second root nearly equal to -1; the amounts by which these roots differ from $+1$ and -1 respectively are infinitesimals of the same order as λ^2. If μ is nearly equal to $+1$, both roots of the quadratic in λ are small; if μ is slightly greater than $+1$, the two roots are conjugate imaginaries; if μ is slightly less than $+1$, they are real and of opposite signs. Also: if μ is nearly equal to -1, both roots of the quadratic in λ are small; if μ is slightly less than -1, the two roots are conjugate imaginaries; if μ is slightly greater than -1, they are real and of opposite signs.

If λ is infinitesimal, then (B) is satisfied by $\mu = 1+\gamma$ and by $\mu = -1+\delta$, where

$$\gamma = -\tfrac{1}{2}\lambda^2 k^2/a^2, \qquad \delta = \tfrac{1}{2}\lambda^2 k^2/a^2,$$

and, as before, $k^2 = m^2+4a^2$.

If $\mu = 1+\gamma$, when γ is infinitesimal, then (B) is satisfied by

$$\lambda_1 = (-2a^2\gamma/k^2)^{\frac{1}{2}}$$

and by

$$\lambda_2 = -(-2a^2\gamma/k^2)^{\frac{1}{2}},$$

with

$$\lambda_1+\lambda_2 = -4a^2\gamma/k^2.$$

If $\mu = -1+\delta$, when δ is infinitesimal, then (B) is satisfied by

$$\lambda_1 = (2a^2\delta/k^2)^{\frac{1}{2}}$$

and by

$$\lambda_2 = -(2a^2\delta/k^2)^{\frac{1}{2}},$$

with

$$\lambda_1+\lambda_2 = -4a^2\delta/k^2.$$

In these relations only the infinitesimals of lowest order have been written down on the right-hand side.

Notice, further, that, when λ is infinitesimal, the value of μ which satisfies (B), and is not nearly equal either to $+1$ or to -1, is also infinitesimal and approximately equal to -2λ (a closer approximation would give $\mu = -2\lambda-6\lambda^3/a^2$); this corresponds to the fact that the tangent of a curve at a point near an inflection has an intersection with the curve also near the inflection.

We can now solve the last problem immediately; we take the parameters of the inflection P and of the points Q and R to be 0, $+1$, and -1 respectively, the parameter of p being an infinitesimal λ. Then, using t, q, t_1, t_2, r, s to denote also the parameters of points, we have, in succession,

$$p = \lambda,$$

$$t_1 = -\tfrac{1}{2}\lambda,$$

$$q = 1-\tfrac{1}{8}\lambda^2 k^2/a^2,$$

$$t = -t_1-4a^2(q-1)/k^2 = \tfrac{1}{2}\lambda+\tfrac{1}{2}\lambda^2,$$

$$r = -1+\tfrac{1}{8}\lambda^2 k^2/a^2,$$

$$t_2 = -t-4a^2(r+1)/k^2 = -\tfrac{1}{2}\lambda-\lambda^2,$$

$$s = -2t_2 = \lambda+2\lambda^2.$$

Now the arc pP is an infinitesimal of order λ while the arc sp is an infinitesimal of order $\lambda+2\lambda^2-\lambda$, and so of order λ^2. Hence, when pP is the principal infinitesimal sp, is an infinitesimal of order 2. This establishes the first application of Zeuthen's rule made in § 13.

19. To establish the second application of Zeuthen's rule made in § 13 we have to solve the following problem*:

The tangent of a curve at an inflection I meets the curve further in three points P, Q, R. A point p is taken on the curve near p; one of the

* The reader can easily supply figures for this and the preceding problem if he wishes.

two tangents from p whose points of contact are near I meets the curve again near Q in a point q; the second tangent from q whose point of contact is near I meets the curve near R in a point r; the second tangent from r whose point of contact is near I meets the curve near P in a point s. If pP is the principal infinitesimal, what is the order of the infinitesimal sp?

The answer to this problem is $\frac{3}{2}$, as was assumed in § 13. It will be sufficient, after so much space has been taken up already in detailed work, merely to indicate how the result is arrived at. Let t_1, t_2, t_3 be the points of contact of the tangents pq, qr, rs respectively; we suppose It_1 to be infinitesimal of the first order; then It_2 and It_3 are infinitesimals of the first order, while t_1t_3 and pP are infinitesimals of the second order. The arc sp is not an infinitesimal of the same order as is the arc t_1t_3, because the curvature at t_1 and t_3 is an infinitesimal of the first order; sp is an infinitesimal of order 3. Hence, if pP is taken as the principal infinitesimal, the order of the infinitesimal sp is $\frac{3}{2}$.

The problem of the in-and-circumscribed polygon for a plane quartic of genus 3.

20. The problem of the in-and-circumscribed polygon can be solved also by correspondence methods; we give here the solution for polygons of four, five, six, etc., sides which are in-and-circumscribed to a plane quartic of genus 3, obtaining the number of quadrilaterals, pentagons, hexagons, etc., in this order step by step. Now an in-and-circumscribed polygon of n sides of a plane quartic of genus 3 is a contact curve of order n. If n is even, the contact curves of order n form sixty-three different systems; each system contains $\infty^{\frac{1}{2}n(n-1)}$ curves and it is therefore to be expected that it contains a finite number of curves with $\frac{1}{2}n(n-1)$ double points, *i.e.* consisting of n lines. The sixty-three systems have like properties, and each of them will contain the same number of in-and-circumscribed polygons of the curve. Hence, if N_n is the number of in-and-circumscribed polygons of n sides, N_n *is a multiple of* 63 *when* n *is even*. We cannot make a statement of this kind when n is odd; there are sixty-four different systems of contact curves of odd order n, each containing $\infty^{\frac{1}{2}n(n-1)}$ curves, but the systems are of two kinds according as the $2n$ points of contact of a curve of the system do not or do lie on a curve of order $\frac{1}{2}(n+1)$; of the sixty-four systems thirty-six are of the first kind, in which the $2n$ points of contact do not lie on a curve of order $\frac{1}{2}(n+1)$, and twenty-eight are of the second kind, in which the $2n$ points of contact do lie on a curve of order $\frac{1}{2}(n+1)$; in the latter case the two remaining intersections of the curve of order $\frac{1}{2}(n+1)$ with the quartic are the two points of contact of one of the bitangents, and the twenty-eight systems of the second kind are associated with the twenty-

eight bitangents in this way*. Each system of the first kind will contain the same number of in-and-circumscribed polygons and each system of the second kind will contain the same number of in-and-circumscribed polygons; but the number of polygons belonging to a system of one kind will not be the same as the number of polygons belonging to a system of the other kind. In the case $n = 3$, however, all the in-and-circumscribed triangles must be contact cubics of the first kind, since a line cannot meet a conic in three points; thus the 288 in-and-circumscribed triangles occur in sets of eight, each set of eight triangles being contact cubics belonging to the same one of the thirty-six systems of the first kind†.

One remark may be made concerning in-and-circumscribed pentagons. Suppose that we have an in-and-circumscribed pentagon which belongs to a system of contact quintics of the second kind; there is a cubic curve $C_3 = 0$ which passes through its five vertices and through the five points of contact of its sides, meeting the quartic again in two points t_1, t_2 which are points of contact of the quartic with one of its bitangents. $t_1 t_2$ meets the cubic in a third point O. Then it is easily seen, from the theory of residual and coresidual sets of points on a plane cubic, that the conic through the five vertices of the pentagon passes through O, and, further, that the conic through the five points of contact of the sides of the pentagon with the quartic curve passes through the point where the cubic is met by its tangent at O. If these two conics are $C_2 = 0$ and $D_2 = 0$ respectively, and if $C_4 = 0$ is the quartic curve, $t = 0$ the bitangent, and $L = 0$ the tangent of $C_3 = 0$ at O, then the two quintic curves

$$tC_2 D_2 = 0 \quad \text{and} \quad LC_4 = 0$$

meet $C_3 = 0$ in the same set of fifteen points. We may, therefore, write

$$tC_2 D_2 + LC_4 \equiv C_3 E_2,$$

where $E_2 = 0$ is the equation of a conic. Thus the three intersections, other than the vertices of the pentagon, of $C_2 = 0$ with the quartic and the three intersections, other than the points of contact of the sides of the pentagon, of $D_2 = 0$ with the quartic all lie on $E_2 = 0$; hence, since this last conic passes through the points of contact of $C_4 = 0$ with $t = 0$, the six points are the points of contact of the quartic with a contact cubic of the second kind.

* Clebsch, "Ueber die Anwendung der Abelschen Functionen in der Geometrie", *J. für Math.*, 63 (1864), 189–243 (212).

† *Cf.* Edge, *loc. cit.*

21. We change the notation which was used in § 4, denoting now the correspondences T, U, V by T_1, T_2, T_3, and the correspondences Q, R, S by S_1, S_2, S_3 respectively. The sets of relations (1) and (2) now become

$$\left.\begin{aligned} 2I+T_1 &\sim 0,\\ 10I+2T_1+S_1 &\sim 0,\\ T_1+2S_1+T_2 &\sim 0,\\ 9S_1+2T_2+S_2 &\sim 0,\\ T_2+2S_2+T_3 &\sim 0,\\ 9S_2+2T_3+S_3 &\sim 0, \end{aligned}\right\} \tag{1'}$$

and

$$\left.\begin{aligned} T_1+2I &\sim 0,\\ S_1+6I &\sim 0,\\ T_2-14I &\sim 0,\\ S_2-26I &\sim 0,\\ T_3+66I &\sim 0,\\ S_3+102I &\sim 0. \end{aligned}\right\} \tag{2'}$$

We can define the analogous correspondences T_4, T_5, ..., and S_4, S_5, ..., which always satisfy the relations

$$T_{n-1}+2S_{n-1}+T_n \sim 0,$$
$$9S_{n-1}+2T_n+S_n \sim 0.$$

From relations such as these, expressing the dependence of the correspondences S and T upon one another, it is found that, when $n \geqslant 3$,

$$S_n+6S_{n-1}+9S_{n-2} \sim 0.$$

This may be regarded as a difference equation

$$\gamma_n+6\gamma_{n-1}+9\gamma_{n-2}=0$$

connecting the valencies γ of the different correspondences S; hence γ_n must be of the form $(-3)^n(a+bn)$, where a and b are constants. Since S_1 has valency $+6$ and S_2 has valency -26 the values of a and b are $-\frac{10}{9}$ and $-\frac{8}{9}$ respectively; hence

$$\gamma_n = -(-3)^{n-2}(8n+10).$$

Each vertex of an in-and-circumscribed polygon of n sides occurs twice among the united points of the correspondence S_n. Thus, when the number of united points of S_n has been obtained, we have to discard the heterotypic solutions and divide the result by $2n$ in order to obtain N_n. The procedure for the lower values of n now follows; it may be continued to any length for higher values of n without difficulty. We use τ $(= 28)$ for the number of bitangents and ι $(= 24)$ for the number of inflections of the curve.

Taking $n = 4$, we find $\gamma_4 = -378$; also the indices of S_4 are each $10 \cdot 9^3$ or 7290, so that the number of united points of S_4 is

$$2 \cdot 7290 - 6 \cdot 378 = 12312.$$

These include the vertices of each in-and-circumscribed quadrilateral, each vertex occurring twice. But they also include other points. Through each inflection of the curve there pass nine tangents, other than the inflectional tangent itself; any one of these has, besides its point of contact, one further intersection, say u_1, with the curve; u_1 is a united point of S_4. If i is the inflection and m the remaining intersection of the inflectional tangent with the curve, then i corresponds to u_1 in S_1, m corresponds to u_1 in S_2, i corresponds to u_1 in S_3, and u_1 is a self-corresponding point in S_4. We have the relation

$$8N_4 + 9\iota = 12312,$$

whence

$$N_4 = 1512.$$

This, then, is the number of in-and-circumscribed quadrilaterals of a plane quartic of genus 3; *there are 24 in-and-circumscribed quadrilaterals belonging to each of the sixty-three systems of contact quartic curves.*

Taking now $n = 5$, we find that $\gamma_5 = 1350$, so that the valency of S_5 is 1350. Its indices being $10 \cdot 9^4$, the number of its united points is

$$2 \cdot 65610 + 6 \cdot 1350 = 139320.$$

These include the vertices of the in-and-circumscribed pentagons, each counted twice. Moreover, there are two kinds of heterotypic solutions. Through the point of contact of a bitangent there pass nine other tangents of the curve each having one intersection, say h_1, with the curve other than its point of contact; through each point such as h_1 there pass nine further tangents of the curve, each having another intersection, say h_2, with the curve. All these points h_2 are united points of S_5. Further, from a vertex of an in-and-circumscribed triangle there can be drawn eight tangents, other than the two sides of the triangle, to the curve; such a tangent

meets the curve again in a point which counts twice among the united points of S_5. If efg is an in-and-circumscribed triangle, suppose that one of the tangents from e, other than ef and eg, meets the curve again in v_1. Then the points e, f, g, e, v_1 correspond to v_1 in the correspondences S_1, S_2, S_3, S_4, S_5 respectively; moreover, we can also arrive at v_1 in the order e, g, f, e, v_1, so that *two* of the points which correspond to v_1 in S_5 coincide with it. We can now write down the equation

$$10N_5+162\tau+48N_3=139320,$$

whence, substituting 28 and 288 for τ and N_3 respectively, we find that

$$N_5=12096.$$

We next find that the valency of S_6 is -4698 and that the number of its united points is 1152792. The heterotypic solutions fall into three different sets. Through each of the points u_1 there pass nine tangents of the curve, apart from the tangent which joins u_1 to an inflection; each such tangent meets the curve further, other than at its point of contact, in a point u_2 which is a united point of S_6. Again, through a vertex of an in-and-circumscribed quadrilateral there pass eight tangents of the curve, apart from the two sides of the quadrilateral which pass through the point; each of these eight tangents meets the curve again, other than at its point of contact, in a point w_1 which occurs twice among the united points of S_6, just as the points v_1 occurred twice among the united points of S_5. Further, each vertex of an in-and-circumscribed triangle is a united point of S_6 and occurs twice. We have then

$$12N_6+81\iota+64N_4+6N_3=1152792,$$

$$N_6=87696.$$

Of these 87696 in-and-circumscribed hexagons there are 1392 belonging to each of the sixty-three systems of contact sextics.

Proceeding now to consider in-and-circumscribed heptagons, we find that the correspondence S_7 is of valency 16038 and has 10725048 united points. Apart from the vertices of the heptagons themselves, the united points of S_7 are of three kinds. Through each of the eighty-one points h_2 associated with a point of contact of a bitangent there pass nine tangents (other than the tangents h_2h_1) each of which has a further intersection h_3 with the curve; the points h_3 are united points of S_7. Through each vertex of an in-and-circumscribed pentagon there pass eight tangents of the curve, apart from the two sides of the pentagon; each of these tangents meets the curve again in a point which occurs twice among the

united points of S_7. Further, through any of the points v_1 associated with the in-and-circumscribed triangles there pass, apart from the tangent which joins v_1 to a vertex of an in-and-circumscribed triangle, nine tangents of the curve; the remaining intersection v_2 of any of these tangents with the curve is a united point of S_7, and, furthermore, is to be counted twice, once for each way in which one can proceed round the in-and-circumscribed triangle from one of its vertices. The equation obtained is

$$14N_7+1458\tau+80N_5+432N_3=10725048,$$

whence

$$N_7=685152.$$

For the correspondence S_8, we find that its valency is -53946 and that the number of its united points is 95335704. The equation for N_8 is found to be

$$16N_8+729\iota+96N_6+8N_4+576N_4+48N_3=95335704.$$

The in-and-circumscribed quadrilaterals occur from two aspects on the left-hand side of this equation. In the first place, each of these vertices occurs twice among the united points of S_8. In the second place, through each of the points w_1 there pass, apart from the tangent which joins w_1 to the vertex of an in-and-circumscribed quadrilateral, nine tangents of the curve; each of these tangents meets the curve again in a point w_2 which is a united point of S_8. The equation gives

$$N_8=5375160.$$

When this number is divided by sixty-three the quotient is 85320, which is therefore the number of in-and-circumscribed octagons belonging to any one of the sixty-three systems of contact curves of order eight.

The procedure can be continued indefinitely, giving

$$N_9=43059744, \quad N_{10}=348636960,$$

and so on.

CONICS IN SPACE, AND THEIR REPRESENTATION BY POINTS IN SPACE OF NINETEEN DIMENSIONS

By J. A. Todd.

[Received 30 April, 1932.—Read 19 May, 1932.]

Introduction.

The present paper has a twofold aim. In the first place, it seeks to extend the well-known idea of Grassmannian varieties, or loci whose points map the linear sub-spaces of some given space, to representations of entities which are not linear spaces, but which may be any algebraical loci whatever. We shall see how a system of coordinates can be defined for such loci, analogous to line-coordinates, which leads to the unexceptional representation of the particular system of loci considered on the points of some manifold in higher space, and we shall see that the ordinary Grassmannian varieties are particular cases of this more general representation. On account of the considerable complexity of the loci involved, the bulk of the paper is concerned with a particular case, namely the representation of the system of ∞^8 conics in ordinary space by the points of a certain locus in [19]; and only a sketch of the general theory is given. We study in detail the representation of conics, and obtain theorems analogous to known results for the geometry of linear spaces. We find, for example, that on our locus there are two fundamental V_7, in terms of which every other V_7 can be expressed; this appears as the generalization of the theorem that every complex of linear spaces is equivalent to a certain multiple of a linear complex.

The second object of the paper is to give an account of the conic as a space element with a view to the study of systems of conics, on which there is a somewhat extensive if unsystematic literature. The present paper

is devoted to the preliminaries of the subject rather than to a detailed account of any particular systems of conics. However, a construct is introduced which does not appear to have been previously considered, namely the *linear complex of conics*, a system of ∞^7 conics containing one member of a general pencil. This is shown to be closely connected with a definite quadratic line-complex, and to be determined by it.

Unlike the case of linear spaces, the totality of conics in space is indeterminate without more precise definition. We can consider (in any number of dimensions) a conic in at least three different ways, each of which leads to a different conception of the totality of conics. We may consider (i) the plane sections of a quadric (degenerate or otherwise), the degenerate conics of the system being then the pairs of intersecting lines and the lines of space repeated; or (ii) we can regard the conic defined tangentially, in which case its degenerate forms are a pair of points or a repeated point; or, finally, (iii) we can adopt a view which comprehends both the former, in which the conic is regarded as having significance both as a locus and as an envelope. In all cases the undegenerate members of the system are unique and correspond one to the other, but each line counted twice, which is regarded as a single conic in case (i), is a doubly infinite set in (ii) or (iii), and so on. In the plane, (i) and (ii) are reciprocal; each leads to the mapping of the conics of the plane on [5] so that the specialized conics correspond to the points of a $M_4{}^3$ and the doubly specialized conics to a double Veronese surface on this. To map the system (iii) it is necessary to consider the locus in [27] represented on [5] by cubic primals passing through the Veronese surface*.

In three dimensions, the conic of case (ii) is simply a tangential quadric whose discriminant vanishes, and as such has been the subject of a paper by Johnson†. We shall be concerned with the case (i), but the relations between our representation and Johnson's will be considered.

The system of coordinates which we find convenient for our representation is essentially due to Cayley‡, although, owing to its early date, the significance of Cayley's work seems to have passed unrecognised. The same representation was arrived at independently by Spottiswoode§,

---

* Study, **12**.

† Johnson, **4**.

‡ Cayley, **1**.

§ Spottiswoode, **10**, **11**.

from another point of view. Spottiswoode's work leads us to consider twenty-one quantities, connected by a single linear relation, as the coordinates of a conic in space. His treatment of these quantities, three of which do not appear in the first account, is unsymmetrical; that of the present work is symmetrical and in consequence more complete.

The present paper begins with a geometrical treatment of the subject, indicating how we are led to the study of a V_8^{92} in [19]. The analytical investigations which follow lead to the same locus and determine its equations. The linear complex of conics is then briefly studied. We then consider the relation of our representation to that given by Johnson, and obtain a representation of our locus on a linear space. Next is proved the fundamental theorem that every complex of conics can be expressed linearly in terms of multiples of two fundamental ones. This is shown to be equivalent to the symbolic formulae of Schubert. Finally, an indication is given of the extension of the theory to other loci than conics.

1. *The representation of the conics in space by points in* [19].

1.1. *The system of conics in a plane.*

The ∞^5 conics in a plane form a linear system and can be mapped by the points of a [5], Σ, in such a way that conics of a pencil are mapped by points of a line. If the conic has the equation

$$\Sigma a_{ij} x_i x_j = 0 \quad (a_{ij} = a_{ji};\ i, j = 1, 2, 3),$$

it can be made to correspond to the point of Σ whose coordinates are a_{ij}. The line-pairs correspond to points for which the determinant $|a_{ij}|$ vanishes, and the repeated lines to points where the determinant is of rank one, namely to the points $A_{ij} = 0$, A_{ij} being the cofactor of a_{ij} in the determinant. These lie on a Veronese surface ${F_2}^4$, and the points at which $|a_{ij}| = 0$ lie on a cubic primal ${M_4}^3$, having this surface double and being the locus of its conic planes.

Dually, regarded as an envelope of equation $\Sigma A_{ij} \xi_i \xi_j = 0$, the conic can be mapped by the point A_{ij} of another [5], Σ'; the point pairs now correspond to the determinantal cubic primal $|A_{ij}| = 0$ and the repeated points of the plane to the double Veronese surface on this.

Since there is a one to one correspondence between the general points of Σ and Σ', there must be a representation of Σ' on Σ; and it is easy to see that in this a prime of Σ' is mapped by a quadric in Σ passing through the Veronese surface F, and conversely. This representation,

of course, gives rise to the well-known Cremona transformation of [5] by means of quadrics through a Veronese surface.

1.2. *The representation of the conics of space.*

We seek now a locus which is in (1, 1) correspondence, without exception, with the conics, degenerate and undegenerate, of space. We shall require the ∞^5 conics in any plane to be mapped linearly on a [5] in the manner just described. The V_8 which we require will therefore be generated by ∞^3 [5], and any pair of these will meet in a point which maps the degenerate conic consisting of the repeated line of intersection of the two planes. This point lies thus on the Veronese surface F in each [5]. The locus of such points on V_8 is a V_4 in unexceptional correspondence with the lines of space, having ∞^3 [5] which meet it in Veronese surfaces. This V_4 is therefore the V_4^{32} of [19] mapped on the quadric of [5] by the totality of its quadric sections. This V_4^{32} contains two systems of Veronese surfaces, arising from the planes of the quadric in [5]; two surfaces of the same system meet in a point, while two of opposite systems meet, if at all, in a conic. The locus of the planes of these conics is a V_7 which is the same for both systems of surfaces and maps the pairs of intersecting lines of space; and the V_8 which we require is the locus of [5] containing one of the systems of Veronese surfaces on this. The [5] containing the other system form a similar locus V_8'; V_8 and V_8' meet in V_7. The locus representing the conics of space is thus capable of the following definition: *it is the locus of the* [5] *which cut out one system of Veronese surfaces on the* V_4^{32} *of* [19] *mapped on a quadric of* [5] *by the totality of its quadric sections.*

1.3. We next prove the following fundamental theorem.

Any irreducible algebraic system of ∞^7 *conics which contains one conic in a general pencil is mapped by a prime section of* V_8.

By a pencil of conics is meant the system of conics lying in a fixed plane and given by an equation of the form $\lambda_1 S_1 + \lambda_2 S_2 = 0$. Our hypothesis is that every such pencil either contains just one conic of the given system or else is entirely contained in it.

In the first place, if the given system contains all the repeated lines of a plane π, then it must contain every conic of π, since any conic in the plane $t = 0$ can be expressed linearly in terms of the squares of the six lines $x = 0$, $y = 0$, $z = 0$, $y + z = 0$, $z + x = 0$, $x + y = 0$.

Consequently, it is impossible for our irreducible system L to contain every repeated line of space. If π is a plane, every repeated line of which does not belong to L, those conics of L which lie in π form a linear system, and of these there are a pencil of line-pairs with vertex at a general point of π; this pencil contains two repeated lines. Thus two repeated lines of L lie in a general pencil, and the lines form a quadratic complex. The conics corresponding to the repeated lines are thus mapped on a prime section of V_4^{32}; and hence, since L is irreducible, it follows that the conics of L are mapped by the section of V_8 by this prime.

In particular, the conics which meet a fixed line of space form a system of the type just described, and thus correspond to the points of a particular prime section of V_8. Thus the order of the locus V_r representing any ∞^r system of conics is equal to the number of conics of the system which meet r lines. In particular, *the order of* V_8 *is* 92, *and the order of the* V_7 *mapping line-pairs is* 140.

We have thus represented the conics of space by the points of a V_8^{92} in [19], on which the line-pairs correspond to a V_7^{140} and the repeated lines to a V_4^{32}. The last locus is *double* on V_7 and V_8.

1.4. *The representation of particular systems of conics*.*

We now consider the representation of some particular systems of conics on V_8.

1.41. *The degenerate conics containing a fixed line* l.

Since two of these conics meet three further lines the representing locus is V_3^2. This contains ∞^1 planes, corresponding to the conics lying in planes through l, all these planes pass through a fixed point on V_4^{32}, mapping the conic consisting of the line l repeated. The planes thus form a quadric point-cone, and the mapping of the conics reduces to the Grassmann mapping of the lines of the special linear complex whose directrix is l.

1.42. *The conics through two points.*

The V_4 which maps this system is of order 4, since four of the conics meet four general lines. It contains ∞^1 [3], corresponding to the conics lying in planes through the two points; and these [3] all pass through the point of V_4^{32} mapping the line joining the points, counted doubly. The [3]

* For the enumerative results quoted in this section see, *e.g.*, Schubert, **6**, 95, 96.

therefore form a cone projecting the rational normal V_3^4 of planes* in [6] from a point outside the [6]. The line-pairs of the system which do not contain the line joining the points are mapped by a V_3^6 lying on V_4^4, having a double point at the vertex, and meeting each [3] in a quadric surface. The line-pairs which contain the join of the two points are mapped by a quadric cone V_3^2, which is the tangent cone to V_3^6 at the vertex of V_4^4.

1.43. *The conics lying on a quadric surface.*

These are in (1, 1) correspondence with the planes of space. Since eight of the conics meet three arbitrary lines it is easily seen that the representing locus must be the Veronese threefold V_3^8 of [9], mapped on [3] by the complete system of quadrics. The conics of V_3^8 correspond to the sections of the quadric by planes of a pencil; the Veronese surfaces to sections by the planes of a star. More generally, the planes of an envelope of class k cut out a system of conics mapped by a surface of order $4k$. For $k = 2$ we have the planes of a quadric envelope cutting out a system mapped by a V_2^8, *i.e.* a prime section. In particular, if the quadric enveloped by the planes coincides with the given quadric the conics become the line-pairs of the system, and V_3^8 meets V_7 in a prime section V_2^8.

1.44. *The conics passing through a point.*

These are mapped on a V_6^{18}, since eighteen of the conics meet six arbitrary lines. This locus lies in [13]. For take six arbitrary lines through the point P. No conics meet all these lines save those which pass through P; on the other hand, any five of the lines determine a quadric cone of vertex P containing infinitely many conics meeting all the five. Now the conics meeting one of the lines are mapped by a prime section of V_8, and this shows that V_6^{18} is the complete intersection of six such primes which are independent. It thus lies in [13]. The line-pairs of the system are mapped by a V_5^{30}, the repeated lines by a V_2^4.

1.45. *The conics having a given line as a chord.*

These are mapped by a V_6^8, the line-pairs of the system by a V_5^{20} and the repeated lines by a V_3^{16}; this last locus is mapped on a quadric cone V_3^2

* The rational normal V_r^n of $[r-1]$ lies in $[n+r-1]$. Its curve sections are rational normal curves. Its equations may be expressed in the form

$$\left\| \begin{matrix} x_1 & x_2 & \dots & x_n \\ x_1' & x_2' & \dots & x_n' \end{matrix} \right\| = 0.$$

by the totality of its quadric sections, and hence lies in [13]. The V_6^8 is the locus of [5] containing one of the two systems of Veronese surfaces on this, and is a cone projecting the rational normal V_5^8 of [4] in [12].

1.46. *The conics whose planes pass through a point.*

These are mapped by a V_7^{34}, since thirty-four of the conics meet seven lines. Any repeated line belongs to the system, so that V_7^{34} contains entirely V_4^{32}.

1.5. *The rationality of V_8 and of its linear sections.*

1.51. *The section of V_8 by a* [14].

Anticipating future consideration of the system, let us refer to an irreducible set of ∞^7 conics containing one member in a general pencil as a *linear complex* of conics. The result of (1.3) shows that the conics of such a system are mapped on a prime section of V_8. Let us consider the section of V_8 by five such primes, *i.e.* by a [14]. The corresponding system of conics contains one conic in each general plane, since the [14] meets a general [5] of V_8 in a point. This system of conics is thus in (1, 1) correspondence with the planes of space, and the corresponding V_3^{92} in [14] is rational. To obtain a representation of V_3^{92}, suppose that the conics corresponding to the points of V_3^{92} lie in a [3] Σ, and that Σ' is another [3]; between Σ and Σ' we can set up a correlation, and any plane of Σ will then correspond to a point of Σ'. We consider the mapping of V_3^{92} on Σ' so obtained.

We denote by ϕ the surfaces in Σ' representing prime sections of V_3^{92}. The points of such a surface ϕ correspond to conics in Σ common to six linear complexes. If l is a line of Σ', the points of l correspond to conics in Σ lying in the planes of a pencil. Since eight of these belong to six linear complexes it follows that the surface ϕ is of the eighth order. Again, to the points of a plane π in Σ' correspond conics in Σ whose planes pass through a fixed point, and to the points in which π meets the residual curve γ of intersection of two surfaces ϕ correspond conics in Σ whose planes pass through a point and which belong to seven linear complexes; of such there are thirty-four. Thus the intersection of two surfaces ϕ consists of a variable part of order 34 and a fixed curve of order 30. Further, since the surfaces ϕ map a V_3^{92}, three of the surfaces have ninety-two variable intersections. It follows, by an application of the formulae for residual intersections of surfaces, that the genera of the fixed and variable curves are respectively 91 and 115, and that the two curves have 180

common points. Hence the theorem:

The section of V_8 by a general [14] *can be mapped on space so that its prime sections correspond to surfaces of order* 8 *passing through a curve of order* 30 *and genus* 91. *Two such surfaces meet residually in a curve of order* 34 *and genus* 115, *meeting the base curve in* 180 *points; three such surfaces have ninety-two variable intersections.*

As a check, we note that the freedom of such surfaces, calculated by the usual formula, is

$$(\tfrac{1}{6}.9.10.11-1)-(8.30-91+1),$$

that is 14, as it should be.

1.52. A plane of Σ' corresponds to a V_2^{34} on V_3^{92}, and as this does not lie in a prime section (V_7^{34} lying in [19] and not in a lower space) there can be no surface of order 7 containing the base curve σ of the surfaces ϕ. The plane maps the ∞^2 conics of the system in Σ whose planes pass through a point P. Of these conics ∞^1 actually pass through P, and these are mapped by the points of a curve in the corresponding plane. This curve meets each surface ϕ in eighteen variable points (since conics through P correspond to a V_6^{18} on V_8); and it is of degree 6, since it can easily be shown by a degeneration argument that there are six conics which belong to five linear complexes, pass through a fixed point, and have a fixed line through the point as a chord. It follows that the conics of the system passing through P are mapped in Σ' by a plane sextic curve passing through the thirty intersections of the plane with σ. Hence (i) the thirty points of any plane section of σ lie on a sextic curve, and (ii) the V_6^{18} mapping conics passing through a point has linear sections of genus 10.

1.53. *Particular cases.*

It may happen that the conics in Σ have one or more directrix lines, the corresponding linear complexes of conics being special. If l is a line of Σ meeting all the conics, and l' is the dually corresponding line in Σ', the line l' is double on the surfaces ϕ. For the points of a line l_1' meeting l' correspond to conics in Σ lying in the planes of a pencil whose vertex l_1 meets l; and the points in which l_1' meets a surface ϕ correspond to conics having the vertex of the pencil as a chord, meeting l, and lying in five linear complexes. Of the eight conics which, in general, satisfy this condition, six pass through the intersection of l and l_1, and the other two

coincide in a conic lying in the plane of l and l_1, which counts doubly because it meets l in two points. Thus two of the intersections of l_1' and ϕ coincide at the point where l_1' meets l', and l' is double on ϕ.

In particular, the system in Σ may be defined by five directrix lines of general position. The base curve σ of the surfaces ϕ then splits up into five double lines and the five pairs of transversals of sets of four of them. The genus of the residual C^{34} is then reduced to 75. This representation of the system of conics meeting five lines has been studied by Schaake*.

1.54. *Other linear sections.*

The sections of V_8 of dimension greater than three consist of a rational family of linear spaces, and are therefore rational. The surface sections of V_8 are birationally equivalent to certain octavic surfaces in space without multiple curves; they have therefore arithmetical and geometrical genus 35, the number of linearly independent adjoint quartic surfaces. The curve sections of V_8 have genus 115. If we denote by p_r the arithmetical genus of the linear V_r section of the locus, p_0 being understood to be the order diminished by one, we have

$$p_0 = 91, \quad p_1 = 115, \quad p_2 = 35, \quad p_3 = p_4 = p_5 = p_6 = p_7 = p_8 = 0.$$

1.6. *Postulation of V_8.*

It has been shown by Severi† that, for an algebraic locus V_d of dimension d, which does not contain multiple parts of this dimension, the postulation $\theta(l)$ for primals *of sufficiently high order* l to contain it is given by the formula

$$(1.61) \quad \theta(l) = (p_0+1)\binom{l+d}{d} - (p_1+p_0)\binom{l+d-1}{d-1} + \ldots + (-)^{d-1}(p_{d-1}+p_{d-2})(l+1) + (-)^d(p_d+p_{d-1}),$$

where p_r denotes the "virtual arithmetical genus" of the section of V_d by $d-r$ primes. If V_d has no multiple parts of dimension $d-1$ these are the actual arithmetical genera of the successive prime sections, p_0 having the conventional meaning explained above.

* Schaake, **5**.

† Severi, **8**, 40.

Applying this to V_8, we find that

$$\theta(l) = 92\binom{l+8}{8} - 206\binom{l+7}{7} + 150\binom{l+6}{6} - 35\binom{l+5}{5}$$

$$= \tfrac{1}{84}\binom{l+5}{5}(23l^3+71l^2+102l+84),$$

and, in particular,

$$\theta(1) = 20, \quad \theta(2) = 189, \quad \theta(3) = 1100.$$

Now, since V_8 actually lies in [19], and in no lower space, its postulation for primes is exactly twenty. This suggests that in the present instance $l = 1$ is a sufficiently high value for the formula to be valid. If this is the case, the number $n(l)$ of linearly independent primals of order l which contain the locus will be given by

$$n(l) = \binom{19+l}{l} - \theta(l),$$

and, using the values already found,

$$n(1) = 0, \quad n(2) = 21, \quad n(3) = 440.$$

We shall verify below that there are actually twenty-one linearly independent quadric primals passing through V_8 by obtaining their equations. The verification of the value of $n(3)$ has not yet been accomplished.

2. *Analytical theory of the conic in space.*

2.1. *Notation and preliminaries.*

We shall now deal with the matter from an analytical point of view, and, in particular, we shall obtain equations for the locus described in the previous section. We shall first standardise a notation and make certain conventions which will be adopted throughout the remainder of the paper. The suffixes used will be throughout capable of the values 1, 2, 3, 4. We shall be much concerned with symbols bearing a pair of suffixes. We shall, in general, assume that the symbols so written are antisymmetrical in the suffixes of a pair, *i.e.* $a_{ii} = 0$, $a_{ij} = -a_{ji}$. Occasionally we shall have to deal with symbols symmetrical in the suffixes of a pair. Such symbols will be distinguished by a bar, *e.g.* $\bar{a}_{ij} = \bar{a}_{ji}$. With this convention fixed it will be unnecessary to state in any given instance whether the

symbols used are symmetrical or antisymmetrical. Symbols bearing two pairs of suffixes will frequently occur. These will be written in the form $a_{ij,\,kl}$ which will imply antisymmetry in the suffixes of a pair, but symmetry as between the two pairs; *i.e.* we shall suppose that

$$a_{ij,\,kl} = a_{kl,\,ij} = -a_{ji,\,kl}.$$

Finally, if i, j denote two *different* suffixes, we shall denote by i', j' the complementary pair, the order being chosen in such a way that i, j, i', j' is an even permutation of 1, 2, 3, 4.

As an instance of the above notation, the identical relation between the six coordinates of a line may be written

$$\Sigma p_{ij} p_{i'j'} = 0, \tag{2.11}$$

the summation being taken either over all permutations of the suffixes or over all the pairs.

We now proceed to define the coordinates of a conic in space, and to study their properties.

2.2. *Cayley's definition.*

The lines which meet a conic form a particular kind of quadratic complex, since two of them lie in any pencil. The equation of this complex may be written in the form

$$\Sigma d_{ij,\,kl} p_{ij} p_{kl} = 0, \tag{2.21}$$

the summation extending to all permutations of the suffixes. In virtue of the identical relation (2.11) satisfied by the coordinates of any line, we may assume that the coefficients of the particular terms $p_{ij} p_{i'j'}$ are connected by the relation

$$\Sigma d_{ij,\,i'j'} = 0. \tag{2.22}$$

The twenty-one distinct coefficients $d_{ij,\,kl}$ in the equation (2.21), subject to (2.22), may be regarded as coordinates of the conic. If the quadratic line-complexes of space, which form a linear system, are mapped on the points of [19], the locus of points representing the particular complexes which consist of the transversal lines of conics is the V_8^{92} considered in § 1. In particular, if the conic degenerates into the repeated line of coordinates q_{ij}, the quadratic complex is the square of the linear complex

$$\Sigma q_{i'j'} p_{ij} = 0,$$

and hence

$$d_{ij,\,kl} = q_{i'j'}q_{k'l'},$$

which are obviously the equations of the V_4^{32} mapped on a quadric of [5] by all its quadric sections.

The idea of representing a curve in space by the complex of lines which meet it is due to Cayley*; and our representation is an obvious consequence of this.

2.3. *Spottiswoode's definition*†.

We can define the coordinates of the conic in a different manner, which, though lacking the generality of Cayley's definition, leads to explicit expressions for the coordinates in a convenient form. We consider the conic as the intersection of a quadric

$$\Sigma \bar{a}_{ij}x_ix_j = 0, \tag{2.31}$$

and a plane

$$\Sigma \alpha_i x_i = 0, \tag{2.32}$$

it being assumed that the quadric does not contain the plane as a part. The quadric is, of course, not completely determinate. If we consider the expressions

$$d_{ij,\,kl} = \bar{a}_{ik}\alpha_j\alpha_l + \bar{a}_{jl}\alpha_i\alpha_k - \bar{a}_{il}\alpha_j\alpha_k - \bar{a}_{jk}\alpha_i\alpha_l, \tag{2.33}$$

it is easy to see that they are independent of the particular quadric containing the conic which we may have selected. Further, as Cayley pointed out‡, they are identical with the coordinates just defined. For the line of coordinates p_{ij} meets the plane (2.32) in the point X of coordinates

$$X_i = \sum_j \alpha_j p_{ij},$$

which lies on the quadric (2.31) if

$$\Sigma \bar{a}_{ij}\alpha_k\alpha_l p_{ik}p_{jl} = 0.$$

---

* Cayley, **1**.

† Spottiswoode, **10, 11**.

‡ Cayley, **2**. For comparison we set out our coordinates with their equivalents in Spottiswoode's notation:

$$d_{23,23};\ d_{31,31};\ d_{12,12};\ d_{14,14};\ d_{24,24};\ d_{34,34} = A;\ B;\ C;\ F;\ G;\ H;$$

$$d_{12,13};\ d_{23,21};\ d_{31,32};\ d_{42,43};\ d_{43,41};\ d_{41,42} = A';\ B';\ C';\ F';\ G';\ H';$$

$$d_{12,14};\ d_{23,24};\ d_{31,34};\ d_{13,14};\ d_{21,24};\ d_{32,34} = L;\ M;\ N;\ L';\ M';\ N';$$

$$d_{23,14};\ d_{31,24};\ d_{12,34} = P;\ Q;\ R.$$

If we compare this with the equation (2.21) of the quadratic complex of lines meeting the conic, we obtain the expressions (2.33) for $d_{ij,kl}$.

Spottiswoode treats the three coordinates $d_{ij,i'j'}$ on a different footing from the rest; in his earlier paper they are not mentioned. In consequence his work lacks symmetry, and many of the relations (discussed below) between the coordinates are overlooked.

2.4. *Relations between the coordinates of a conic.*

The twenty-one coordinates just found are connected by a large number of identities, which are the equations of the locus V_8^{92}. In order to obtain these identities it is convenient to use the expressions (2.33) for the coordinates, and to employ a symbolic notation. To this end we write

$$\bar{a}_{ij} = l_i l_j = m_i m_j = \ldots, \tag{2.41}$$

where the symbols l_i, m_i, ... are equivalent, but have actual meaning only when they occur in combinations $l_i l_j$, etc., and in manipulating the symbols we never allow any letter to occur to higher aggregate degree than the second, so that $\bar{a}_{ij}^2$ would be written $l_i l_j m_i m_j$ and not $l_i^2 l_j^2$. This is the familiar symbolic notation used in the theory of algebraic invariants*.

We shall also write

$$\lambda_{ij} = a_i l_j - a_j l_i, \quad \mu_{ij} = a_i m_j - a_j m_i, \ldots. \tag{2.42}$$

Substituting from (2.41) in (2.33), we obtain the coordinates of the conic expressed in the concise symbolic form

$$d_{ij,kl} = \lambda_{ij}\lambda_{kl} = \mu_{ij}\mu_{kl} = \ldots. \tag{2.43}$$

Now, from the form of the expressions (2.42), it is clear that

$$\theta(\lambda, \mu) \equiv \sum_{(ij)} \lambda_{ij}\mu_{i'j'} = 0. \tag{2.44}$$

On multiplying this by any expression bilinear in the λ and μ we obtain, in virtue of (2.43), a relation between the coordinates. Taking the bilinear form to be $\lambda_{kl}\mu_{mn}$ (in terms of which all others can be expressed) we obtain the relations

$$Q_{kl,mn} \equiv \sum_{(ij)} d_{kl,ij} d_{mn,i'j'} = 0 \tag{2.45}$$

for all k, l, m, n.

* *Cf.* Grace and Young, 3, 9 *et seq.*

The equations (2.45) represent twenty-one linearly independent quadrics which pass through V_8, the existence of which was anticipated in (1.6). Their independence follows by inspection of their formal expressions. The three quadratic relations given by Spottiswoode* are the three expressed by $Q_{kl,k'l'}=0$.

It is easily seen that there are no other quadratic relations between the coordinates, since any such relation must arise from (2.44), the only relation holding identically between the λ and μ. These relations, however, do not completely define V_8. For we saw in (1.2) that the V_7^{140} on V_8 which maps the line-pairs of space contains two systems of [5] meeting it in cubic primals; one of these systems generates V_8 and the other a similar locus V_8'. Any quadric primal containing V_7^{140} must contain both V_8 and V_8'. In fact, as we shall see later, the intersection of the quadrics $Q_{kl,mn}$ consists of the *two* loci V_8 and V_8'.

To define V_8 it is necessary to introduce certain cubic primals which contain it. To obtain these we notice that

$$\alpha_i\lambda_{jk}+\alpha_j\lambda_{ki}+\alpha_k\lambda_{ij}=0,$$

and hence, by eliminating α_i, α_j, α_k from this and two similar expressions, we get

$$\Delta_{ijk}\equiv\begin{vmatrix}\lambda_{jk} & \lambda_{ki} & \lambda_{ij}\\ \mu_{jk} & \mu_{ki} & \mu_{ij}\\ \nu_{jk} & \nu_{ki} & \nu_{ij}\end{vmatrix}=0. \tag{2.46}$$

It follows that

$$\begin{vmatrix}d_{pq,jk} & d_{pq,ki} & d_{pq,ij}\\ d_{rs,jk} & d_{rs,ki} & d_{rs,ij}\\ d_{tu,jk} & d_{tu,ki} & d_{tu,ij}\end{vmatrix}=0 \tag{2.47}$$

for all values of the suffixes. This leads to eighty cubic relations between the coordinates. Spottiswoode gives the cases (*a*) $pq, rs, tu = jk, ki, ij$ and (*b*) $pq, rs, tu = 41, 42, 43$; $i, j, k = 1, 2, 3$. These relations, together with 420 of the form $d_{pq,rs}Q_{kl,mn}=0$, give 500 cubic relations between the coordinates. These are not all linearly independent. For, if ijk and pqr are two different triads of suffixes, the expression

$$\Delta_{ijk}\Delta_{pqr}=0$$

* Spottiswoode, **11**.

can be evaluated in two ways; likewise, from the relation

$$\mu_{pq}\nu_{rs}\theta(\lambda,\mu)\theta(\lambda,\nu)=0$$

it follows that $$\sum_{(ij)} d_{pq,ij}Q_{rs,i'j'}=\sum_{(ij)} d_{rs,ij}Q_{pq,i'j'}.$$

There are also other linear relations between the cubics of a more complicated nature, but we do not enter into a discussion of these here.

We conclude this section by remarking that the quartic identities given by Spottiswoode are deducible from the symbolic relation

$$\Delta(\lambda,\mu,\nu,\rho)\equiv\begin{vmatrix} 0 & \lambda_{34} & \lambda_{42} & \lambda_{23} \\ \mu_{43} & 0 & \mu_{14} & \mu_{31} \\ \nu_{24} & \nu_{41} & 0 & \nu_{12} \\ \rho_{32} & \rho_{13} & \rho_{21} & 0 \end{vmatrix}=0$$

and that they are consequences of the quadratic relations, since we have

$$\Sigma\Delta(\lambda,\mu,\nu,\rho)\equiv 2\,[\theta(\lambda,\mu)\theta(\nu,\rho)+\theta(\lambda,\nu)\theta(\mu,\rho)+\theta(\lambda,\rho)\theta(\mu,\nu)],$$

the summation being over all permutations of λ, μ, ν, ρ.

3. *The linear complex of conics.*

3.1. *Analytical theory.*

The linear complex of conics has already been defined as a system of conics which contains one member of a general pencil. We saw in (1.3) that such a system was mapped on a prime section of V_8. Thus the linear complex can be defined analytically as the aggregate of conics whose coordinates satisfy a linear relation

$$\Sigma a_{ij,kl}d_{ij,kl}=0, \tag{3.11}$$

where, in virtue of (2.22), we may suppose that

$$\Sigma a_{ij,i'j'}=0. \tag{3.12}$$

Now the system of conics which lie on a quadric surface is mapped on V_8 by a $V_3{}^8$ lying in [9]; and the prime section of this, a surface of order 8, corresponds to the set of conics on the quadric cut out by the planes of a quadric envelope. Thus the linear complex of conics associates with any quadric surface a quadric envelope which cuts out the conics of

the surface belonging to the complex. Let us examine this correspondence in detail.

We consider a general prime section of V_8. It follows by a simple counting of constants that the general section will not contain any of the generating [5] of V_8 nor any of its $V_3{}^8$. Consequently, a general linear complex has no *total* planes or quadrics, *i.e.* planes or quadrics every conic of which belongs to the complex.

Now let K denote the quadric envelope corresponding to the quadric Q. If the section of Q by a plane π belongs to the complex Γ, then π is a plane of K. If Q' is a second quadric whose section by π belongs to the complex, π also touches the corresponding quadric envelope K'. Then every quadric of the pencil determined by Q and Q' meets π in a conic of the system (since two of them do so), so that π belongs to the envelope associated with any one of these quadrics. In other words, *the quadrics of a pencil have for corresponding envelopes quadrics of a tangential pencil, and the association of point-quadric and envelope determined by the complex* Γ *is linear.*

In symbols, the quadric Q, of equation

$$\Sigma \bar{q}_{ij} x_i x_j = 0, \tag{3.13}$$

is associated with the envelope K whose tangential equation is

$$\Sigma \bar{\kappa}_{ab} \xi_a \xi_b = 0, \tag{3.14}$$

and the coefficients $\bar{\kappa}_{ab}$ are linear functions of the $\bar{q}_{ij}$, *i.e.*

$$\bar{\kappa}_{ab} = \sum_{ij} c_{ab,ij} \bar{q}_{ij}, \tag{3.15}$$

where the coefficients $\bar{c}$ are assumed to be symmetrical in the suffixes of each pair, but, at present, not necessarily symmetrical for interchange of pairs. We shall now consider the nature of these coefficients.

The most general relation of the form (3.15) involves 100 coefficients of the type $\bar{c}_{ab,ij}$. In the present instance, the relation between the quadrics Q and the envelopes K is such that the quadrics Q which correspond to the ∞^8 envelopes which touch π all meet π in one of the ∞^4 conics of Γ which lie in π*. It is clear that a necessary and sufficient condition for this to happen is that this system of quadrics Q should contain every plane-pair in which π is one of the planes, for only in this case can the system Q meet π in ∞^4 conics only.

---

* We are assuming the general case in which the equations (3.15) are soluble for the $\bar{q}$ in terms of the $\bar{\kappa}$. This will happen unless the complex contain total planes.

Now, if u_i are the coordinates of the plane π, the envelopes to which π belongs are those for which

$$\Sigma \bar{\kappa}_{ab} u_a u_b = 0,$$

and hence the corresponding quadrics Q satisfy

$$\sum_{a,b,i,j} \bar{c}_{ab,ij} \bar{q}_{ij} u_a u_b = 0. \tag{3.16}$$

If this system of quadrics contains every plane-pair to which π belongs, it will be satisfied by $\bar{q}_{ij} = \frac{1}{2}(u_i v_j + u_j v_i)$, where v_j, v_j are arbitrary. This means that, for each fixed j, we must have the relations

$$\sum_{a,b,i} \bar{c}_{ab,ij} u_a u_b u_i = 0.$$

Equating coefficients of like polynomials in the u to zero, we are led to relations between the $\bar{c}$ which can be expressed in the form

$$\left.\begin{array}{c} \bar{c}_{ab,cd} = \bar{c}_{cd,ab} \\ \bar{c}_{ab,cd} + \bar{c}_{ac,bd} + \bar{c}_{ad,bc} = 0. \end{array}\right\} \quad \text{for all } a, b, c, d. \tag{3.17}$$

In virtue of these relations the number of effectively independent homogeneous constants $\bar{c}_{ab,ij}$ in the collineation (3.15) is twenty, agreeing with the fact that the freedom of the linear complex of conics is nineteen.

If the equation of the complex of conics is

$$\Sigma a_{ij,kl} d_{ij,kl} = 0,$$

it follows, by using the expressions (2.33) for the coordinates and comparing with (3.16), that we have

$$\bar{c}_{ij,kl} = a_{ik,jl} + a_{il,jk}, \tag{3.18}$$

which lead, conversely, to

$$a_{ij,kl} = \bar{c}_{ik,jl} - \bar{c}_{il,jk}. \tag{3.19}$$

These relations enable us to pass from the equation of the complex to that of the collineation, and *vice versa*.

3.2. *Connection with the theory of the quadratic line-complex.*

With every linear complex of conics is associated a perfectly definite quadratic line-complex, namely that formed by the double lines of the

complex; and, conversely, every quadratic line-complex may be so derived. In fact, a prime section of V_8 in [19] meets the V_4^{32} representing the double lines in a V_3^{32}, mapped by a quadric section of the quadric in [5]; and, conversely, every quadric section of the quadric in [5] determines a prime in [19]. We have seen that the equations of V_4^{32} are

$$d_{ij,\,kl} = q_{i'j'}q_{k'l'},$$

so that the quadratic line-complex associated with the linear complex of conics

$$\Gamma \equiv \Sigma a_{ij,\,kl}d_{ij,\,kl} = 0 \tag{3.21}$$

has for its equation

$$C \equiv \Sigma a_{ij,\,kl}p_{i'j'}p_{k'l'} = 0. \tag{3.22}$$

We can at once make deductions from this. It follows from (3.1), for example, *that every quadratic line-complex determines a linear relation between the quadric surfaces and quadric envelopes of space. If the quadric surface degenerate into a repeated plane, the corresponding quadric envelope degenerates into the planes passing through the lines of the complex lying in this plane, which envelop a conic.*

Consider now a general linear complex of conics Γ. The conics of Γ lying in a plane π form a linear system and are in general out-polar to a definite conic γ. The lines of the associated quadratic complex C which lie in π are the tangents to γ. If the plane is a singular plane of C, the conic γ becomes a point-pair, and all the conics of Γ in π have a common pair of conjugate points. If γ further degenerates into a repeated point, this point is a base-point for the conics of Γ lying in π. *Hence, for* ∞^2 *planes in space, which touch a Kummer surface, the system of conics of* Γ *possesses a pair of conjugate points, and in the sixteen tropes of this surface the system has a base-point.*

Each species of quadratic line-complex determines a projectively distinct type of linear complex of conics, for which the singular surface takes various degenerate forms. Suppose, in particular, that the line-complex consists of the lines which meet a certain conic γ of coordinates $D_{ij,\,kl}$. Its equation is then

$$\Sigma D_{ij,\,kl}p_{ij}p_{kl} = 0,$$

and the associated linear complex of conics Γ has for its equation

$$\Sigma D_{ij,\,kl}d_{i'j',\,k'l'} = 0. \tag{3.23}$$

If π is a plane which meets γ in two points, the repeated lines of Γ which lie in π form two flat pencils whose vertices are these points; and the conics of π belonging to Γ are just those for which these points are conjugate. *Thus* (3. 23) *is the condition satisfied by the coordinates d, D of two conics which meet the line of intersection of their planes in two pairs of points which harmonically separate one another.* It is easily verified that it is satisfied by the coordinates of any two coplanar conics*.

4. *Connection with the alternative representation.*

4.1. *The conic as a degenerate quadric envelope.*

We now consider the consequences of regarding the conic in space as a degenerate quadric envelope. The quadric envelope whose tangential equation is

$$\Sigma \bar{y}_{ij} \xi_i \xi_j = 0$$

can be mapped by the point $\bar{y}_{ij}$ of [9]; and the conics arise if the determinant $|\bar{y}_{ij}|$ vanishes. The conic splits up into a point-pair or a repeated point if the determinant has rank two or one. *Hence the conics of space, regarded as degenerate tangential quadrics, can be mapped by the points of a* V_8^4 *of* [9], *the symmetrical determinantal primal. The point-pairs correspond to the points of a* V_6^{10}, *double on* V_8^4, *and the repeated points to a* V_3^8, *quadruple for* V_6^{10} *and triple for* V_8^4. The V_3^8 is the Veronese manifold, mapped on space by all quadrics, the V_6^{10} is the locus of its conic planes, and the V_8^4 is the locus of the [5] containing its Veronese surfaces. The generating [5] of V_8^4 correspond to the systems of conics lying in the various planes of space.

Johnson's paper† deals in detail with systems of conics from this point of view. In this paper we shall only be concerned with the relations between his mapping of the conics and the one which we have described above.

4.2. *Connection between the representations.*

Since to a general point of V_8^4, mapping an undegenerate conic, there corresponds a unique and definite point on V_8^{92}, it follows that V_8^{92} can be mapped on V_8^4. The exceptional elements in this transformation can

---

* This relationship of two conics is also considered by Johnson, **4**, 352. Considering the conics as degenerate quadric envelopes, the invariant (3. 23) is the invariant Φ of the pair of quadrics.

† Johnson, **4**.

only be the points whose correlate on the other locus is indeterminate, namely the points of the V_3^8 which maps repeated points, and the points of the V_4^{32} which maps repeated lines. Further, since the conics of a plane are mapped by a [5] in each of the representations, it follows that these [5] are transformed one into the other, and this transformation is obviously the one indicated in (1 . 1), namely by quadrics passing through a Veronese surface. We infer, therefore, that V_8^{92} *can be represented on* V_8^4 *so that its prime sections correspond to the sections of* V_8^4 *by quadrics passing through its triple* V_3^8. This suggests that the system of quadrics in [9] passing through a V_3^8 is worth examination, and we proceed to investigate the system.

4 . 3. *The system of quadrics in* [9] *passing through* V_3^8.

We shall require to make use of the two following intersection theorems, which are due to Severi*:

(i) In $[r]$, if Φ_h is a locus of dimension $h < \frac{1}{2}r$, of order μ_0, and classes $\mu_1, \mu_2, \ldots, \mu_h$; and if $r-1$ primals of orders $n_1, n_2, \ldots, n_{r-1}$ are drawn through Φ_h, they intersect further in a curve of order ϵ_0, rank ϵ_1, meeting Φ_h in d points, where

$$\epsilon_0 = n_1 n_2 \ldots n_{r-1} - \{\mu_0 s'_{h-1} + (\mu_0 - \mu_1) s'_{h-2} + (\mu_0 - \mu_1 + \mu_2) s'_{h-3} + \ldots\},$$

$$d = \mu_0 s_h' - \mu_1 s'_{h-1} + \ldots + (-)^h \mu_h,$$

$$\epsilon_1 = \epsilon_0 s_1' - hd,$$

s_i' being the sum of products of $n_1-1, n_2-1, \ldots, n_{r-1}-1$, taken i at a time.

(ii) If r primals of orders $n_1, n_2, \ldots, n_r$ are drawn through Φ_h, they intersect further in a number of points equal to

$$n_1 n_2 \ldots n_r - \{\mu_0 s_h + (\mu_0 - \mu_1) s_{h-1} + \ldots + [\mu_0 - \mu_1 + \ldots + (-)^h \mu_h]\},$$

where s_i is the sum of products of $n_1-1, n_2-1, \ldots, n_r-1$, taken i at a time.

Consider now the locus V_3^8. Its quadric sections are mapped in ordinary space by the totality of quartic surfaces, of which there are ∞^{34}. Since there are ∞^{54} quadrics in [9] it follows that V_3^8 lies on ∞^{19} quadrics. By means of these the [9] is mapped on a V_9 of [19]. Using the first of the

* Severi, 7.

formulae just quoted, it is easily seen that eight of the quadrics have as residual intersection a curve of order 92 and rank 460, meeting V_3^8 in ninety-two points, and that nine of the quadrics have ninety-two residual intersections. *Hence the quadrics through* V_3^8 *map the prime sections of a locus* V_9^{92} *in* [19], *whose curve sections have genus* 139. We shall refer to this locus simply as V_9.

If, as in (2.2), we map the quadratic line-complexes of space by the points of [19], the points of V_9 map the complexes which consist of the aggregate of tangents to the quadrics of space. That V_9 is of order 92 then corresponds to the known result that there are ninety-two quadrics in space which touch nine lines of general position.

A primal in [9] of order N which passes i times through the V_3^8 will be mapped on V_9 by a V_8 of order $92(N-i)$, as is easily seen by counting the intersections of the primal with a residual curve of intersection of the quadrics. In particular, we verify that the quartic V_8^4, passing triply through V_3^8, maps a V_8^{92}. The section of V_9 by an m-ic primal is mapped in [9] by a primal of order $2m$ which passes m times through the V_3^8.

It is not difficult to see what corresponds on V_9 to the neighbourhood of points of the V_3^8. The neighbourhood of a point P of V_3^8 is mapped by a [5] of V_9, all points near P which lie in the same [4] through the tangent [3] at P to V_3^8 corresponding to the same point of this [5]. Any two of these [5] meet in a point, through which pass ∞^1 other [5], mapping the neighbourhoods of all the points on one of the conics of V_3^8. The common point of these [5] maps the plane of the conic. Now the conic planes of V_3^8 generate the V_6^{10} which represents point-pairs; and this is mapped on the double V_4^{32} of V_8^{92}. It is now clear that the system of [5] under discussion is simply the set of generating [5] of the locus which we have called V_8'; the second system of [5] which meet V_4^{32} in Veronese surfaces. Also, since the neighbourhood of a base locus can be considered formally as a primal of order 0 passing -1 times through the locus, it is clear that the quartic primal with V_3^8 triple, together with the neighbourhood of V_3^8, make up a locus representing a complete quadric intersection of V_9, *i.e. there exist quadrics whose intersection with* V_9 *consists of* V_8 *and* V_8'.

Incidentally, we may notice the following simple numerical check. The rank of a composite curve, consisting of two curves of ranks r_1 and r_2 meeting in i points, is r_1+r_2+2i. Apply this to the curve sections of two particular intersections of V_9 and a quadric, (i) when the intersection consists of two prime sections, and (ii) when it consists of V_8 and V_8', we have, in the first case, $r_1 = r_2 = 460$, $i = 92$, and, in the second case, $r_1 = r_2 = 412$, $i = 140$; we find, in each case, the rank of the composite curve to be 1104.

The primes in [9] correspond to loci V_8^{92} on V_9, which we may denote by U_8. The residual intersection of V_9 by a quadric through U_8 is a locus U_8', of the same character, mapped by cubic primals on which V_3^8 is double. These cubic primals form a homaloidal system, and determine a symmetrical cubic Cremona transformation of the [9], in which the quartic V_8^4 is closed up into the neighbourhood of the V_3^8 of the reverse transformation, and the quadrics through V_3^8 are transformed into quadrics through the second V_3^8. If we choose the transformation so that the V_3^8 are the same for the direct and reverse transformations, a collineation in [19] is determined which leaves the locus V_9 fixed but interchanges V_8 and V_8', U_8 and U_8'.

4.4. *Analytical theory.*

The results of the last section may be obtained very simply by analysis. The equations of the V_3^8 are

$$\| \bar{y}_{ij} \|_1 = 0,$$

the notation implying, as usual, the vanishing of the two-rowed minors in the determinant. The quadrics which pass through V_3^8 are linearly expressible in terms of these two-rowed minors, *i.e.* of the quantities

$$\begin{vmatrix} \bar{y}_{ik} & \bar{y}_{il} \\ \bar{y}_{jk} & \bar{y}_{jl} \end{vmatrix},$$

so that the parametric equations of V_9 are

$$d_{i'j',k'l'} = \bar{y}_{ik}\bar{y}_{jl} - \bar{y}_{il}\bar{y}_{jk}, \tag{4.41}$$

taking the d as coordinates in the [19].

To obtain equations for V_9, we consider the determinant

$$\Delta(ijkl) \equiv \begin{vmatrix} \bar{y}_{i1} & \bar{y}_{i2} & \bar{y}_{i3} & \bar{y}_{i4} \\ \bar{y}_{j1} & \bar{y}_{j2} & \bar{y}_{j3} & \bar{y}_{j4} \\ \bar{y}_{k1} & \bar{y}_{k2} & \bar{y}_{k3} & \bar{y}_{k4} \\ \bar{y}_{l1} & \bar{y}_{l2} & \bar{y}_{l3} & \bar{y}_{l4} \end{vmatrix},$$

and expand it in terms of its two-rowed minors. Using (4.41) we have, with an obvious change in the meaning of the suffixes,

$$\Delta(ijkl) = \sum_{(p,q)} d_{ij,pq} d_{kl,p'q'} = \sum_{(p\ q)} d_{ik,pq} d_{lj,p'q'} = \sum_{(p,q)} d_{il,pq} d_{jk,p'q'},$$

so that the equations of V_9 are

$$Q_{ij,\,kl} = Q_{ik,\,lj} = Q_{il,\,jk}, \tag{4.42}$$

where, as in (2.4), we have $Q_{ij,\,kl} = \sum_{(p,q)} d_{ij,\,pq} d_{kl,\,p'q'}$.

Since $Q_{ii,\,jk}$ is zero, these can be written in the alternative form

$$Q_{ij,\,ik} = 0\,; \quad Q_{12,\,34} = Q_{13,\,42} = Q_{14,\,23}.$$

[The conventions of (2.1) are still supposed to apply.]

The equations of V_9 are given by (4.42), and it follows that V_9 is cut in $V_8 + V_8'$ by any quadric $Q_{ij,\,kl}$ which has four different suffixes. This leads back to the equations (2.45) which completely determine V_8 and V_8'.

If $\bar{Y}_{ij}$ is the cofactor of $\bar{y}_{ij}$ in the determinant $|\bar{y}_{ij}|$, the equations

$$\sigma \bar{y}'_{ij} = \bar{Y}_{ij} \tag{4.43}$$

determine a symmetrical Cremona transformation of the [9] in which to the prime $\Sigma \bar{a}_{ij} \bar{y}_{ij} = 0$ corresponds the cubic primal $\Sigma \bar{a}_{ij} \bar{Y}_{ij} = 0$ having $V_3{}^8$ double. Since

$$\begin{aligned} \sigma^2(\bar{y}'_{ik}\bar{y}'_{jl} - \bar{y}'_{il}\bar{y}'_{jk}) &= \bar{Y}_{ik}\bar{Y}_{jl} - \bar{Y}_{il}\bar{Y}_{jk} \\ &= |\bar{y}_{\alpha\beta}|(\bar{y}_{i'k'}\bar{y}_{j'l'} - \bar{y}_{i'l'}\bar{y}_{j'k'}), \end{aligned}$$

this transformation corresponds to the collineation in [19] whose equations are

$$d'_{ij,\,kl} = \rho d_{i'j',\,k'l'}, \tag{4.44}$$

which leaves fixed V_9 but interchanges V_8 and V_8'. The fixed spaces of the collineation are the [10] $d_{ij,\,ik} = d_{i'j',\,i'k'}$ and the [8] $d_{ij,\,kl} + d_{i'j',\,k'l'} = 0$. (All permutations of the suffixes are to be made.)

We saw in (2.47) that the locus V_8 lies on the cubic primals of equation

$$\begin{vmatrix} d_{pq,\,jk} & d_{pq,\,ki} & d_{pq,\,ij} \\ d_{rs,\,jk} & d_{rs,\,ki} & d_{rs,\,ij} \\ d_{tu,\,jk} & d_{tu,\,ki} & d_{tu,\,ij} \end{vmatrix} = 0.$$

The collineation which interchanges V_8 and V_8' transforms primals of this type into primals of the form

$$\begin{vmatrix} d_{ab,\,wx} & d_{ab,\,wy} & d_{ab,\,wz} \\ d_{cd,\,wx} & d_{cd,\,wy} & d_{cd,\,wz} \\ d_{ef,\,wx} & d_{ef,\,wy} & d_{ef,\,wz} \end{vmatrix} = 0,$$

which contain V_8'. It follows that the cubic primals

$$(4.45) \qquad \begin{vmatrix} d_{pq,jk} & d_{pq,ki} & d_{pq,ij} \\ d_{pr,jk} & d_{pr,ki} & d_{pr,ij} \\ d_{ps,jk} & d_{ps,ki} & d_{ps,ij} \end{vmatrix} = 0$$

contain both V_8 and V_8'; and it is easy to see, by substituting from (4.41), that they also contain V_9.

4.5. *The representation of V_8 on a linear space.*

It was shown in (4.2) that the V_8 in [19] can be mapped on V_8^4 in [9] so that prime sections of the former correspond to sections of the latter by quadrics passing through the triple V_3^8. Now V_8^4 is a rational locus whose representation on [8] is derived directly by projection from a point on the triple locus. In this way we can get a representation of V_8 on [8] which we now consider.

If the V_8^4 $|\bar{y}_{ij}| = 0$ is projected from the point $\bar{Y}_{44}$, which lies on the triple locus, on to the prime $\bar{y}_{44} = 0$, the triple V_3^8 is projected into a V_3^7 in the prime, whose equations are

$$(4.51) \qquad \left\| \begin{matrix} \bar{x}_{11} & \bar{x}_{12} & \bar{x}_{13} & \bar{x}_{14} \\ \bar{x}_{12} & \bar{x}_{22} & \bar{x}_{23} & \bar{x}_{24} \\ \bar{x}_{13} & \bar{x}_{23} & \bar{x}_{33} & \bar{x}_{34} \end{matrix} \right\|_1 = 0,$$

and the equations connecting the $\bar{y}$ and the $\bar{x}$ are

$$(4.52) \qquad \bar{y}_{44} = -\Delta, \quad \bar{y}_{ij} = X\bar{x}_{ij} \quad (ij \neq 44),$$

where

$$(4.53) \qquad \Delta \equiv \begin{vmatrix} \bar{x}_{11} & \bar{x}_{12} & \bar{x}_{13} & \bar{x}_{14} \\ \bar{x}_{12} & \bar{x}_{22} & \bar{x}_{23} & \bar{x}_{24} \\ \bar{x}_{13} & \bar{x}_{23} & \bar{x}_{33} & \bar{x}_{34} \\ \bar{x}_{14} & \bar{x}_{24} & \bar{x}_{34} & 0 \end{vmatrix}, \quad X \equiv \begin{vmatrix} \bar{x}_{11} & \bar{x}_{12} & \bar{x}_{13} \\ \bar{x}_{12} & \bar{x}_{22} & \bar{x}_{23} \\ \bar{x}_{13} & \bar{x}_{23} & \bar{x}_{33} \end{vmatrix}.$$

The V_6^{10}, locus of conic planes of V_3^8, projects from $\bar{Y}_{44}$ into a V_6^6, and Δ, X touch along this V_6^6, in fact, $X = 0$ is the condition for Δ, regarded as a quadratic form in x_{i4}, to be a perfect square. The section of V_8^4 by the prime $\Sigma \bar{a}_{ij}\bar{y}_{ij} = 0$ is mapped by the quartic primal

$$-\bar{a}_{44}\Delta + X\Sigma\bar{a}_{ij}\bar{x}_{ij} = 0 \quad (ij \neq 44),$$

which passes doubly through V_3^7; all these quartic primals touch Δ and each other along V_6^6.

If no two of the suffixes are 4, we have

$$\bar{y}_{ik}\bar{y}_{jl}-\bar{y}_{il}\bar{y}_{jk}=X^2(\bar{x}_{ik}\bar{x}_{jl}-\bar{x}_{il}\bar{x}_{jk}),$$

while

$$\bar{y}_{i4}\bar{y}_{j4}-\bar{y}_{ij}\bar{y}_{44}=X(X\bar{x}_{i4}\bar{x}_{j4}+\Delta\bar{x}_{ij}),$$

so that the sections of V_8^4 by quadrics through V_3^8 are mapped by the family of quintic primals

$$(4.54)\qquad \Sigma a_{ij,\,kl}X(\bar{x}_{ik}\bar{x}_{jl}-\bar{x}_{il}\bar{x}_{jk})+\Sigma a_{i4,\,j4}(X\bar{x}_{i4}\bar{x}_{j4}+\Delta\bar{x}_{ij})=0,$$

which have V_3^7 triple and touch Δ along the V_6^6.

Conversely, the quintic primals are characterized by this property. For the quintics which touch Δ along V_6^6 all have an equation of the form

$$X\Phi_2+\Delta\Phi_1=0,$$

where Φ_2, Φ_1 denote polynomials of the degree indicated by their suffixes. We must adjust these so that V_3^7 is triple on the quintic. Now on V_3^7, which is given parametrically by the relations $x_{ij}=t_it_j$, the expressions $\partial X/\partial\bar{x}_{ij}$, $\partial\Delta/\partial\bar{x}_{ij}$ vanish, so that V_3^7 is double on X and Δ. For it to be triple on the quintic we must have

$$\frac{\partial^2}{\partial\bar{x}_{ij}\partial\bar{x}_{kl}}(X\Phi_2+\Delta\Phi_1)=0\quad(\text{on } V_3^7),$$

so that, omitting vanishing terms,

$$(4.55)\qquad \Phi_2\frac{\partial^2X}{\partial\bar{x}_{ij}\partial\bar{x}_{kl}}+\Phi_1\frac{\partial^2\Delta}{\partial\bar{x}_{ij}\partial\bar{x}_{kl}}=0.$$

On V_3^7, however, the derivatives of X and Δ are easily found, and the only non-vanishing derivatives are the following:

$$\frac{\partial^2X}{\partial\bar{x}_{ii}\partial\bar{x}_{jj}}=\bar{x}_{kk},\quad \frac{\partial^2X}{\partial\bar{x}_{ii}\partial\bar{x}_{jk}}=-2\bar{x}_{jk},\quad \frac{\partial^2X}{\partial\bar{x}_{ij}^2}=-2\bar{x}_{kk},\quad \frac{\partial^2X}{\partial\bar{x}_{ij}\partial\bar{x}_{ik}}=+2\bar{x}_{jk},$$

$$\frac{\partial^2\Delta}{\partial\bar{x}_{ii}\partial\bar{x}_{jj}}=-\bar{x}_{k4}^2,\quad \frac{\partial^2\Delta}{\partial\bar{x}_{ii}\partial\bar{x}_{jk}}=+2\bar{x}_{j4}\bar{x}_{k4},\quad \frac{\partial^2\Delta}{\partial\bar{x}_{ij}^2}=2\bar{x}_{k4}^2,\quad \frac{\partial^2\Delta}{\partial\bar{x}_{ij}\partial\bar{x}_{ik}}=-2\bar{x}_{j4}\bar{x}_{k4},$$

where (i, j, k) is a permutation of (1, 2, 3). This leads to the relations

$$\left.\begin{aligned}\Phi_2\bar{x}_{ii}-\Phi_1\bar{x}_{i4}^2=0\\ \Phi_2\bar{x}_{ij}-\Phi_1\bar{x}_{i4}\bar{x}_{j4}=0\end{aligned}\right\}\quad(i, j=1, 2, 3),$$

holding on V_3^7. If then we put

$$\Phi_1 = \Sigma \bar{h}_{ij} \bar{x}_{ij} \quad (i, j = 1, 2, 3, 4;\ ij \neq 44)$$

and substitute $\bar{x}_{ij} = t_i t_j$, we find for Φ_2 the expression

$$\Phi_2 = Q + \Sigma \bar{h}_{ij} \bar{x}_{i4} \bar{x}_{j4},$$

where Q is a quadric passing through V_3^7, whence it follows that the quintic primal is of the form

$$XQ + \Sigma \bar{h}_{ij} (\Delta \bar{x}_{ij} + X \bar{x}_{i4} \bar{x}_{j4})$$

comprised in (4.54).

We have thus established *a representation of* V_8^{92} *on* [8] *in which prime sections correspond to quintic primals having a triple* V_3^7 *and touching along the* V_6^6 *locus of chords of this.*

5. *Complexes and systems of conics in space.*

5.1. *Complexes of conics and their representation.*

We now return to the consideration of the conic as a locus of points. By a *complex* of conics in space will be understood an algebraic system of freedom seven, and we shall always suppose that the complex is free from reducible parts of dimension less than seven. (The term complex has been applied to systems of freedom three by some writers; the present use of the term follows the generally accepted terminology in the geometry of systems of linear spaces.) In an arbitrary pencil of conics there will lie a finite number of conics of the complex. We shall denote this number by m and call m the *order* or *degree* of the complex. The term *general pencil* is here to be understood not merely as a pencil not specially associated with the particular complex, but to denote *a pencil of conics no member of which is a repeated line.* This restriction is desirable since a repeated line is of indeterminate plane, and, in general, it is only by considering the particular complex under discussion that we are able to decide how often such a degenerate conic is to be regarded as belonging to a pencil in any assigned plane through the line.

In the geometry of linear spaces, the fundamental basis for the discussion of complexes is a pair of theorems due to Severi*. These state,

* Severi, **9**, 96.

first, that there exist no complexes of degree zero, and, secondly, that every complex is coresidual to a multiple of an arbitrary linear complex, and is therefore represented by a single algebraic equation, or by a complete intersection of the Grassmannian locus with a primal. In the case of conics in space, neither of these theorems holds. However, we may deduce results which correspond to them and which play a fundamental part in the theory of systems of conics.

Let us consider, in the first place, whether it is possible to have complexes of degree zero. Let Γ^0 be such a complex. Since it is of degree zero, and contains no conics in a general pencil, it must contain every conic which lies in the plane of any one of its members which is not a repeated line. Accordingly Γ^0 consists of the aggregate of conics which lie in the planes of some doubly infinite system. Conversely, the aggregate of such conics evidently forms a complex of degree zero. The class n of the envelope of the planes will be called the *class* of the complex. Any line of space lies in n of the planes, and thus any repeated line counts n times in the complex. We have the result that *any complex of conics of degree zero and class n consists of the aggregate of conics in the planes of an envelope of class n.*

If the system is of class 1, the planes of the conics all pass through a point. The corresponding system of conics, Σ, is mapped on V_8 by a V_7^{34} S. For any complex of degree zero and class n, we have the equivalence

$$\Gamma^0 = n\Sigma,$$

or, for the corresponding loci on V_8,

$$G^0 \equiv nS.$$

The locus G^0 passes through V_4^{32}, and is composed of a set of ∞^2 [5]. The locus V_4^{32} is simple on S and hence n-ple on the locus representing a complex of degree zero and class n.

If a complex Γ^m of degree m greater than zero contains all the conics of ∞^2 planes, then Γ^m is reducible and we can remove a complex of order zero from Γ^m, leaving a complex Γ'^m still of degree m but containing at most ∞^1 total planes.

Now let Γ^m, Γ'^m be any two complexes of the same degree m, and let G^m, G'^m be the loci on V_8 which map them. On V_8 we construct a system of ∞^7 lines of which one passes through a general point. To do this, consider the section of V_8 by an arbitrary [14], which meets a general [5] of V_8 in a point, and consider the stars of lines in the [5] of V_8

which pass through the points of intersection of the [5] with the given [14]. This set of lines has the required property.

A general line of this system meets G^m and G'^m in two sets of m points $A_1, \ldots, A_m$; $B_1, \ldots, B_m$. Let x be a coordinate of position on the line; then we can construct the rational function of x which vanishes at the points $A_1, \ldots, A_m$ and becomes infinite at $B_1, \ldots, B_m$. Now the line lies in a unique [5] of V, since we may suppose the [14] to be so general that it does not intersect V_4^{32}. The lines of the system lying in this [5] form a star. As the line varies in the star, the function of x which we have just obtained becomes a rational function of position in the [5] which becomes zero on G^m and infinite on G'^m. Further, since by construction the function is of order m, it can have no other zero in the [5] without vanishing everywhere in the [5].

Now let the [5] vary among the ∞^3 [5] of V_8, and we obtain a rational function of position on V_8, zero on G^m and infinite on G'^m. If there are any further loci of poles or zeros of the function, they must be composed entirely of generating [5] of V_8, and hence correspond to complexes of conics of degree zero. Expressing the fact that the loci of poles and zeros of a rational function are coresidual, we have the relation of equivalence

$$G^m + G^0 \equiv G'^m + G'^0,$$

where G^0, G'^0 map some complexes of degree zero; if the classes of these are n and n', this relation can be written in the form

$$G^m \equiv G'^m + (n' - n)\, S.$$

This relation of equivalence connects any two G^m, or the corresponding complexes of conics of degree m. In particular, we may take G'^m to be a complete intersection of V_8 with a primal of order m, so that $G'^m \equiv mL$, where L denotes a prime section. We then have the relation, for any G^m on V_8,

$$G^m \equiv mL + nS,$$

where n is an integer, positive, negative, or zero. We may call n the *class* of the complex Γ^m associated with G^m. We have thus proved that *every V_7 on V_8 is coresidual to multiples, with integral coefficients, of two loci L, S, which are respectively the prime section of V_8 and the V_7^{34} mapping conics whose planes pass through a fixed point.*

For complexes of degree 0, $n > 0$; for complete intersections of V_8, $n = 0$.

The above argument is similar to the one used by Severi to establish the equivalence of any complex of linear spaces to a multiple of a linear

complex. As in Severi's paper, we infer that the intersections of V_8 with primals of given order form a complete linear system, so that any complex of class zero is mapped by a complete intersection.

We shall now illustrate the above theorem by considering a few particular cases. The representative locus of a complex of order m and class n is of order $92m+34n$.

(i) The aggregate of line-pairs forms a complex of degenerate conics of degree 3. It is mapped by V_7^{140}. The class of the complex is determined by the equation $140 = 92.3+34n$, so that $n = -4$. Thus *the line-pairs in space form a complex of conics of order* 3 *and class* -4, and the residual intersection of V_8 by a cubic primal through V_7^{140} maps a set of conics lying in the planes of a quartic envelope.

(ii) The ∞^7 conics which touch a given plane form a complex of order 2. Since 116 of them meet seven lines* they are mapped on V_8 by a V_7^{116}, and the class of the system is given by $116 = 92.2+34n$ or $n = -2$. *The conics touching a plane therefore form a complex of order* 2 *and class* -2.

(iii) As a final example, consider the conics which have a tangent belonging to a given congruence of lines of order r and class s. The order of the complex is evidently $2s$. If we consider the degenerate congruence consisting of r point-stars of lines and s planes of lines, the conics in question are the conics which touch the s planes and the conics whose planes pass through the r points, the latter counting doubly since two tangents to each such conic lie in the congruence. The representing locus on V_8 is thus of order $s.116+2r.34$, and so the class of the system is $2(r-s)$. In particular, for a linear congruence, $r = s = 1$, and we get a complex of order 2 and class 0, while for a congruence composed of the axes of a cubic developable $r = 3$, $s = 1$, and the complex has order 2 and class 4; this giving an example of an irreducible complex of conics of class greater than zero.

5.2. *Connection with Schubert's symbolic calculus*†.

The theory of the last section is in intimate connection with the symbolic calculus of Schubert, as applied to the conic in space. In fact,

* Schubert, **6**, 95.

† Schubert, **6**, 93.

the relations between the condition symbols which are obtained by Schubert are precisely expressions of the equivalence relations of the previous section.

Schubert's condition symbols are the five following:

μ, the condition that the plane of the conic should pass through a given point;

ν, the condition that the conic should meet a line;

ρ, the condition that the conic should touch a plane;

δ, the condition that the conic should be a line-pair;

η, the condition that the conic should degenerate tangentially into a point-pair.

Any linear relation between these symbols, of the form

$$p\mu+q\nu+r\rho+s\delta+t\eta=0,$$

implies that, for any simply infinite system of conics of which μ' satisfy condition μ, ν' satisfy condition ν, and so on, we have the relation

$$p\mu'+q\nu'+r\rho'+s\delta'+t\eta'=0.$$

The aggregate of conics satisfying one of these conditions corresponds to a V_7 on V_8, and the number of intersections of this V_7 with any curve on V_8 is the number of conics of the system represented by the curve which satisfy the condition in question. Any relation of equivalence between these loci will thus lead to a relation between the corresponding condition symbols.

We have now to identify the loci on V_8 which correspond to the aggregate of conics satisfying these conditions. The system of conics satisfying μ is evidently mapped on a V_7^{34} S; that corresponding to ν is mapped on a prime section L. The locus corresponding to ρ is the locus R, which is the V_7^{116} mentioned in the last section, while that corresponding to δ is the locus V_7^{140}, which we may denote by D. We have obtained in the last section the order and class of the complexes corresponding to R and D, and hence we have the relations of equivalence on V_8:

$$R\equiv 2L-2S, \quad D\equiv 3L-4S.$$

The case of the condition η is somewhat more complicated, since, in our consideration of the conic as a locus of points, the question of

tangentially degenerate members does not arise. A tangentially degenerate conic must be regarded as a repeated line, corresponding to a point on V_4^{32}, but each such point maps a triple infinity of degenerate conics, since it is possible in ∞^3 ways to associate with a given line a plane passing through it and a pair of points on it. Thus we must regard the ∞^7 tangentially degenerate conics as corresponding to a V_7 on V_8 of order zero, which is, in fact, the locus V_4^{32}, every point of which counts ∞^3 times. For any curve on V_8, the number of its intersections with V_4^{32} then represents the number of tangentially degenerate conics of the corresponding system. (We may note that a similar ambiguity occurs in the case of conics in a plane, and can be avoided by considering a locus in [27] in which every conic of the plane having a definite point and tangential equation corresponds to a point on the locus, and every point of the locus maps a conic unique in both senses. To get a similar locus in which the tangentially degenerate conics in space are considered on the same footing as line-pairs, we should consider the transform of V_8 by cubic primals through V_4^{32}, or of the $V_8{}^4$ in [9] by cubic primals through its triple $V_3{}^8$.)

Let us now use the equivalence relations written above to obtain relations between the condition symbols. The first relation can be put in the form

$$2L \equiv R + 2S,$$

which expresses the fact that a quadric can be drawn through R to meet V_8 in a locus equivalent to $2S$. Since R contains V_4^{32} this quadric must pass through V_4^{32}. Of its $2\nu'$ intersections with any curve C, η' lie on V_4^{32}, and of the rest ρ' lie on R and $2\mu'$ on a locus equivalent to $2S$. Since this result holds independently of the curve, we deduce that

$$2\nu = \eta + \rho + 2\mu.$$

Again, the second equivalence can be put in the form

$$3L \equiv D + 4S,$$

so that a cubic primal through D (hence having V_4^{32} double) meets V_8 further in a locus equivalent to $4S$. Hence, considering intersections with a curve, we have

$$3\nu = 2\eta + \delta + 4\mu.$$

Thus we have the relations

$$2\nu = \rho + 2\mu + \eta, \quad 2\rho = \delta + \nu, \tag{5.21}$$

established by Schubert*, here deduced as consequences of the general theorem of (5.1).

5.3. *Further study of the representation of V_8 on* [8].

We may now consider further the representation of V_8 on [8] obtained in (4.5), in which prime sections of V_8 are mapped by quintic primals having in common a triple V_3^7 and touching along the V_6^6 locus of chords (or conic planes) of this the cubic cone $X = 0$, which is the cone projecting the V_6^6 from the unique plane lying on V_3^7. A general [5] of V_8 is mapped by a [5] meeting V_3^7 in a Veronese surface and V_6^6 in a cubic primal passing doubly through this surface. The residual intersection of such a [5] with the system of quintic primals consists of the quadrics passing through the Veronese surface. A pencil of conics, mapped by a line on V_8, corresponds to a conic lying in this [5], trisecant of V_3^7. Particular prime sections of V_8 are mapped by quadrics passing simply through V_3^7, the rest of the quintic primal consisting of the cubic cone $X = 0$. The ∞^3 cubic primals through V_6^6 represent the ∞^3 V_7^{34} on V_8, and correspond to complexes of conics of order 0 and class 1. The locus V_4^{32} is exceptional; each of its points is transformed into a whole plane, meeting V_3^7 in a conic and lying on V_6^6. The V_7^0 on V_8, consisting of V_4^{32} counted ∞^3 times, and mapping tangentially degenerate conics, corresponds to the neighbourhood of the locus V_6^6. On V_8, if tangentially degenerate members are included, there are *three* V_7 in terms of which all others can be expressed, namely the prime section, V_7^{34}, and V_7^0; in [8] there are also three, namely the prime, and the neighbourhoods of V_6^6 and V_3^7.

6. *Extensions of the previous theory.*

While it is not the purpose of the present paper to consider in great detail the generalizations of the foregoing theory, it seems worth while to indicate their nature, more particularly as the work is, in many respects, closely parallel to what has preceded.

We consider in $[r]$ an aggregate of loci V_k^m which form an irreducible system. It is a single condition for an $[r-k-1]$ to intersect such a locus, and the ones which do so form a complex of order m. The coefficients in the equation of this may be taken as coordinates of the V_k^m; if the aggregate of m-ic complexes is mapped linearly on a space of appropriate dimension, the particular complexes under consideration are mapped by

* Schubert, **6**, 92.

the points of a locus in (1, 1) correspondence with the V_k^m of $[r]$. For curves in space this is Cayley's representation.

The general theory presents considerable complications, since, for general types of loci, there are no reasonably simple systems (linear, etc.) which would give rise to simple constructs on the representing locus. A considerable simplification, however, is possible in the case (which includes the one considered in the early part of this paper) in which V_k^m lies in a $[k+1]$ of the $[r]$, and in this is a primal of general character. For, in this case, we clearly have on our representing locus a system of linear spaces whose freedom is that of $[k+1]$ in $[r]$ and whose dimension is that of m-ic primals in $[k+1]$. These linear spaces generate the locus. Particular V_k^m consist of a $[k]$ repeated m times, and the locus of such points, in general multiple on the representative locus, is the locus mapped on the Grassmann variety for $[k]$ in $[r]$ by all its m-ic sections.

In this simplified case, too, the analysis can be generalized, and coordinates can be defined analogous to those of Spottiswoode. For consider, first, the case in which $k = r-2$. Then the V_k^m can be regarded as the intersection of a prime of equation

$$a_i x_i = 0$$

(using the summation convention), with the m-ic primal

$$a_{i_1 i_2 \dots i_m} x_{i_1} x_{i_2} \dots x_{i_m} = 0,$$

which may be written in the symbolic form $(l_i x_i)^m = 0$, with

$$a_{i_1 i_2 \dots i_m} = l_{i_1} l_{i_2} \dots l_{i_m}.$$

If, as in § 2, we put

$$\lambda_{ij} = l_i a_j - l_j a_i,$$

we may take as coordinates of the V_k^m the quantities

$$d_{i_1 j_1, i_2 j_2, \dots, i_m j_m} = \lambda_{i_1 j_1} \lambda_{i_2 j_2} \dots \lambda_{i_m j_m}$$

depending on m pairs of suffixes $i_1 j_1, i_2 j_2, \dots, i_m j_m$, capable of values 1, 2, ..., $r+1$; the coordinates being antisymmetrical in the suffixes of each pair but symmetrical between the m pairs.

These coordinates are identical with the ones previously defined, the formal algebraic proof of this statement being exactly parallel to that in § 2.

It may be noticed that these d are the non-vanishing components of a covariant tensor of rank $2m$, transforming according to the tensor law,

for, if we make any change of coordinates (in this case linear),

$$x_i' = c_{ik} x_k = \frac{\partial x_i'}{\partial x_k} \partial x_k,$$

we find that

$$a_i' = \frac{\partial x_k}{\partial x_i'} a_k, \quad l_i' = \frac{\partial x_k}{\partial x_i'} l_k,$$

whence

$$\lambda_{ij} = \frac{\partial x_k}{\partial x_i'} \frac{\partial x_l}{\partial x_j'} \lambda_{kl},$$

and

$$d'_{i_1 j_1, \ldots, i_m j_m} = \frac{\partial x_{k_1}}{\partial x'_{i_1}} \frac{\partial x_{l_1}}{\partial x'_{j_1}} \cdots \frac{\partial x_{k_m}}{\partial x'_{i_m}} \frac{\partial x_{l_m}}{\partial x'_{j_m}} d_{k_1 l_1, \ldots, k_m l_m}.$$

This suggests that these coordinates have some fundamental importance. In the case of the conic in space, the twenty-one non-vanishing components have the same properties as regards symmetry of suffixes as the well-known Riemann-Christoffel tensor.

Let us now take the case $k < r-2$. The ${V_k}^m$ is now the section of the primal

$$(l_i x_i)^m = 0$$

by a $[k+1]$ defined by $r-k-1$ primes

$$a_i^{(s)} x_i = 0 \quad (s = 1, 2, \ldots, r-k-1).$$

An $[r-k-1]$ of coordinates $p_{i_1 i_2 \ldots i_{(r-k)}}$ meets this $[k+1]$ in a point of coordinates

$$x_i = h_{j^{(1)} \ldots j^{(r-k-1)}} p_{ij^{(1)} \ldots j^{(r-k-1)}},$$

where $h_{j^{(1)} \ldots j^{(r-k-1)}}$ denotes the determinant in the matrix $\| a_i^{(s)} \|$ formed from the $j^{(1)}, \ldots, j^{(r-k-1)}$-th columns. The equation of the complex of $[r-k-1]$ meeting the ${V_k}^m$ is obtained by expressing the fact that this point lies on the m-ic primal; and the coordinates of ${V_k}^m$ are obtained by gathering up coefficients of the various distinct products of the p. In the symbolic notation, if $\lambda_{uv \ldots w}$ is the determinant formed by taking the $r-k$ columns $u, v, \ldots, w$ from the matrix $\| a_i^{(s)} \|$ and adding the row of quantities $l_u, l_v, \ldots, l_w$ at the top, the coordinates of the ${V_k}^m$ are expressible in the form

$$d_{i_1 j_1^{(1)} \ldots j_1^{(r-k-1)}, \ldots, i_m j_m^{(1)} \ldots j_m^{(r-k-1)}} = \lambda_{i_1 j_1^{(1)} \ldots j_1^{(r-k-1)}} \ldots \lambda_{i_m j_m^{(1)} \ldots j_m^{(r-k-1)}},$$

and depend on m sets each of $r-k$ suffixes; it is not difficult to see that these transform according to the tensor law.

For such loci we may expect to prove results analogous to those of (5.1) by the use of similar methods. It looks as though we should find

that all complexes of V_k^m were dependent on two, as for conics in space, but we do not enter into a discussion here.

The determination of the fundamental numbers for complexes of V_k^m, and of the orders of the representative loci, present difficult enumerative problems the majority of which are at present unsolved. And when the simplification which we have been discussing does not occur, the theory of the corresponding locus seems still more difficult. Even the case of the twisted cubic in [3] has a very forbidding appearance. These facts afford justification for a detailed study of a simple case, such as has been undertaken in the previous pages.

References.

1. A. Cayley, "On a new analytical representation of curves in space", *Quart. J. of Math.*, 3 (1860), 225-236; *Papers*, 4, 446-455.
2. ———, "Addition to Mr. Spottiswoode's paper", *Proc. London Math. Soc.* (1), 10 (1879), 194-196; *Papers*, 11, 82-83.
3. J. H. Grace and A. Young, *Algebra of invariants* (Cambridge, 1903).
4. R. A. Johnson, "The conic as a space element", *Trans. Amer. Math. Soc.*, 15 (1914), 335-368.
5. G. Schaake, "The complex of the conics which cut five given straight lines", *Proc. Nat. Acad. of Sciences* (Amsterdam), 26 (1923), 513-521.
6. H. Schubert, *Kalkül der abzählende Geometrie* (Leipzig, 1879).
7. F. Severi, "Sulle intersezioni delle varietà algebriche e sopra i loro caratteri e singolarità proiettive", *Mem. di Torino* (2), 52 (1902), 61-118.
8. ———, "Fondamenti per la geometria sulle varietà algebriche", *Rend. di Palermo*, 28 (1909), 33-87.
9. ———, "Sulla varietà che rappresenta gli spazi subordinati di data dimensione, immersi in uno spazio lineare", *Annali di Mat.* (3), 24 (1915), 89-120.
10. W. Spottiswoode, "The eighteen coordinates of a conic in space", *Report of British Association* (Dublin, 1878), 462.
11. ———, "The twenty-one coordinates of a conic in space", *Proc. London Math. Soc.* (1), 10 (1879), 185-194.
12. E. Study, "Abbildung der Mannigfaltigkeit aller Kegelschnitte eines Ebene auf einen Punktraum", *Math. Annalen*, 40 (1892), 551-558.

A SPECIAL TYPE OF QUINTIC SYMMETROID (II)

By WILLIAM P. MILNE.

[Received 11 March, 1932.—Read 28 April, 1932.]

Introduction.

In a series of papers Mr. L. Roth has obtained the geometrical properties of the quintic symmetroid Γ, defined as the locus of a point $P \equiv (x_1, x_2, x_3, x_4)$ of three-dimensional space for which the hyperquadric

$$x_1 S_1 + x_2 S_2 + x_3 S_3 + x_4 S_4 = 0 \tag{1}$$

is a hypercone, where S_1, S_2, S_3, S_4 are four linearly independent hyperquadrics in four-dimensional space. Using Roth's results, I studied* the particular case when the linear system (or web) (1) contains a hyperquadric S_4 consisting of two quadritangent hyperplanes α, β of the curve C common to the three hyperquadrics S_1, S_2, S_3. Among other properties I showed that the equation to the symmetroid Γ can be expressed in the form

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0, \tag{2}$$

where a, b, h are each of the first order; f, g of the second order; and c of the third order in x_1, x_2, x_3, x_4. In the present paper, I propose to show, among other developments, that it is not necessary to postulate the existence of two quadritangent hyperplanes in order to obtain the result (2). It is sufficient if S_4 is an 8-tangent hyperquadric of the curve C; in other words, if S_1, S_2, S_3, S_4 intersect in eight pairs of coincident points. In the case when S_4 consists of two quadritangent hyperplanes α, β, two contact quartic surfaces of Γ (*e.g.* $bc-f^2=0$ and $ca-g^2=0$) each degenerate into two quadric cones. This does not happen in the more

* W. P. Milne, *Proc. London Math. Soc.* (2), 33 (1931), 165–176.

general case when S_4 is an 8-tangent hyperquadric of C and the web (1) does not contain two quadritangent hyperplanes.

1. *On a certain sextic curve K_0 of genus 2.*

I have shown* that to every system of 8-tangent hyperquadrics of the curve C corresponds a unique quartic hypersurface Φ having five nodal points and containing C as a double curve. This hypersurface Φ can be regarded as the envelope of the system of hyperquadrics which are the reciprocals of a fixed hyperquadric Σ with respect to a pencil of hyperquadrics $S+\lambda S'$. Φ can also be regarded as the locus of points whose polar planes with respect to $S+\lambda S'$ cut Σ in pairs of straight lines. Two 8-tangent hyperquadrics of C are said to be of the same family when a hyperquadric not containing C passes through the points of contact of each with C. There are 1023 families of 8-tangent hyperquadrics of C. The points of contact of all members of the same family defining Φ are the points of contact of the polar hypercones with respect to $S+\lambda S'$ of lines on Σ, where the polar hypercone of a line l with respect to the pencil $S+\lambda S'$ is defined to be the locus of a point whose polar plane with respect to the pencil $S+\lambda S'$ intersects l. These systems of eight points of contact can also be regarded as the eight residual points of intersection with C of the web of hyperquadrics through the five nodes of Φ and through the points of contact with C of any assigned polar hypercone of the above system. If we take the equation to the hyperquadric Σ in the form

$$x^2+yt+zu=0,$$

the polar hypercone of the line $x=y=z=0$ is

$$\Omega\equiv TU'-T'U\equiv(TU)=0, \tag{3}$$

where X, X' are the polar hyperplanes of the point (1, 0, 0, 0, 0) with respect to S, S' and similarly with respect to the other vertices. The general equation to the web of hyperquadrics through the five nodes of Φ and the points of contact of $\Omega\equiv(TU)$ with C is†

$$\lambda\{(YT)+(ZU)\}+\mu(XT)+\nu(XU)+\rho(TU)=0. \tag{4}$$

All members of the web (4) contain a sextic curve K_0 in common, and, in fact, constitute the general system of hyperquadrics containing K_0.

---

* W. P. Milne, *Proc. London Math. Soc.* (2), 26 (1927), 119–134; W. P. Milne, *Proc. London Math. Soc.* (2), 28 (1928), 484–492.

† W. P. Milne, *Proc. London Math. Soc.* (2), 28 (1928), 492.

The sextic K_0 is the curve of intersection of the cubic scroll

$$\frac{X}{X'} = \frac{T}{T'} = \frac{U}{U'} \qquad (5)$$

with the hyperquadric

$$(YT)+(ZU) = 0. \qquad (6)$$

We can easily obtain the general character and properties of the sextic curve K_0 as follows. A cubic scroll can be mapped on a plane by means of conics passing through a fixed point of the plane, which we take as one of the vertices of the triangle of reference $\xi = \eta = 0$. We obtain

$$x = \xi^2, \quad y = \eta^2, \quad z = \eta\zeta, \quad t = \zeta\xi, \quad u = \xi\eta.$$

Substituting these expressions in the equation to any hyperquadric, we obtain a plane quartic curve having a node at $\xi = \eta = 0$. *K_0 is therefore a sextic curve of genus* 2. The generators of the cubic scroll map into the pencil of lines through $\xi = \eta = 0$, and hence are bisecants of K_0. The point $\xi = \eta = 0$ corresponds to the directrix of the scroll (*i.e.* the unique line lying on the scroll which all the other generators intersect). It thus becomes evident that there exist corresponding pairs of points on K_0, which are mapped on the (ξ, η, ζ) plane into the two points in which a line through the node of the quartic meets the quartic again. On the other hand, they are the pairs of points in which the generators of the scroll meet K_0. Plainly, there are six cases of coincidence of these pairs of corresponding points on K_0, which map on the (ξ, η, ζ) plane into the six points of contact of the tangents from the node to the quartic curve.

The polar plane with respect to Σ of all points on the line $x = y = z = 0$ is the plane $y = z = 0$. Hence the polar planes of all points on $y = z = 0$ with respect to $S+\lambda S'$ intersect on the line

$$X+\lambda X' = 0, \quad T+\lambda T' = 0, \quad U+\lambda U' = 0, \qquad (8)$$

which is a generator of the cubic scroll (5). The reciprocal of Σ with respect to $S+\lambda S'$ is

$$(X+\lambda X')^2+4(Y+\lambda Y')(T+\lambda T')+4(Z+\lambda Z')(U+\lambda U') = 0. \qquad (9)$$

The plane

$$T+\lambda T' = U+\lambda U' = 0 \qquad (10)$$

obviously intersects the hyperquadric (9) in two coincident straight lines, viz. the line (8) taken twice over. The plane (10) is the polar plane of the line $x = y = z = 0$ with respect to $S+\lambda S'$, and is a generating plane of the hypercone $\Omega \equiv (TU) = 0$. The hypercone Ω and the hyperquadric (9) will therefore define a pencil of hyperquadrics, which in addition to Ω

possesses four other hypercones coinciding into two pairs, their vertices lying on the line (8). We propose to show that *the two vertices of such coincident hypercones constitute corresponding points on the sextic curve* K_0.

Let $$(x_1, y_1, z_1, t_1, w_1) \tag{11}$$

be the vertex of Ω. Let $X_1, \ldots$ be the expressions resulting from the insertion of $(x_1, y_1, z_1, t_1, u_1)$ in $X, \ldots$. Thus

$$T_1 = T_1' = U_1 = U_1' = 0. \tag{12}$$

The enveloping hypercone from the point (11) to the hyperquadric (9) is therefore

$$\begin{aligned}(T+\lambda T')\,[&(X_1+\lambda X_1')^2(Y+\lambda Y')-(Y_1+\lambda Y_1')^2(T+\lambda T')\\ &-(X_1+\lambda X_1')(Y_1+\lambda Y_1')(X+\lambda Y')\\ &-(Y_1+\lambda Y_1')(Z_1+\lambda Z_1')(U+\lambda U')]\\ +(U+\lambda U')\,[&(X_1+\lambda X_1')^2(Z+\lambda Z')-(Z_1+\lambda Z_1')^2(U+\lambda U')\\ &-(X_1+\lambda X_1')(Z_1+\lambda Z_1')(X+\lambda X')\\ &-(Y_1+\lambda Y_1')(Z_1+\lambda Z_1')(T+\lambda T')] = 0. \end{aligned} \tag{13}$$

Also $$\Omega \equiv (TU) \equiv (T+\lambda T')\,U' - (U+\lambda U')\,T' = 0. \tag{14}$$

Hence the polar hyperplane of the point (11) with respect to the hyperquadric (9) intersects (13) and (14) in two quadrics cutting each other in a twisted cubic and the line (8). Two cones containing this twisted cubic have their vertices at the points where the line (8) cuts

$$\begin{aligned}T'\,[&(X_1+\lambda X_1')^2(Y+\lambda Y')-(Y_1+\lambda Y_1')^2(T+\lambda T')\\ &-(X_1+\lambda X_1')(Y_1+\lambda Y_1')(X+\lambda X')\\ &-(Y_1+\lambda Y_1')(Z_1+\lambda Z_1')(U+\lambda U')]\\ +U'\,[&(X_1+\lambda X_1')^2(Z+\lambda Z')-(Z_1+\lambda Z_1')^2(U+\lambda U')\\ &-(X_1+\lambda X_1')(Z_1+\lambda Z_1')(X+\lambda X')\\ &-(Y_1+\lambda Y_1')(Z_1+\lambda Z_1')(T+\lambda T')] = 0,\end{aligned}$$

i.e. $$T'(Y+\lambda Y')+U'(Z+\lambda Z') = 0,$$

i.e., using (8), $$(YT)+(ZU) = 0,$$

which establishes the proposition.

Another noteworthy property of K_0 is the following:

The cubic scroll (5) *is a contact surface of the quartic hypersurface* Φ, *the curve of contact being the sextic* K_0.

The equation to Φ is

$$\{(YT)+(ZU)\}^2-(XY)(XT)-(XZ)(XU)-4(YZ)(TU)=0. \quad (15)$$

Hence we see that the cubic scroll $(XT)=(XU)=(TU)=0$ touches Φ where it is met by $(YT)+(ZU)=0$, *i.e.* along the curve K_0.

Plainly, also, K_0 passes through the vertex of Ω, since (12) satisfies every member of the web (4).

2. *The Jacobian of the web of hyperquadrics through the eight pairs of coincident points common to* C *and* Ω.

Roth has shown* that in general the symmetroid Γ, as defined in the introduction to this paper, is a surface of order 5 possessing twenty nodes corresponding to the twenty line-hypercones of the web (1), and eight further nodes corresponding to the eight points of contact of Ω with C. Let O be the point of Γ corresponding to Ω. To every line through O will correspond a pencil of hyperquadrics of the web (1), one member of which is Ω. To the four points other than O in which the line cuts Γ will correspond the other four members of the pencil that are hypercones. Consider the pencil defined by Ω and the hyperquadric (9). We have shown that the four hypercones of the pencil, other than Ω, consist of two coincident pairs. Hence *the line through* O *corresponding to the pencil defined by* Ω *and the hyperquadric* (9) *touches* Γ *at two points other than* O. Let us denote by K the curve on Γ described by these two points of contact. There will therefore be a (1, 1) correspondence between the points of K and K_0, showing that K *will be of genus* 2. Furthermore, to the points in which a plane cuts K will correspond the nodes in those hypercones of the web (1) that belong to a net and lie on K_0, *i.e.* the five nodes of Φ. Hence K *is of order* 5. Now it is known that a quintic of genus 2 in three-dimensional space lies on a quadric, the generators of one system of which are trisecants of the curve. We have shown that in the present case the trisecants meet at a point of the curve. Hence K *lies on a quadric cone* k_2, *whose vertex is* O, *the point of* Γ *corresponding to* Ω, *and* k_2 *is a contact quadric cone of* Γ. It is also known that a quintic of genus 2 in three-

* L. Roth, *Proc. Camb. Phil. Soc.*, 25 (1929), 268–271; *Proc. London Math. Soc.* (2), 30 (1930), 118–126, 425–432.

dimensional space is the residual curve of intersection of a quadric and a cubic surface having a line in common and that a linear system of ∞^5 cubic surfaces contain the curve. Hence any one of these cubic surfaces containing K intersects Γ in a residual curve of order 10, along which Γ is touched by a quartic surface Ψ. This cubic surface also cuts Ψ in a conic along which Ψ is touched by a plane. Thus Ψ is a quartic surface possessing a trope, and hence has an equation of the form

$$\Psi \equiv ac - g^2 = 0, \tag{16}$$

where a, g, c are of orders 1, 2, 3 respectively in the coordinates. The general equation to any cubic surface containing the conic $a = g = 0$ is

$$af - gh = 0, \tag{17}$$

where h, f are of orders 1, 2 respectively in the coordinates. Plainly,

$$2fgh - af^2 - ch^2 = 0 \tag{18}$$

is a quintic surface touching Ψ along the curve in which Ψ is met by the cubic surface (17). Hence

$$b(ac - g^2) + (2fgh - af^2 - ch^2) = 0 \tag{19}$$

is the general equation to the quintic surface touching Ψ along the curve in which it is met by the cubic surface (17), b being linear in the coordinates. Thus we see that *the symmetroid* Γ *can have its equation expressed in the form*

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0,$$

where a, b, h *are each of the first order*; f, g *of the second order*; *and* c *of the third order respectively in the coordinates.*

3. *Properties of the Jacobian and the symmetroid.*

There is no need to give in detail the properties of the symmetroid that spring immediately from its equation having the above determinantal form, since these were fully examined in my previous paper on "A special type of quintic symmetroid"*. We discuss at present only those properties that differ in the present more general type of symmetroid.

* W. P. Milne, *Proc. London Math. Soc.* (2), 33 (1931), 165–176.

The polar surface of O with respect to Γ is of order 4 and hence intersects Γ in a curve of order 20, which has already been shown to consist in part of the quintic curve K. There is therefore a residual curve K' of order 15. To K corresponds a sextic curve K_0 on the Jacobian J. Since to the curve of intersection of the polar quartic of O with Γ corresponds the curve of intersection of Ω with J, and since J is of order 10, we see that *to K' corresponds on J a curve K_0' of order* 14.

Let P be a point in which K meets K'. The polar surface of O touches Γ at P and hence the tangent plane at P to Γ intersects Γ in a quintic curve having a cusp at P, such that the cuspidal tangent passes through O. But if a tangent from O to Γ meets Γ in three coincident points, it must perforce meet Γ in four coincident points, since every line joining O to a point of K meets Γ in two pairs of coincident points. Hence the line OP must touch K at P. Thus *K' intersects K at the six points where the tangents to K pass through O.* But K' intersects k_2 in thirty points, which include (i) two coincident points at its vertex O, (ii) six pairs of coincident points at the six points where the tangents to K pass through O, and (iii) the eight nodes of Γ corresponding to the eight hypercones of the web (1), whose vertices lie at the points of contact of C with Ω. We are left with eight further intersections of K and K', which are therefore situated at nodes of Γ corresponding to eight line-hypercones of the web (1). *There are eight line-hypercones of the web* (1), *whose axes meet once each of the curves K_0 and K_0'. There remain twelve line-hypercones, of the web* (1), *whose axes are each bisecants of K_0'*, as is evident from the fact that the polar surface of O with respect to Γ must intersect Γ at each node in a curve having a double point thereat, whence K' must have a double point at each node of Γ through which K does not pass. *The twenty nodes of Γ corresponding to line-hypercones of the web* (1) *are therefore divided into two types, eight through which K passes, and twelve through which K does not pass. At each of the latter K' possesses a double point.*

4. *The contact quadric cones of the special symmetroid* Γ.

We proceed now to consider once again the more specialized case, when the web (1) contains two quadritangent hyperplanes α and β of C. We know from my previous paper* that α intersects the Jacobian surface J in two twisted cubics V_0, V_0', and the plane cubic σ in which the plane $\alpha\beta$ cuts J. Similarly we know that β intersects J in two twisted cubics

* W. P. Milne, *Proc. London Math. Soc.* (2), 33 (1931), 165–176.

W_0, W_0', and in σ. Furthermore it is known that K_0 cuts α in its four points of contact with C and in two further points. Now a hypercone which belongs to the web (1) and has its vertex on α must have its vertex lying on one or other of the two twisted cubics V_0, V_0' lying in α. Hence, since every point of K_0 is the vertex of a hypercone of the web (1), the two points in which K_0 intersects α must be either on V_0 or V_0'. We propose to show that they cannot both lie on the same twisted cubic V_0. For, if so, V_0 would be the twisted cubic through the six points in which α meets K_0, and hence would be the curve of intersection of α with the cubic scroll containing K_0. The twisted cubic V_0 would therefore lie on Ω which contains the cubic scroll, and hence would be part of the curve of intersection of Ω and J, since V_0 is known to lie on J. This is impossible, since it is known that the tangent planes to Γ at all points of V do not pass through O, which is not coincident with the trinode D of Γ. Hence *K_0 intersects V_0, V_0', W_0, W_0' each in one point in addition to the points of contact of α and β with C.* It follows that *K intersects V, V', W, W' each in one ordinary point of Γ, and hence that k_2 touches each of the cones v_2, v_2', w_2, w_2' at one ordinary point of Γ.* Now V intersects k_2 in ten points, four of which are nodes of Γ corresponding to the four points of contact of α with C, and two are coincident points where v_2 touches k_2. We are left with four points of intersection which must therefore be nodes of Γ, since we have seen that v_2 and k_2 cannot touch at these points. Hence *K intersects V, V', W, W' each in nine points.*

We have seen that an ∞^5 linear system of cubic surfaces contain K, hence a unique cubic surface can in general be made to contain K and pass through five further arbitrarily assigned points. Let us choose D, the trinode of Γ, which is known also to be a trinodal point on V, as one of these five points together with four other arbitrarily chosen points on V (other than its nine intersections with K). The unique cubic surface Q, containing K and passing through these five points, intersects V (i) in the nine points it has in common with K, (ii) in three points at the trinode of V, (iii) in four ordinary points of V, *i.e.* sixteen points in all. Hence Q must contain V. Thus, *a unique cubic surface Q can be found to contain both K and V.* Q must plainly have a node at D, the trinode of V, such that the nodal cone of Q at D contains the three trinodal tangents of V at D. The contact cone v_2 plainly touches the trinodal cone of Γ along the three trinodal tangents of V at D. Since k_2, v_2 are each contact cones of Γ, of which K, V are the curves of contact, it is evident that Q, which contains both K and V, must intersect Γ in a residual curve of order 5, which is the curve of contact of a quadric cone with Γ. This residual curve of order 5 plainly has a trinode at D, the trinodal tangents being the rays

of contact of the quadric cone containing the curve with the trinodal cone of Γ. The nodal cone of Q plainly contains the trinodal tangents of this residual curve of order 5. Now we know* that four enveloping quadric cones v_2, v_2', w_2, w_2' can be drawn from D to Γ, such that the trinodal tangents of V, W lie on the same quadric cone and that this property is not possessed by either V, V', or V, W'. Hence *the cubic surface Q containing K and V contains also W; and similarly the cubic surface Q' containing K, V' contains also W'.*

It is important also to state the matter from another standpoint.

If α, β are two quadritangent hyperplanes of C, the locus of points corresponding to those members of the web (1) *which are hypercones is a quintic symmetroid Γ having a trinode at D. The enveloping cone from D resolves itself into four quadric cones v_2, v_2', w_2, w_2', whose curves of contact are V, V', W, W', each of which possesses a trinode at D. The trinodal tangents of V, W, and V', W' each lie on quadric cones, namely the nodal cones of uniquely defined cubic surfaces Q and Q' containing V, W, and V', W' respectively. Q and Q' each intersect Γ in the residual quintic curve K, which is the curve of contact with Γ of a contact quadric cone k_2 whose vertex is O. To O corresponds a hypercone of the web* (1) *which is the polar hypercone of a line on Σ with respect to the pencil $S+\lambda S'$.*

To every plane corresponds a net of hyperquadrics of the web (1). Let, in particular, γ be the plane corresponding to the net of hyperquadrics containing C. γ intersects Γ in a quintic curve having the sections of γ with the quadric cones v_2, v_2', w_2, w_2', k_2 each as contact conics of the plane quintic. To the points of γk_2 correspond those hyperquadrics containing C that envelop the quartic hypersurface Φ, whose five nodes are the vertices of the five hypercones of the net corresponding to the five points in which the conic of γk_2 touches the quintic $\gamma\Gamma$. γv_2 and $\gamma v_2'$ are the contact conics of $\gamma\Gamma$ corresponding to those hypercones of the C-net whose vertices lie in the quadritangent hyperplane α, and similarly with regard to γw_2, $\gamma w_2'$, and β. A cubic curve γQ passes through the points of contact of γv_2, γw_2, γk_2 with $\gamma\Gamma$, and similarly $\gamma Q'$ passes through the points of contact of $\gamma v_2'$, $\gamma w_2'$, γk_2 with $\gamma\Gamma$. Hence we have the following result:

If α, β are two quadritangent hyperplanes of C, which together constitute a degenerate contact hyperquadric of C defining the quartic hypersurface Φ†, and if to α correspond the two contact conics γv_2, $\gamma v_2'$ of $\gamma\Gamma$ and to β the two contact conics γw_2, $\gamma w_2'$ of $\gamma\Gamma$, two cubic curves γQ and $\gamma Q'$ pass through

* W. P. Milne, *Proc. London Math. Soc.* (2), 33 (1931), 165-176.

† W. P. Milne, *Proc. London Math. Soc.* (2), 26 (1927), 119-134; 28 (1928), 484-492.

the points of contact of γv_2, γw_2, *and* $\gamma v_2'$, $\gamma w_2'$ *respectively, and each intersect* $\gamma\Gamma$ *in five residual points that are the points of contact of a contact conic* γk_2, *to the points of contact of which with* $\gamma\Gamma$ *correspond the five nodes of* Φ.

It has already been seen* that the cubic surface Q contains the curves K, V, W and intersects the cones k_2, v_2, w_2 in three coplanar lines residual to K, V, W respectively, and similarly with regard to Q' and K, V', W'. The two planes thus defined were shown to be the two planes containing the line OD and touching k_2, and the lines residual to K on k_2 with respect to Q and Q' are the generators of contact with k_2 of these two planes. Also it has been shown that one of these generators intersects K in the points where it is met by V, W respectively, and similarly with regard to V', W'.

Thus the points where K_0 *intersects* V_0, W_0 *on* J (*other than the points of contact with* C *of the quadritangent hyperplanes* α, β) *are corresponding points on* K_0, *and hence the line joining them is a generator of the cubic scroll and lies upon* Ω; *similarly with regard to* V_0', W_0'.

It remains to point out that the eight points of contact of Ω with C are not eight points of general character upon the cubic scroll. For it has been seen that the reciprocal of Σ with respect to any member of the pencil $S+\lambda S'$ cuts the cubic scroll in a generator and hence in a residual quintic curve through the eight points of contact of Ω with C. It is known that any hyperquadric cuts the cubic scroll in a sextic curve that maps itself upon the plane of (ξ, η, ζ), according to the scheme (7), into a quartic curve having a node at $\xi = \eta = 0$. Since a generator of the cubic scroll maps into a line through $\xi = \eta = 0$, it is evident that the quartic curve will degenerate into a line and a cubic curve each passing through $\xi = \eta = 0$. Thus all quintics of the above system through the eight points of contact of Ω with C map on the plane of (ξ, η, ζ) into cubic curves through $\xi = \eta = 0$, and are therefore of genus 1. The eight points of contact of Ω with C therefore map into a system of eight points on the plane of (ξ, η, ζ), which, together with $\xi = \eta = 0$, form an associated system of nine points. *Every such quintic curve of genus* 1 *through the eight points of contact of* Ω *with* C *and lying upon the cubic scroll intersects the directrix of the cubic scroll in one point.*

* W. P. Milne, *Proc. London Math. Soc.* (2), 33 (1931), 165–176.

RELATION BETWEEN THE QUADRITANGENT HYPERPLANES OF NOETHER'S CANONICAL CURVE FOR $p = 5$ AND THE 5-TANGENT CONICS OF THE PLANE QUINTIC CURVE

By WILLIAM P. MILNE.

[Received 11 May, 1932.—Read 19 May, 1932.]

Introduction.

Some years ago I obtained* the complete family of hyperquadrics of the same system which touch at eight points Noether's canonical curve C for $p = 5$, where C is the curve of intersection of three hyperquadrics in space of four dimensions, two 8-tangent hyperquadrics being regarded as of the same system, when another hyperquadric not belonging to the net containing C passes through the eight points of contact of both. I showed that to each such system of 8-tangent hyperquadrics corresponds a unique quartic hypersurface Φ, containing C as a double curve and possessing five nodal points, lying on the decimic curve J, the Jacobian of the net of hyperquadrics containing C (*i.e.* the locus of the vertices of the hypercones of the net). It was also shown that Φ may be regarded as the envelope of the reciprocal of a fixed hyperquadric Σ with respect to a pencil of hyperquadrics $S+\lambda S'$.

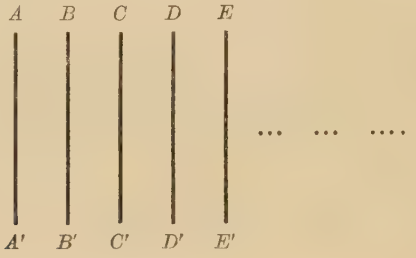
By considering the net of hyperquadrics $\xi U+\eta V+\zeta W=0$ containing C and regarding (ξ, η, ζ) as the coordinates of a point in a plane π, I also proved† that the general plane quintic curve can be regarded as the discriminant Δ of a net of hyperquadrics in space of four dimensions, and that to the 1023 systems of 8-tangent hyperquadrics of C correspond 1023 contact conics $\bar{K}$ of the discriminant-quintic Δ. It was also proved that to the 496 quadritangent hyperplanes of C correspond in pairs A, A', etc. $2\times 496 = 992$ further 5-tangent conics of Δ, where A, A' together

---

* W. P. Milne, *Proc. London Math. Soc.* (2), 28 (1928), 484–492.

† W. P. Milne, *Journal London Math. Soc.*, 2 (1927), 79–84.

constitute the locus of a point (ξ, η, ζ) in π, the hyperquadric of the net containing C corresponding to which touches the quadritangent hyperplane α. These 992 contact conics of Δ it will be convenient to represent in pairs on parallel lines, thus:



(1)

If a quartic passes through the points of contact of any four contact conics of Δ, we shall say that they form a " congruent tetrad " of contact conics. Thus the pair of contact conics A, A' is congruent with B, B' and C, C', etc.

We shall also say that the 1023 contact conics $\bar{K}$ constitute Group I, and the 992 contact conics A, A', B, B', etc., constitute Group II. The Groups I and II together account for all the 2015 contact conics of Δ as revealed by the Abelian characteristics of the half-periods for $p = 6$, it being noted that two coincident lines together constitute a degenerate contact conic, thus giving $2^{p-1}(2^p-1) = 2^5(2^6-1) = 2016 = 2015+1$ for the general case $p = 6$.

Later on, F. P. White* obtained another interesting grouping of the 5-tangent conics of the quintic by considering those planes containing a fixed line on the cubic hypersurface in space of four dimensions that cut the cubic hypersurface in a residual conic which degenerates into two straight lines. The point of intersection of such a plane with a fixed plane generates a quintic curve. White then obtained the following configuration of 5-tangent conics:

Given a 5-tangent conic of the quintic, 495 cubics pass through the points of contact of this conic with the quintic, so as to cut the quintic in ten residual points at which it is touched by two further 5-tangent conics.

This configuration of 5-tangent conics can also be regarded as a particular case of the configuration (1). For the configuration (1) can be regarded as established from a study of the characteristics of the half-periods of the Abelian integrals of the first kind for the general case of

* F. P. White, *Proc. London Math. Soc.* (2), 30 (1930), 347-358.

$p = 6$. If we then regard A' as consisting of two coincident straight lines, we see that the quartic through the points of contact of A, A', B, B' with Δ degenerates into one of the two coincident straight lines constituting A', together with a cubic through the points of contact of A, B, B'. Proceeding in this way, all the 495 cubics through the points of contact of A and two further 5-tangent conics are accounted for.

We shall designate three 5-tangent conics through whose points of contact with the quintic a cubic passes as a "triadic set" of three 5-tangent conics. More generally, if a set of $2n+1$ contact conics of the quantic is such that n cubics pass through the points of contact of any member of the set and also through the points of contact of two further members of the set, we shall designate such a set as a "triadic set" of $2n+1$ contact conics of the quintic.

1. *Pairs of quadritangent hyperplanes of C of the same system.*

A fundamental problem in the study of the canonical curve C for $p = 5$ is plainly the configuration of pairs of quadritangent hyperplanes that constitute degenerate 8-tangent hyperquadrics of the same system. We propose to deal with this problem in the present paper in terms of the 5-tangent conics of the plane quintic curve. We shall show that the configuration of quadritangent hyperplane-pairs of C belonging to the same system corresponds to those 5-tangent conics of Δ that are common to the above grouping obtained by White in studying the cubic hypersurface and to the grouping (1) dealt with by myself*. We shall require the results obtained by me in two papers on "a special type of quintic symmetroid"†. Let α and β be two quadritangent hyperplanes of C. Let Δ be the corresponding discriminant quintic of the net of hyperquadrics containing C as explained in my first paper*. It was there proved that to α correspond two 5-tangent conics A, A' and to β correspond B, B'. It was also proved† that a cubic U passes through the points of contact of A and B, and intersects Δ in five residual points which are the points of contact of Δ with a 5-tangent conic $\bar{K}$. Similarly a cubic U' passes through the points of contact of A', B', and $\bar{K}$. A quartic passes through the points of contact of A, A', B, B', it will frequently be described as the quartic $AA'BB'$. The cubic U will also be described as the cubic $AB\bar{K}$. To the five points of contact of $\bar{K}$ with Δ correspond

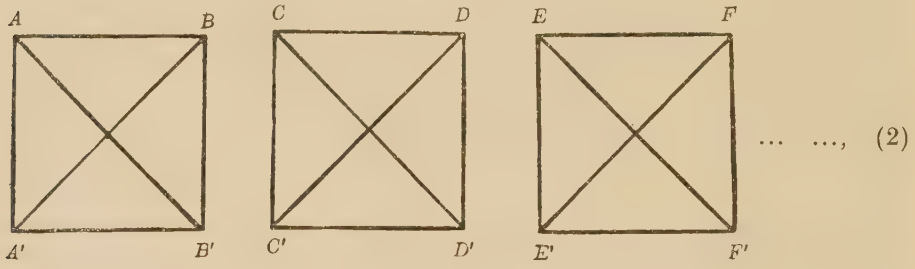
* See foot-note †, p. 217.

† W. P. Milne, *Proc. London Math. Soc.* (2), 33 (1932), 165-176; W. P. Milne, *Proc. London Math. Soc.* (2), 36 (1933), 207-216.

the five nodal points of the quartic hypersurface Φ defined by the degenerate 8-tangent hyperquadric α, β of C. We know that there are ${}_{496}C_2$ pairs of quadritangent hyperplanes α, β of C, which on being regarded as degenerate 8-tangent hyperquadrics of C define 1023 quartic hypersurfaces Φ. Hence the same quartic hypersurface Φ will be defined by ${}_{496}C_2 \div 1023 = 120$ pairs of quadritangent hyperplanes α, β, where each such pair belongs to the same system of 8-tangent hyperquadrics.

To each of the 120 *pairs* α, β *of quadritangent hyperplanes of* C *belonging to the same system of* 8*-tangent hyperquadrics of* C*, and hence defining the same quartic hypersurface* Φ*, correspond* 120 *pairs of cubics* U, U' *each passing through the points of contact of the* 5*-tangent conic* $\bar{K}$ *(whose five points of contact with* Δ *correspond to the five nodes of* Φ*) and each intersecting* Δ *respectively in ten residual points, which are the points of contact of two* 5*-tangent conics* A, B *and* A', B'*, where the points of contact of these four conics with* Δ *lie on a quartic.*

Viewed from a more general standpoint, this immediately leads to a more general grouping of the 5-tangent conics of Δ. Let $\bar{L}$ be a degenerate 5-tangent conic of Δ consisting of two coincident straight lines. Then, if we combine the cubic $\bar{K}AB$ with one of the two coincident straight lines together constituting $\bar{L}$, we obtain a degenerate quartic curve $\bar{K}\bar{L}AB$ passing through the points of contact of these four conics. Generalizing this point of view, we begin with the configuration (1), and use the characteristics of the Abelian half-periods for the general case of $p = 6$ to obtain a second configuration consisting of 120 squares,



... ..., (2)

such that the points of contact of four conics represented by pairs of points lying on sides of the squares parallel to AA' or on one of the remaining 256 lines of the configuration (1) lie on a quartic, while four conics represented by pairs of points on sides of the squares parallel to AB or on one of other 256 lines $\bar{K}\bar{L}$ (regarded as parallel to AB and $\bar{L}$ being regarded as undegenerate) also lie on a quartic. Finally, any quartic through the points of contact of any pair of contact conics represented by a pair of

points at the ends of a diagonal of any one of the above squares, *e.g.* A, B', cuts Δ in ten residual points which are the points of contact with Δ of a quartic that may degenerate into a pair of contact conics represented by points on a diagonal (*e.g.* A', B, or C, D', or C', D) or into one or other of 256 pairs of conics, the above three sets of 256 pairs of conics being entirely distinct from one another and having no members in common. Thus the number of contact conics of Δ accounted for is

$$4\times 120+2\times 256+2\times 256+2\times 256 = 2016,$$

which exhausts the entire set of 5-tangent conics. Among these 2016 contact conics is one which degenerates into a pair of coincident straight lines, for the particular case of $p = 6$ representing the plane quintic curve.

In the particular case which we have had under consideration, viz. that of the 120 *hyperplane-pairs of C belonging to the same system, one of the* 256 *pairs of contact conics defined by quartics through the points of contact of conics represented by pairs of points lying on sides of the squares parallel to AB contains the degenerate conic* $\bar{L}$ *consisting of two coincident straight lines. The two pairs of conics represented by points on the sides parallel to AA′ of the same one of the* 120 *squares correspond to the* 120 *quadritangent hyperplane-pairs of C belonging to the same system of* 8*-tangent hyperquadrics.*

Pairs of contact conics of Δ congruent with those pairs represented by A, B and sides of the squares (2) parallel to AB and similarly with regard to the diagonals of the squares (2) belong to Group I; and if we reject the degenerate contact conic consisting of two coincident straight lines, we are left with $2\times 256+2\times 256-1 = 1023$ contact conics belonging to Group I, which we know to be correct. Thus, of the 495 cubics through the points of contact of $\bar{K}$ with Δ that pass through the points of contact of two further 5-tangent conics belonging to Group II, $2\times 120 = 240$ cubics U have been accounted for. *There remain* 255 *cubics* $\bar{U}$ *that pass through the points of contact of* $\bar{K}$ *and two further conics of the Group I.*

It is plain that *no quartic through the points of contact of a pair of* 5*-tangent conics A, A′ of the system* (1) *corresponding to a quadritangent hyperplane* α *can pass through the points of contact of a* 5*-tangent conic* $\bar{G}$ *of Group I.* For, if so, the quartic would cut Δ in five residual points which would also be the points of contact of a 5-tangent conic $\bar{G}'$. Thus the pair of 5-tangent conics $\bar{G}$, $\bar{G}'$ would together correspond to a quadritangent hyperplane of C. We know that all such conics belong to Group II.

Let us now isolate the quadritangent hyperplane α, to which correspond the two 5-tangent conics A, A'. To β correspond B, B'.

A cubic passes through the points of contact of A and B. Similar results hold with regard to the quadritangent hyperplanes γ, δ, ϵ, etc. Hence, *since there are* 495 *quadritangent hyperplanes of* C *other than* α, *if we isolate the* 5*-tangent conic* A, *there are* 495 *other* 5*-tangent conics belonging to Group II, the points of contact of each of which together with those of* A *lie on a cubic.* It follows immediately that, *if we isolate the* 5*-tangent conic* A *belonging to Group II, there exist* 495 *contact conics* K *of Group I the points of contact of each of which together with those of* A *lie on a cubic.*

Thus the configuration of 495 pairs of 5-tangent conics described by White is accounted for, both in the case when a conic of Group I and a conic of Group II are isolated.

We have now discussed in some detail the case of when the degenerate 5-tangent conic of Δ consisting of two coincident straight lines is represented by a point which is not the corner of a square in the configuration (2), on account of its importance with regard to the configuration of 120 hyperplane-pairs of the same system of C.

We shall now discuss the case when the point representing the degenerate 5-tangent conic consisting of two coincident straight lines lies at the corner of a square of the configuration (2) considered as a general system established by the study of the characteristics of $p = 6$, and is in fact A'. Since, in the configuration (2), a quartic passes through the points of contact of A, B, A', B' with Δ, where A' consists of two coincident straight lines, we see that the quartic $ABB'A'$ will degenerate into the cubic ABB' and one of the lines constituting A'. Similarly we have the cubics ACC', ADD', BCD', $BC'D$, $B'C'D'$, $B'CD$, which is a triadic set of seven contact conics. Hence, *if, in the configuration* (2), *the conic* A' *consists of two coincident straight lines, the conics* A, B, B', C, D, D', C' *form a triadic set of seven contact conics and similarly with regard to all the other squares of* (2). *Given three* 5*-tangent conics* A, B, B' *such that a cubic passes through their points of contact, there are* 119 *other tetrads of conics* C, D, D', C', *etc., each forming with* A, B, B' *a triadic set of seven contact conics. Through the five points of contact of each of* A, B, B' *pass* 256 *cubics* (*other than those belonging to the triadic sets*), *on each of which lie the points of contact of other two* 5*-tangent conics.*

I was first informed in conversation of the existence of triadic sets of seven 5-tangent conics of the plane quintic curve some years ago by Mr. F. P. White.

We note in passing that the congruences of a square of (2) can be represented analytically as follows:

$$A \equiv (x, y), \quad B \equiv (-x, y), \quad A' \equiv (x, -y), \quad B' \equiv (-x, -y),$$

by manipulating the signs of $(\pm x, \pm y)$. If A is taken to correspond to B, we change the sign of x and keep y unaltered. Changing the sign of x for A', we obtain B'. Similarly, if A is taken to correspond to B', we change the signs of both x and y. If we change the signs of both x and y for B, we obtain A', and so on.

2. *Some geometrical properties.*

Following Hesse's* reasoning in the usual way, we see that to the twenty points of intersection of the Jacobian J of the net of hyperquadrics containing C correspond in the plane π the twenty points in which a quartic curve meets the discriminant quintic Δ, and conversely. Hence, since a fixed cubic V and an arbitrary line l in π can together be regarded as constituting a net of plane quartics, we see that to the points of intersection of the quartic lV with Δ correspond the points of intersection of a net of hyperquadrics with J, viz. (i) the fifteen fixed points on J corresponding to the fifteen points in which V cuts Δ, (ii) the five variable points on J which are the five vertices of a pencil of hyperquadrics of the net containing C corresponding to the five points in which l cuts Δ. Now, it is known that, in general, nine points on Δ define a cubic V uniquely, and hence the remaining six points of intersection of V with Δ are, in general, known, if the original nine points are assigned. If the nine corresponding points on J are regarded as given, and if five points are taken on J which are the vertices of the hypercones of a pencil of the net of hyperquadrics containing C, we have $9+5=14$ points on J which uniquely define a hyperquadric. This hyperquadric cuts J in the $2\times 10-14=6$ residual points corresponding to the six residual points in which V intersects Δ. Also, to the fifteen points of intersection of a cubic V with Δ correspond fifteen points on J which lie also on the curve of intersection C' of three hyperquadrics and are such that every hyperquadric containing C' cuts J in five residual points that are the vertices of five hypercones of a pencil of hyperquadrics belonging to the net containing C.

Let us consider now the particular case in which V is the cubic $U\equiv \bar{K}AB$. We have seen† that to the points of A, B correspond the points of two twisted cubics A_3, B_3 lying in the quadritangent hyperplanes α, β respectively, and each passing through five of the ten points of intersection of α, β with J respectively corresponding to the five points

* O. Hesse, *Journal für Math.*, 49 (1855), 243–264.

† See foot-note †, p. 217 and foot-note †, p. 219.

of contact of each of A, B with Δ. [This result is obtained by considering the locus of a point (ξ, η, ζ) in π, the hyperquadrics of the net containing C corresponding to which touch α or β.] Hence, *to the fact that a cubic* $U \equiv \bar{K}AB$ *passes through the points of contact of the* 5*-tangent conics* $\bar{K}$, A, B *with* Δ *corresponds the fact that a net of hyperquadrics passes through the five nodes of* Φ *defined by the hyperplane-pair* α, β *and through the ten points of intersection with* J *of the twisted cubics* A_3 *and* B_3, *together with the fact that every such hyperquadric cuts* J *in five residual points that are the five vertices of a pencil of hypercones belonging to the net of hyperquadrics containing* C. *There are* 240 *such nets of hyperquadrics corresponding to the* 2×120 *cubics* U *through the isolated* 5*-tangent conic* $\bar{K}$.

Thus, given the hyperplane-pair α, β regarded as a degenerate 8-tangent hyperquadric of C, we obtain directly the five nodes of the corresponding quartic hypersurface Φ. For, let α intersect J in the ten points in which it is met by the twisted cubics A_3, A_3', which together constitute the locus of the vertices of the cones of the net of quadrics in α through the eight points (*i.e.* the four pairs of coincident points) in which the quadritangent hyperplane α meets C; and similarly with regard to B_3 and B_3'. We know that hyperquadrics through the ten points in which A_3, B_3 meets J, and through the five vertices of the hypercones of any pencil of hyperquadrics belonging to the net containing C, cut J in the five nodes of Φ; and similarly with regard to A_3', B_3'.

It is noteworthy that, although twelve points in general define a curve C' which is the curve of intersection of three hyperquadrics, *such a curve* C' *intersects* J *in the ten points where it is met by* A_3, B_3 *and in the five nodes of* Φ, *i.e. fifteen points in all.*

Corresponding to the fact that there are 255 *cubics through the points of contact of* $\bar{K}$ *and two other* 5*-tangent conics belonging to Group I, there are* 255 *nets of hyperquadrics which contain the nodes of* Φ *and of two further similar quartic hypersurfaces* Φ_1 *and* Φ_2, *and which each in addition cut* J *in five residual points which are the vertices of the five hypercones of a pencil of hyperquadrics belonging to the net containing* C. *Also a curve* C' (*which is the curve of intersection of three hyperquadrics*) *passes through the nodes of* Φ, Φ_1, Φ_2.

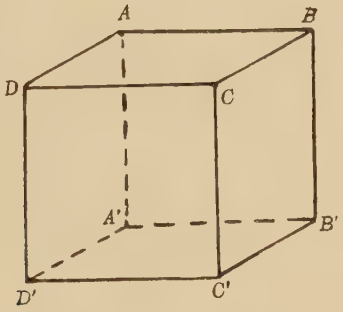
3. *Two pairs of quadritangent hyperplanes of* C *belonging to the same system of* 8*-tangent hyperquadrics.*

Let us consider now two of the squares of the configuration (2), and for convenience let us take the first two squares $ABB'A'$ and $CDD'C'$. Their

characteristics for the general case $p = 6$ may be exemplified as follows:

$$\begin{array}{cccc} A & B & C & D \\ \left[\begin{smallmatrix}1&0&0&1&1&1\\0&0&0&1&0&0\end{smallmatrix}\right] & \left[\begin{smallmatrix}0&0&0&1&1&0\\1&1&0&1&0&1\end{smallmatrix}\right], & \left[\begin{smallmatrix}1&0&0&1&1&1\\0&0&1&1&0&0\end{smallmatrix}\right] & \left[\begin{smallmatrix}0&0&0&1&1&0\\1&1&1&1&0&1\end{smallmatrix}\right] \\ \left[\begin{smallmatrix}0&0&0&1&1&0\\1&0&0&1&0&1\end{smallmatrix}\right] & \left[\begin{smallmatrix}1&0&0&1&1&1\\0&1&0&1&0&0\end{smallmatrix}\right], & \left[\begin{smallmatrix}0&0&0&1&1&0\\1&0&1&1&0&1\end{smallmatrix}\right] & \left[\begin{smallmatrix}1&0&0&1&1&1\\0&1&1&1&0&0\end{smallmatrix}\right]. \\ A' & B' & C' & D' \end{array}$$

We can now form a cube



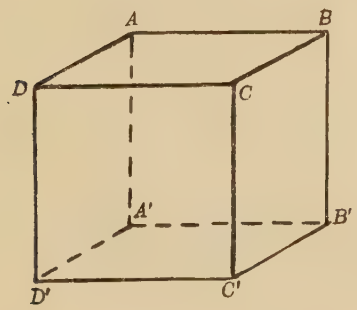
possessing the property that four 5-tangent conics form a congruent tetrad (*i.e.* a quartic passes through their points of contact with Δ) if they are represented by two pairs of points lying

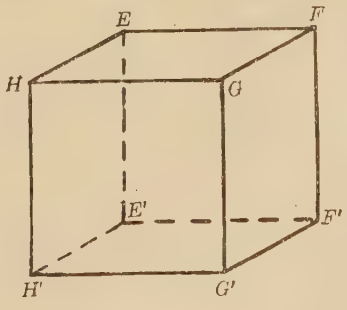
(i) on two parallel edges,

(ii) on two diagonals of parallel faces,

(iii) on two diagonals of the cube.

By making the necessary calculations on the characteristics, we find that, out of the configuration (2), we can form twenty-eight such cubes





(3)

possessing the property that four 5-tangent conics form a congruent tetrad if they are represented by two pairs of points lying

(i) on two parallel edges (*e.g.* AB, HG),

(ii) on two diagonals of parallel faces (*e.g.* $A'B$, HG'),

(iii) on two diagonals of cubes (*e.g.* $A'C$, $H'F$).

From the squares of the configuration (2) we have used $2\times 28 = 56$ squares as opposite faces (parallel to $ABB'A'$) of the cubes of the configuration (3). We are thus left with $120-56 = 64$ squares in the configuration (2). Thus the configuration (1) has been used to form the edges AA', BB', etc., of twenty-eight cubes and sixty-four squares, leaving 256 lines as before. If we treat the faces of the cubes of the configuration (3) parallel to $ADD'A'$ in precisely the same way, we get a residual set of sixty-four squares made up of $2\times 64 = 128$ of the above 256 lines of the configuration (1). These two sets of sixty-four squares contain no elements in common. Finally, we can, by employing precisely the same procedure, obtain a system of sixty-four squares residual to the diagonal rectangles of the cubes (*e.g.* $ACC'A'$). Thus *the* 496 *lines* AA', ..., *of the configuration* (1) *have been grouped as edges of twenty-eight cubes together with three sets of sixty-four squares residual to the facial squares and diagonal rectangles of the cubes* (3) *as above explained, the cubes and the three sets of residual squares containing no elements in common.*

More generally, a cube of the configuration (3) contains seven basic tetrads of congruent 5-tangent conics, no two of the basic tetrads being mutually congruent [*i.e.* satisfying the conditions of two tetrads of the configuration (2)]. This result is easily established as follows. If we isolate any point A' of the cube $ABCDA'B'C'D'$, and pair off A' with each of the seven other letters in turn (*e.g.* C), and thereafter take the three pairs of points congruent with the chosen pair (viz. AC', BD', $B'D$ in the case of $A'C$), we obtain 7×3 congruent pairs. But in this enumeration, a tetrad such as $A'B'CD$ enters three times, since it is derived from either $A'B'$ or $A'C$ or $A'D$. Hence the total number of basic congruent tetrads each containing A' as a member is $(7\times 3)\div 3 = 7$. They are $A'ABB'$, $A'ACC'$, $A'ADD'$, $A'BCD'$, $A'BC'D$, $A'B'CD$, $A'B'C'D'$. Each such congruent tetrad yields sixty-four residual tetrads as above explained. Thus the total number of 5-tangent conics is accounted for, viz. $8\times 28+7\times 4\times 64 = 2016$ (including the degenerate case of two coincident straight lines).

Let us now consider the effect of regarding as the degenerate 5-tangent conic consisting of two coincident straight lines, either (i) a member of one of the residual sets of sixty-four tetrads, or (ii) a member represented by one of the letters of the twenty-eight cubes (3).

(i) Let $\bar{K}\bar{L}\bar{M}\bar{N}$ be a tetrad of 5-tangent conics belonging to Group I and congruent with $ABCD$, of which $\bar{N}$ degenerates into two coincident straight lines. $\bar{K}$, $\bar{L}$, $\bar{M}$ will then be a triad of 5-tangent conics through the points of contact of which with Δ a cubic passes. Let AB, DC, $A'B'$, $D'C'$ be pairs of conics congruent with $\bar{K}\bar{N}$, *i.e.* $\bar{K}AB$, $\bar{K}DC$, $\bar{K}A'B'$, $\bar{K}D'C'$ will be cubics through the points of contact of the corresponding conics. Plainly, therefore, the pairs of quadritangent hyperplanes $\alpha\beta$ and $\gamma\delta$ of C belong to the same family of 8-tangent hyperquadrics defining the same quartic hypersurface Φ whose five nodes correspond to the five points of contact of $\bar{K}$ with Δ. Similarly $\beta\gamma$, $\alpha\delta$ are two pairs of quadritangent hyperplanes of C belonging to the same family of 8-tangent hyperquadrics and defining a quartic hypersurface Φ_1 whose five nodes correspond to the points of contact of $\bar{L}$ with Δ. In the same way, let $\gamma\alpha$, $\beta\delta$ define Φ_2 whose five nodes correspond to the points of contact of $\bar{M}$ with Δ.

We, therefore, have this result:

If α, β, γ, δ are four quadritangent hyperplanes of C through whose points of contact with C passes a hyperquadric not containing C, and if we take them in three pairs (i) $\alpha\beta$, $\gamma\delta$, (ii) $\beta\gamma$, $\alpha\delta$, (iii) $\gamma\alpha$, $\beta\delta$ *and regard them in each case as pairs of degenerate hyperquadrics of the same system, there will be defined three corresponding quartic hypersurfaces* (i) Φ, (ii) Φ_1, (iii) Φ_2 *to whose nodes correspond the points of contact of three 5-tangent conics of Δ, whose points of contact lie on a cubic.*

We have seen that to $\bar{K}$ correspond 120 quadritangent hyperplane-pairs $\alpha\beta$ of the same system. Taking two such pairs $\alpha\beta$ and $\gamma\delta$ each defining $\bar{K}$ and remembering that there are 255 cubics through the points of contact of $\bar{K}$ with Δ and passing also through the points of contact of two further conics $\bar{L}$, $\bar{M}$ belonging to Group I, we see that *there are* $_{120}C_2 \div 255 = 28$ *tetrads of quadritangent hyperplanes α, β, γ, δ through whose points of contact with C passes a hyperquadric not containing C and defining the same triad of 5-tangent conics $\bar{K}$, $\bar{L}$, $\bar{M}$ through the points of contact of which with Δ passes a cubic.*

We thus see that *there are twenty-eight tetrads*

$$\alpha_1\beta_1\gamma_1\delta_1, \quad \alpha_2\beta_2\gamma_2\delta_2, \quad \ldots, \quad \alpha_{28}\beta_{28}\gamma_{28}\delta_{28}$$

of quadritangent hyperplanes of C such that a hyperquadric not containing C passes through their points of contact with C, and such that two pairs from two tetrads with both letters the same or both letters different belong to the same system of 8-tangent hyperquadrics.

$\bar{K}\bar{L}\bar{M}\bar{N}$ belongs to a set of sixty-four mutually congruent tetrads and hence, *in the case where $\bar{N}$ consists of two coincident straight lines, we have a configuration of sixty-three tetrads of 5-tangent conics belonging to Group I each of which forms with $\bar{K}\bar{L}\bar{M}$ a triadic set of seven 5-tangent conics of Δ.*

Also $\bar{K}\bar{L}\bar{M}\bar{N}$ is congruent with all tetrads of 5-tangent conics of Δ represented by points on the faces of the cubes parallel to $ABCD$. Hence all tetrads of 5-tangent conics of Δ belonging to Group II and represented by points on the faces of the cubes of (3) *parallel to $ABCD$ form with $\bar{K}\bar{L}\bar{M}$ triadic sets of seven 5-tangent conics of Δ.*

Thus the 119 congruent tetrads which with $\bar{K}\bar{L}\bar{M}$ form triadic sets of seven 5-tangent conics are accounted for, viz. sixty-three tetrads belonging to Group I and $2 \times 28 = 56$ tetrads belonging to Group II.

(ii) Next let A' consist of two coincident straight lines. It will be evident immediately that A, B, C, D, B', C', D' *of the cube $ABCDA'B'C'D'$ of* (3) *constitute a triadic set of seven 5-tangent conics of Δ, the constituent triads being ABB', $B'C'D'$, $B'CD$, ACC', ADD', BCD', $BC'D$. To each of these seven triads correspond seven groups of sixty-four tetrads forming with each of them respectively a triadic set of seven 5-tangent conics of Δ. Each group of eight conics of the remaining twenty-seven cubes of* (3) *forms with A, B, C, D, B', C', D' a triadic system of fifteen contact conics of Δ.*

We note once again that the congruences of a cube of (3) can be represented analytically as follows:

$$A \equiv (x, y, z), \quad B \equiv (-x, y, z), \quad C \equiv (-x, -y, z), \quad D \equiv (x, -y, z),$$

$$A' \equiv (x, y, -z), \quad B' \equiv (-x, y, -z), \quad C' \equiv (-x, -y, -z), \quad D' \equiv (x, -y, -z),$$

by manipulating the signs of $(\pm x, \pm y, \pm z)$. If A is taken to correspond to C, we change the signs of x and y, and leave the sign of z unaltered. Performing the same operation, we see that B corresponds to D, A' to C', B' to D'. Thus every set of four congruent pairs can be written down from the diagram.

4. *A certain set of eight quadritangent hyperplanes of C.*

Let us now combine together two of the cubes of the configuration (3) and let us take for convenience the first two, $ABCDA'B'C'D'$ and

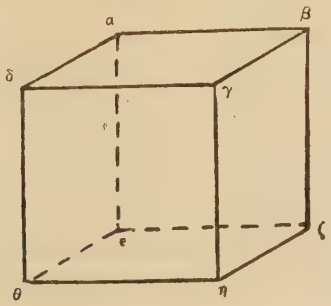
$EFGHE'F'G'H'$. Let the pairs of 5-tangent conics AA', BB', CC', DD', EE', FF', GG', HH' correspond to the quadritangent hyperplanes $\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta, \theta$ respectively, We have just seen that the two tetrads of quadritangent hyperplanes

$$\alpha \quad \beta \quad \gamma \quad \delta$$
$$\epsilon \quad \zeta \quad \eta \quad \theta$$

possess the property that any pair of hyperplanes selected from the top line and the other pair also from the top line together with the corresponding pairs obtained by taking the letters respectively below them in the second line, constitute four hyperplane-pairs all belonging to the same system of 8-tangent hyperquadrics of C, and hence defining the same quartic hypersurface Φ whose five nodes correspond to the points of contact with Δ of a 5-tangent conic. Thus $\alpha\beta$, $\gamma\delta$, $\epsilon\zeta$, $\eta\theta$ will define $\bar{K}$; $\beta\gamma$, $\alpha\delta$, $\zeta\eta$, $\epsilon\theta$ will define $\bar{L}$; $\gamma\alpha$, $\beta\delta$, $\eta\epsilon$, $\zeta\theta$ will define $\bar{M}$. We wish to consider the configuration of 5-tangent conics defined by combining $\alpha, \beta, \gamma, \delta$ respectively with $\epsilon, \zeta, \eta, \theta$ or with these letters simultaneously interchanged in pairs (precisely as in dealing with the problem of cross-ratios), so as to obtain in each case four hyperplane pairs belonging to the same system of 8-tangent hyperquadrics.

α	β	γ	δ
ϵ	ζ	η	θ
ζ	ϵ	θ	η
η	θ	ϵ	ζ
θ	η	ζ	ϵ

The same result can be represented diagrammatically by placing the letters at the corners of a cube, thus:



(4)

Then four hyperplane-pairs belonging to the same system of 8-tangent hyperquadrics can be obtained by taking four pairs of letters

(i) on parallel edges,

(ii) on the diagonals of two opposite faces,

(iii) on the four diagonals of the cube.

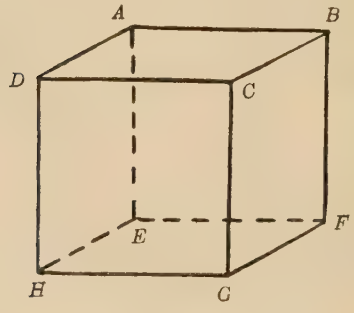
Taking α with each of the other seven hyperplanes of the group in turn to form hyperplane-pairs and so to define seven systems of 8-tangent hyperquadrics of C, we obtain in the usual way seven corresponding 5-tangent conics of Δ, and we shall show that they belong to a triadic set of seven 5-tangent conics.

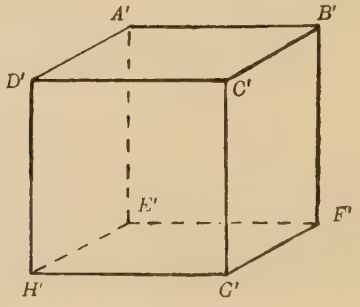
Let us begin, therefore, by combining two of the twenty-eight cubes of (3), and calculating the necessary characteristics for the general case of $p = 6$. We exemplify two such cubes of (3) as follows:

$$\begin{matrix} A & B & C & D \\ \left[\begin{matrix}1&0&0\\0&0&0\end{matrix}\begin{matrix}1&1&1\\1&0&0\end{matrix}\right] & \left[\begin{matrix}0&0&0\\1&1&0\end{matrix}\begin{matrix}1&1&0\\1&0&1\end{matrix}\right] & \left[\begin{matrix}1&0&0\\0&0&1\end{matrix}\begin{matrix}1&1&1\\1&0&0\end{matrix}\right] & \left[\begin{matrix}0&0&0\\1&1&1\end{matrix}\begin{matrix}1&1&0\\1&0&1\end{matrix}\right], \\ \left[\begin{matrix}0&0&0\\1&0&0\end{matrix}\begin{matrix}1&1&0\\1&0&1\end{matrix}\right] & \left[\begin{matrix}1&0&0\\0&1&0\end{matrix}\begin{matrix}1&1&1\\1&0&0\end{matrix}\right] & \left[\begin{matrix}0&0&0\\1&0&1\end{matrix}\begin{matrix}1&1&0\\1&0&1\end{matrix}\right] & \left[\begin{matrix}1&0&0\\0&1&1\end{matrix}\begin{matrix}1&1&1\\1&0&0\end{matrix}\right], \\ A' & B' & C' & D' \end{matrix}$$

$$\begin{matrix} E & F & G & H \\ \left[\begin{matrix}1&0&0\\0&0&0\end{matrix}\begin{matrix}1&1&1\\0&1&0\end{matrix}\right] & \left[\begin{matrix}0&0&0\\1&1&0\end{matrix}\begin{matrix}1&1&0\\0&0&1\end{matrix}\right] & \left[\begin{matrix}1&0&0\\0&0&1\end{matrix}\begin{matrix}1&1&1\\0&1&0\end{matrix}\right] & \left[\begin{matrix}0&0&0\\1&1&0\end{matrix}\begin{matrix}1&1&0\\1&1&1\end{matrix}\right] \\ \left[\begin{matrix}0&0&0\\1&0&0\end{matrix}\begin{matrix}1&1&0\\0&1&1\end{matrix}\right] & \left[\begin{matrix}1&0&0\\0&1&0\end{matrix}\begin{matrix}1&1&1\\0&1&0\end{matrix}\right] & \left[\begin{matrix}0&0&0\\1&0&1\end{matrix}\begin{matrix}1&1&0\\0&1&1\end{matrix}\right] & \left[\begin{matrix}1&0&0\\0&1&1\end{matrix}\begin{matrix}1&1&1\\0&1&0\end{matrix}\right] \\ E' & F' & G' & H' \end{matrix} \qquad (5)$$

Then, to form congruences, we construct the following cubes:





(6)

If we take the eight pairs of points on the eight edges parallel to AB, we get eight pairs of 5-tangent conics, the points of contact of two pairs with Δ lying on a quartic. Proceeding in this way and forming the seven systems of congruences defined by $ABCDEFGH$, each system containing four pairs of mutually congruent conics, and taking the corresponding elements from $A'B'C'D'E'F'G'H'$, we obtain seven systems each containing eight mutually congruent pairs. Let us take now any such system of eight mutually congruent pairs, *e.g.*

$$AF, \quad BE, \quad CH, \quad DG,$$
$$A'F', \quad B'E', \quad C'H', \quad D'G',$$

and let us interchange the second letters of each pair thus:

$$AF', \quad BE', \quad CH', \quad DG',$$
$$A'F, \quad B'E, \quad C'H, \quad D'G.$$

We thus obtain a new system of eight mutually congruent pairs. Proceeding in this way and including the case AA', BB', etc., we obtain $7+7+1=15$ systems of eight mutually congruent pairs of 5-tangent conics, where the leading pairs of conics can be regarded as the pairs formed by A' with the remaining fifteen conics of the set.

By manipulating the signs of $(\pm x, \pm y, \pm z, \pm t)$, we can represent the sixteen points diagrammatically as before:

$$A \equiv (x, y, z, t), \quad B \equiv (-x, y, z, t), \quad C \equiv (-x, -y, z, t),$$
$$D \equiv (x, -y, z, t), \quad E \equiv (x, y, -z, t), \quad F \equiv (-x, y, -z, t),$$
$$G \equiv (-x, -y, -z, t), \quad H \equiv (x, -y, -z, t),$$
$$A' \equiv (x, y, z, -t), \quad B' \equiv (-x, y, z, -t), \quad C' \equiv (-x, -y, z, -t),$$
$$D' \equiv (x, -y, z, -t), \quad E' \equiv (x, y, -z, -t). \quad F' \equiv (-x, y, -z, -t),$$
$$G' \equiv (-x, -y, -z, -t), \quad H' \equiv (x, -y, -z, -t). \qquad (7)$$

If we combine A with F', we change the signs of all the letters of A except y. Doing this in each case, we obtain the following pairs congruent with AF', viz. BE', CH', DG', EB', FA', GD', HC', and so on.

It will be found, on calculating the general characteristics for $p=6$, that, out of the twenty-eight cubes of the configuration (3), six pairs of cubes (5) possess the property of being mutually congruent. Thus, of the twenty-eight cubes of the configuration (3), $6\times 2=12$ cubes are used to form these six pairs of cubes typified by (5), leaving $28-12=16$ cubes

as a remainder. This can be done in seven ways according to the above seven methods of grouping $ABCDEFGH$. Thus we obtain seven groups each containing sixteen cubes of the configuration (3), and possessing no elements in common. The number of 5-tangent conics thus accounted for is $6\times 8\times 2+7\times 8\times 16 = 992$, which constitutes the total number of 5-tangent conics in Group II. These congruences may be regarded as obtained by pairing A with any of the seven letters B, C, D, E, F, G, H; and to every one of the seven pairs of letters AB, AC, etc., corresponds a residue of sixteen cubes, all of whose members belong to Group II. More generally, to every one of the pairs $AA', AB', \ldots, AH'$ corresponds similarly a residue of sixteen sets of cubes, containing no elements either in common with each other or with the groups of sixteen cubes already dealt with, and hence all belonging to Group I. Thus $8\times 8\times 16 = 1024$ contact conics of Group I are accounted for. We have thus a central grouping of six sets of sixteen points of the type $(\pm x, \pm y, \pm z, \pm t)$ representing sixteen contact conics $A, B, C, \ldots, H, A', B', C', \ldots, H'$ together with fifteen groups each consisting of sixteen cubes of the type of the configuration (3) giving $6\times 16+15\times 16\times 8 = 2016$ contact conics in all. This result can, of course, be obtained by studying the characteristics for the general case $p = 6$ and ignoring for the moment the existence of a degenerate contact conic consisting of two coincident straight lines.

We have now to consider two cases, according as the contact conics degenerating into two coincident straight lines belong to one of the residual sets of 16 cubes or to the central nucleus of six sets of sixteen points of the type $(\pm x, \pm y, \pm z, \pm t)$.

(i) Let a member of the eight groups of sixteen cubes containing the 5-tangent conics of Group I represent a 5-tangent conic degenerating into two coincident straight lines. The cube to which this 5-tangent conic belongs will, therefore, denote this degenerate conic together with seven other 5-tangent conics forming a triadic system of seven contact conics. *The other fifteen cubes of the set represent groups of eight contact conics each forming with these seven contact conics a triadic set of fifteen contact conics.*

Hence *there exist groups of eight quadritangent hyperplanes of C of the type* (4) *which can be arranged in seven different ways into four hyperplane-pairs belonging to the same system of* 8*-tangent hyperquadrics, and defining seven* 5*-tangent conics which belong to a triadic set. There are six such sets of eight quadritangent hyperplanes of C defining the same triadic set of seven contact conics.*

Let the contact conic degenerating into two coincident straight lines belong to a cube of Group I congruent with $ABCDEFGH$ and $A'B'C'D'E'F'G'H'$. We thus see that *$ABCDEFGH$ and $A'B'C'D'E'F'G'H'$ each constitute sets of eight contact conics of Group II forming along with the given triadic set of Group I a triadic set of fifteen contact conics of* Δ.

Thus the $15+2\times 6=27$ sets of eight contact conics forming with the given triadic set of seven contact conics a triadic set of fifteen contact conics are all accounted for.

The fact that six sets of eight quadritangent hyperplanes yield the same triadic set of seven contact conics can be seen otherwise. Taking two of the twenty-eight sets of four quadritangent hyperplanes defining the same triadic set of three contact conics, we ought to obtain ${}_{28}C_2$ tetrads of 5-tangent conics which with the given triad form a triadic set of seven contact conics. Instead, we have seen that there are sixty-three such tetrads belonging to Group I, whence ${}_{28}C_2 \div 63 = 6$ sets of eight quadritangent hyperplanes yield the same triadic set of seven contact conics.

(ii) Next consider the configuration (5) and let A' consist of two coincident straight lines. Plainly, as before, $ABCDA'B'C'D'$ will yield a triadic set of seven contact conics. If we add $EFGHE'F'G'H'$, we plainly obtain a triadic set of fifteen contact conics. If we take now with (5) one of the other five members of the type (7) that together constitute the complete set of six groups of (5), *we obtain five triadic sets of thirty-one contact conics. Given a triadic set of fifteen contact conics, there are five sets of sixteen contact conics which with the given set of fifteen contact conics form triadic sets of thirty-one contact conics.*

5. *A certain set of sixteen quadritangent hyperplanes of* C.

Let us proceed as before and group together two of the six configurations (4) of eight quadritangent hyperplanes. We obtain a set of sixteen quadritangent hyperplanes. If any one of the sixteen quadritangent hyperplanes be isolated, it can be paired off with any one of the other fifteen hyperplanes of the set to form a degenerate 8-tangent hyperquadric of C. Thus *the sixteen quadritangent hyperplanes of the group can be divided in fifteen ways into eight hyperplane-pairs, all belonging to the same system of 8-tangent hyperquadrics. Each of these fifteen sets of 8-tangent hyperplane-pairs defines a 5-tangent conic of* Δ, *and we obtain a triadic set of fifteen contact conics of* Δ *belonging to Group I.*

Let us also group together any two of the six configurations (5), thus obtaining a new configuration of thirty-two contact conics of the type

$(\pm x, \pm y, \pm z, \pm t, \pm u)$, and leaving a residue of four configurations of the type $(\pm x, \pm y, \pm z, \pm t)$ each containing sixteen contact conics. Regarding the thirty-two contact conics as being composed of sixteen pairs of conics of the type (1) in the usual way, and pairing A with any one of fifteen letters B, C, etc., and omitting A', B', etc., for the moment, we obtain $15\times4\times16$ residual conics belonging to Group II. Thus we have $32+15\times4\times16=992$ contact conics of Group II accounted for. Thus *in Group II there are fifteen mutually exclusive configurations of four sets of contact conics each consisting of sixteen members residual to the central group of thirty-two contact conics.*

From the general standpoint of $p=6$, to each of the thirty-one ways of dividing the group of thirty-two conics into sixteen congruent pairs correspond four residual configurations of the type (7) each containing sixteen contact conics. Thus the total number of 5-tangent conics accounted for is $32+4\times16\times31=2016$.

(i) If the degenerate 5-tangent conic consisting of two coincident straight lines lies in one of the above sets of sixteen contact conics belonging to Group I, we obtain a triadic set of fifteen contact conics of Δ. The central group of thirty-two contact conics will divide itself into two groups of sixteen contact conics, each forming with the triadic set of fifteen a new triadic set of thirty-one contact conics.

To a triadic set of fifteen contact conics of Group I corresponds a single group of sixteen quadritangent hyperplanes of C defining it. There remain three sets of sixteen contact conics of Group I forming with the above triadic set of fifteen contact conics triadic sets of thirty-one contact conics.

Thus the five sets of sixteen contact conics which with the given triadic set of fifteen contact conics form triadic sets of thirty-one contact conics are accounted for, viz. two sets of sixteen in Group II and three sets of sixteen in Group I.

(ii) *If, however, the 5-tangent conic consisting of two coincident straight lines belongs to the central group of thirty-two contact conics, we obtain a triadic set of thirty-one contact conics. Through the points of contact of any one of the thirty-one contact conics with Δ pass in addition thirty-two cubics on which lie the points of contact of two further 5-tangent conics with Δ, not belonging to the central set of thirty-one.*

TRANSFORMATIONS OF WELL-POISED HYPERGEOMETRIC SERIES

By W. N. BAILEY.

[Received 4 May, 1932.—Read 19 May, 1932.]

1. In a paper* published in 1926, Whipple investigated the relations giving a well-poised hypergeometric series of the type ${}_7F_6$ in terms of a Saalschützian series ${}_4F_3$. He also gave relations connecting two well-poised series ${}_7F_6$, both finite and infinite, and relations connecting two Saalschützian series ${}_4F_3$. In all these formulae there is the restriction that a certain number, sometimes a parameter and sometimes a linear combination of parameters, must be a negative integer.

The object of this paper is to find the corresponding results when this restriction is removed. I also give relations connecting well-poised series of the type ${}_9F_8$.

2. We begin with the formula†

$$(2.1)\quad {}_7F_6\left[\begin{matrix} a, & 1+\frac{1}{2}a, & c, & d, & e, & f, & g; \\ \frac{1}{2}a, & 1+a-c, & 1+a-d, & 1+a-e, & 1+a-f, & 1+a-g \end{matrix}\right]$$

$$= \frac{\Gamma(1+a-e)\,\Gamma(1+a-f)\,\Gamma(1+a-g)\,\Gamma(1+a-e-f-g)}{\Gamma(1+a)\,\Gamma(1+a-f-g)\,\Gamma(1+a-e-g)\,\Gamma(1+a-e-f)}$$

$$\times {}_4F_3\left[\begin{matrix} e, & f, & g, & 1+a-c-d; \\ 1+a-c, & 1+a-d, & e+f+g-a \end{matrix}\right]$$

$$+ \frac{\Gamma(1+a-c)\,\Gamma(1+a-d)\,\Gamma(1+a-e)\,\Gamma(1+a-f)\,\Gamma(1+a-g)}{\Gamma(1+a)\,\Gamma(e)\,\Gamma(f)\,\Gamma(g)\,\Gamma(1+a-c-d)}$$

$$\times \frac{\Gamma(2+2a-c-d-e-f-g)\,\Gamma(e+f+g-1-a)}{\Gamma(2+2a-c-e-f-g)\,\Gamma(2+2a-d-e-f-g)}$$

$$\times {}_4F_3\left[\begin{matrix} 1+a-f-g, & 1+a-e-g, & 1+a-e-f, & 2+2a-c-d-e-f-g; \\ 2+2a-c-e-f-g, & 2+2a-d-e-f-g, & 2+a-e-f-g \end{matrix}\right].$$

---

* Whipple, **6**.

† Bailey, **3**, and Bailey, **1** (4·1).

It is convenient to write $W(a;\ c,\ d,\ e,\ f,\ g)$ for the series ${}_7F_6$ on the left of (2.1).

Now transform the series

$$W(1+2a-e-f-g;\quad c,\ d,\ 1+a-f-g,\ 1+a-e-g,\ 1+a-e-f)$$

by (2.1), and we obtain the same series ${}_4F_3$ as occur in (2.1). We thus find that

$$(2.2)\quad W(a;\ c,\ d,\ e,\ f,\ g)$$

$$= \frac{\Gamma(1+a-c)\,\Gamma(1+a-d)\,\Gamma(2+2a-e-f-g)\,\Gamma(2+2a-c-d-e-f-g)}{\Gamma(1+a)\,\Gamma(1+a-c-d)\,\Gamma(2+2a-c-e-f-g)\,\Gamma(2+2a-d-e-f-g)}$$

$$\times W(1+2a-e-f-g;\ c,\ d,\ 1+a-f-g,\ 1+a-e-g,\ 1+a-e-f).$$

On duplicating this formula we obtain

$$(2.3)\quad W(a;\ c,\ d,\ e,\ f,\ g)$$

$$= \frac{\Gamma(1+a-c)\,\Gamma(1+a-d)\,\Gamma(1+a-e)\,\Gamma(1+a-f)}{\Gamma(1+a)\,\Gamma(g)\,\Gamma(2+2a-d-e-f-g)\,\Gamma(2+2a-c-e-f-g)}$$

$$\times \frac{\Gamma(3+3a-c-d-e-f-2g)\,\Gamma(2+2a-c-d-e-f-g)}{\Gamma(2+2a-c-d-f-g)\,\Gamma(2+2a-c-d-e-g)}$$

$$\times W(2+3a-c-d-e-f-2g;\ 1+a-c-g,\ 1+a-d-g,\ 1+a-e-g,\ 1+a-f-g,\ 2+2a-c-d-e-f-g).$$

If g or $1+a-e-f$ is a negative integer, (2.2) gives a relation between a finite and an infinite well-poised series, whereas, if c is a negative integer, the formula reduces to one connecting two terminating series. Similarly (2.3) gives, in certain circumstances, a relation between two series, only one of which is finite.

The formulae (2.2) and (2.3) do not generalize all Whipple's formulae of this type. Apparently the generalizations of the other formulae are not of such a simple form.

3. The series on the right of (2.3) can be transformed by means of (2.1), and we thus derive the formula

$$(3.1)\quad {}_7F_6\left[\begin{matrix} a, & 1+\frac{1}{2}a, & c, & d, & e, & f, & g; \\ \frac{1}{2}a, & 1+a-c, & 1+a-d, & 1+a-e, & 1+a-f, & 1+a-g & \end{matrix}\right]$$

$$= \frac{\Gamma(1+a-c)\,\Gamma(1+a-d)\,\Gamma(1+a-e)\,\Gamma(1+a-f)}{\Gamma(1+a)\,\Gamma(g)\,\Gamma(1+a-c-f)\,\Gamma(1+a-d-f)}$$

$$\times \frac{\Gamma(g-f)\,\Gamma(2+2a-c-d-e-f-g)}{\Gamma(1+a-e-f)\,\Gamma(2+2a-c-d-e-g)}$$

$$\times {}_4F_3\left[\begin{matrix} 1+a-c-g, & 1+a-d-g, & 1+a-e-g, & f; \\ 2+2a-c-d-e-g, & 1+a-g, & 1+f-g & \end{matrix}\right]$$

$$+ \frac{\Gamma(1+a-c)\,\Gamma(1+a-d)\,\Gamma(1+a-e)\,\Gamma(1+a-g)\,\Gamma(f-g)\,\Gamma(2+2a-c-d-e-f-g)}{\Gamma(1+a)\,\Gamma(f)\,\Gamma(1+a-c-g)\,\Gamma(1+a-d-g)\,\Gamma(1+a-e-g)\,\Gamma(2+2a-c-d-e-f)}$$

$$\times {}_4F_3\left[\begin{matrix} 1+a-c-f, & 1+a-d-f, & 1+a-e-f, & g; \\ 2+2a-c-d-e-f, & 1+a-f, & 1+g-f. & \end{matrix}\right].$$

The relations (2.1) and (3.1) appear to be the only formulae of their type. If, in (2.1), e or $1+a-c-d$ is a negative integer, the second series on the right disappears, while if $1+a-f-g$ is a negative integer the first series on the right disappears from the equation. We thus have three distinct relations from (2.1) which express a well-poised series ${}_7F_6$ in terms of a terminating Saalschützian ${}_4F_3$; in one case the ${}_7F_6$ is finite, and in the other two infinite. Similarly, from (3.1), we can derive two distinct relations of a similar type in one of which the ${}_7F_6$ is finite and in the other infinite. These are all the relations of this kind obtained by Whipple, apart from those which can be derived from them by the mere reversal of series.

A relation connecting four Saalschützian series ${}_4F_3$ is obtained by equating the right-hand sides of (2.1) and (3.1). By taking the parameters in different ways, it is found that there are three distinct formulae of this type. In certain cases when a parameter is a negative integer these formulae reduce to relations connecting two terminating Saalschützian series, and in other cases they become relations connecting three series, one of which is finite and the other two infinite. For example, in the formula obtained directly from (2.1) and (3.1), if f is a negative integer we obtain a relation between two finite series, whereas if $1+a-c-d$ is a negative integer the relation connects a finite series with two infinite series.

4. We now consider well-poised series of the type ${}_9F_8$, and it is convenient to write

$$V(a;\ b,\ c,\ d,\ e,\ f,\ g,\ h)$$

$$\equiv \frac{\Gamma(1+a)\,\Gamma(b)\,\Gamma(c)\,\Gamma(d)\,\Gamma(e)\,\Gamma(f)\,\Gamma(g)\,\Gamma(h)}{\Gamma(1+a-b)\,\Gamma(1+a-c)\,\Gamma(1+a-d)\,\Gamma(1+a-e)\,\Gamma(1+a-f)\,\Gamma(1+a-g)\,\Gamma(1+a-h)}$$

$$\times {}_9F_8\left[\begin{matrix} a,\ 1+\tfrac{1}{2}a,\ b,\ c,\ d,\ e,\ f,\ g,\ h; \\ \tfrac{1}{2}a,\ 1+a-b,\ 1+a-c,\ 1+a-d,\ 1+a-e,\ 1+a-f,\ 1+a-g,\ 1+a-h \end{matrix}\right].$$

It is known that*

$$(4.1)\quad \operatorname{cosec}(b-a)\pi\,.\{V(a;\ b,\ c,\ d,\ e,\ f,\ g,\ h) - V(2b-a;\ b,\ b+c-a,\ b+d-a,\ b+e-a,\ b+f-a,\ b+g-a,\ b+h-a)\}$$

$$= \frac{\Gamma(c)\,\Gamma(d)\,\Gamma(e)\,\Gamma(f+b-a)\,\Gamma(g+b-a)\,\Gamma(h+b-a)\operatorname{cosec}(1+a-f-g-h)\pi}{\Gamma(1+a-d-e)\,\Gamma(1+a-c-e)\,\Gamma(1+a-c-d)\,\Gamma(1+a-g-h)\,\Gamma(1+a-f-h)\,\Gamma(1+a-f-g)}$$

$$\times\{V(1+2a-c-d-e;\ b,\ 1+a-d-e,\ 1+a-c-e,\ 1+a-c-d,\ f,\ g,\ h) - V(2b-2a-1+c+d+e;\ b,\ b-a+c,\ b-a+d,\ b-a+e,\ 1+a-g-h,\ 1+a-f-h,\ 1+a-f-g)\}$$

provided that

$$(4.11)\qquad 2+3a = b+c+d+e+f+g+h.$$

By repeating this transformation, we find that the left-hand side of (4.1) is equal to

$$(4.2)\quad \frac{\Gamma(c)\,\Gamma(d)\,\Gamma(e)\,\Gamma(f)\,\Gamma(c+b-a)\,\Gamma(d+b-a)}{\Gamma(1+a-c-g)\,\Gamma(1+a-d-g)\,\Gamma(1+a-e-g)\,\Gamma(1+a-f-g)}$$

$$\times \frac{\Gamma(e+b-a)\,\Gamma(f+b-a)\operatorname{cosec}(g-h)\pi}{\Gamma(1+a-c-h)\,\Gamma(1+a-d-h)\,\Gamma(1+a-e-h)\,\Gamma(1+a-f-h)}$$

$$\times\{V(b-g+h;\ b,\ 1+a-c-g,\ 1+a-d-g,\ 1+a-e-g,\ 1+a-f-g,\ h,\ h+b-a) - V(b+g-h;\ b,\ 1+a-c-h,\ 1+a-d-h,\ 1+a-e-h,\ 1+a-f-h,\ g,\ g+b-a)\}$$

provided that the condition (4.11) is satisfied.

---

* See Bailey, **1**, §9. The formula is not stated explicitly there, but the parameters of the series are given.

The formulae (4.1) and (4.2) reduce considerably when h is a negative integer $-n$, in which case we obtain identities equivalent to

$$(4.3)\quad {}_9F_8\left[\begin{matrix} a,\ 1+\tfrac{1}{2}a,\ b,\ c,\ d,\ e,\ f,\ g,\ -n; \\ \tfrac{1}{2}a,\ 1+a-b,\ 1+a-c,\ 1+a-d,\ 1+a-e,\ 1+a-f, \\ 1+a-g,\ 1+a+n \end{matrix}\right]$$

$$= \frac{(1+a)_n(1+k-e)_n(1+k-f)_n(1+k-g)_n}{(1+k)_n(1+a-e)_n(1+a-f)_n(1+a-g)_n}$$

$$\times {}_9F_8\left[\begin{matrix} k,\ 1+\tfrac{1}{2}k,\ k+b-a,\ k+c-a,\ k+d-a,\ e,\ f,\ g,\ -n; \\ \tfrac{1}{2}k,\ 1+a-b,\ 1+a-c,\ 1+a-d,\ 1+k-e, \\ 1+k-f,\ 1+k-g,\ 1+k+n \end{matrix}\right],$$

where $k = 1+2a-b-c-d$; and

$$(4.4)\quad {}_9F_8\left[\begin{matrix} a,\ 1+\tfrac{1}{2}a,\ b,\ c,\ d,\ e,\ f,\ g,\ -n; \\ \tfrac{1}{2}a,\ 1+a-b,\ 1+a-c,\ 1+a-d,\ 1+a-e, \\ 1+a-f,\ 1+a-g,\ 1+a+n \end{matrix}\right]$$

$$= \frac{(1+a)_n(1+a-b-c)_n(1+a-b-d)_n(1+a-b-e)_n(1+a-b-f)_n(g)_n}{(1+a-b)_n(1+a-c)_n(1+a-d)_n(1+a-e)_n(1+a-f)_n(g-b)_n}$$

$$\times {}_9F_8\left[\begin{matrix} b-g-n,\ 1+\tfrac{1}{2}(b-g-n),\ b,\ 1+a-c-g,\ 1+a-d-g, \\ 1+a-e-g,\ 1+a-f-g,\ b-a-n,\ -n; \\ \tfrac{1}{2}(b-g-n),\ 1-g-n,\ b+c-a-n,\ b+d-a-n, \\ b+e-a-n,\ b+f-a-n,\ 1+a-g,\ 1+b-g \end{matrix}\right].$$

In both these formulae there is the restriction that

$$2+3a = b+c+d+e+f+g-n.$$

The identity (4.3) is already known* and (4.4) can be deduced from it by duplication. From (4.3) we can†, by a limiting process, deduce (2.1), and similarly we can derive (3.1) from (4.4). Also, if in (4.3) we substitute the value of g in terms of the other parameters and then let n tend to infinity, we obtain (2.2), and if in (4.4) we substitute for b and let n tend to infinity, we obtain (2.3).

5. The above results have an interesting connection with results which have recently been obtained giving the partial sums of hypergeometric series in terms of infinite series. Several formulae have been given for the sum of n terms of the series ${}_2F_1(\alpha, \beta;\ \gamma)$, and I have given‡ an expression

* Bailey, **2** (8.6).

† Bailey, **3**.

‡ See Bailey, **4**, where references to other papers are given.

for the sum of n terms of the series ${}_3F_2(x, y, z; v, w)$, where

$$v+w=x+y+z+1,$$

in terms of a well-poised series ${}_7F_6$. This well-poised series can be transformed in various ways by (2.1) and (3.1), and so the sum of n terms of the series ${}_3F_2$ can be expressed in several ways in terms of two series ${}_4F_3$. One such formula has been given by Darling*. These results can also be obtained from the four-term relations between Saalschützian ${}_4F_3$ referred to in § 3.

More general results can be derived from (4.1) and (4.2) by putting $c=1-n$, where n is a positive integer, and then letting b tend to $a+n$. We thus obtain two formulae for the sum of n terms of the series

$${}_7F_6\left[\begin{matrix} a, & 1+\frac{1}{2}a, & d, & e, & f, & g, & h; \\ & \frac{1}{2}a, & 1+a-d, & 1+a-e, & 1+a-f, & 1+a-g, & 1+a-h \end{matrix}\right],$$

where $1+2a=d+e+f+g+h$, in terms of two infinite well-poised series†. The formulae are naturally complicated, and probably the only interest lies in the fact of their existence.

References.

1. W. N. Bailey, "Some transformations of generalized hypergeometric series, and contour integrals of Barnes's type", *Quart. J. of Math.* (Oxford series), 3 (1932), 168–182.
2. ———, "Some identities involving generalized hypergeometric series", *Proc. London Math. Soc.* (2), 29 (1929), 503–516.
3. ———, "An extension of Whipple's theorem on well-poised hypergeometric series", *Proc. London Math. Soc.* (2), 31 (1930), 505–511.
4. ———, "On one of Ramanujan's theorems", *Journal London Math. Soc.*, 7 (1932), 34–36.
5. H. B. C. Darling, "On certain relations between hypergeometric series of higher orders", *Proc. London Math. Soc.* (2), 34 (1932), 323–339.
6. F. J. W. Whipple, "Well-poised series and other generalized hypergeometric series", *Proc. London Math. Soc.* (2), 25 (1926), 525–544.

* Darling, 5, p. 335.

† In one case one of the series is of the type ${}_7F_6$ and one of the type ${}_9F_8$. In the other case both series are ${}_9F_8$.

ON THE REPRESENTATIONS OF A NUMBER AS A SUM OF SQUARES AND PRODUCTS

(*First paper.*)

By A. PAGE.

[Received 13 May, 1932.—Read 19 May, 1932.]

1. This paper deals with the number $\nu(n)$ of solutions of the equation

$$n = \sum_{u=1}^{r} x_u^2 + \sum_{v=1}^{s} y_v z_v$$

in integers x_u, y_v, and z_v, with $y_v > 0$, $z_v > 0$, where n is any given integer greater than or equal to 2, and $r+2s \geqslant 5$, $r > 0$, $s > 0$. The method is a direct application of the well-known Hardy-Littlewood method, and leads to the following asymptotic formula for $\nu(n)$:

$$\nu(n) = \pi^{\frac{1}{2}r} n^{\frac{1}{2}r+s-1} \sum_{t=0}^{s} \log^t n \sum_{m=0}^{s-t} c_{m,t} f^{(m)}(s) + O(n^{r_1} \log^{s_1} n),$$

where the c's are constants; $f^{(m)}(s)$ denotes the m-th derivative of $f(u)$ for $u = s$; $r_1 = \max(\frac{1}{4}r+\frac{1}{2}s,\ \frac{1}{2}r+s-2)$; $s_1 = s$ unless $r+2s = 8$, in which case

$$s_1 = s+1;$$

and $f(u)$ is the singular series. We show that $f(u)$ may be expressed as a simple function of the divisors of n when r is even, and as a series of the Smith-Minkowski type when r is odd.

In the case $s = 0$, the results are well known. The case $r = 0$ has been investigated by Estermann in a recent paper "On the representations of a number as the sum of three or more products"*.

* T. Estermann, *Proc. London Math. Soc.* (2), 34 (1932), 190–195.

2. *Application of the Farey-dissection.*

Let

$$f(\tau) = \sum_{m=-\infty}^{\infty} e^{2\pi i m^2 \tau} \quad [I(\tau) > 0], \tag{1}$$

and

$$\phi(\tau) = \sum_{n=1}^{\infty} d(n)\, e^{2\pi i n \tau} \quad [I(\tau) > 0], \tag{2}$$

where $d(n)$ is the number of positive divisors of n. Then it is obvious that

$$f^r(\tau)\,\phi^s(\tau) = \sum_{n=s}^{\infty} \nu(n)\, e^{2\pi i n \tau},$$

and hence

$$\nu(n) = \int_{\tau_0}^{\tau_0+1} f^r(\tau)\,\phi^s(\tau)\, e^{-2\pi i n \tau}\, d\tau, \tag{3}$$

where τ_0 is any number with $I(\tau_0) > 0$.

Putting

$$N = [\sqrt{n}], \quad \tau_0 = \frac{1}{N+1} + \frac{i}{n}, \tag{4}$$

and applying the Farey-dissection of order N, we obtain, from (3),

$$\nu(n) = \sum_{k=1}^{N} \sum_{h}{}' \int_{i/n+h/k-\vartheta_1}^{i/n+h/k+\vartheta_2} f^r(\tau)\,\phi^s(\tau)\, e^{-2\pi i n \tau}\, d\tau, \tag{5}$$

where the dash indicates that h runs through the positive integers less than or equal to k with $(h, k) = 1$, and $h/k - \vartheta_1$, $h/k + \vartheta_2$ are the medians between h/k and its neighbours in the Farey series. It is known that

$$(2kN)^{-1} \leqslant \vartheta_r \leqslant k^{-1}(N+1)^{-1} \quad (r = 1, 2). \tag{6}$$

Writing

$$\xi_k = e^{2\pi i/k}, \tag{7}$$

we have, by (5),

$$\nu(n) = \sum_{k=1}^{N} \sum_{h}{}' \xi_k^{-nh}\, a_{k,h}, \tag{8}$$

where

$$a_{k,h} = \int_{i/n-\vartheta_1}^{i/n+\vartheta_2} f^r\left(\frac{h}{k}+y\right) \phi^s\left(\frac{h}{k}+y\right) e^{-2\pi i n y}\, dy. \tag{9}$$

3. *Investigation of* $f(h/k+y)$ *and* $\phi(h/k+y)$.

Throughout paragraphs 3 and 4, let

$$k \leqslant \sqrt{n}, \quad y = i/n + \vartheta, \quad -k^{-1}n^{-\frac{1}{2}} < \vartheta < k^{-1}n^{-\frac{1}{2}}. \tag{10}$$

3.1. *By* (1),

$$\begin{aligned} f(h/k+y) &= \sum_{m=-\infty}^{\infty} e^{2\pi i m^2(h/k+y)} \\ &= \sum_{j=0}^{k-1} \sum_{l=-\infty}^{\infty} e^{2\pi i(h/k+y)(lk+j)^2} \\ &= \sum_{j=0}^{k-1} e^{2\pi i h j^2/k} \sum_{l=-\infty}^{\infty} e^{2\pi i y(lk+j)^2}. \end{aligned} \tag{11}$$

Now, by the formula for transformation of theta functions*,

$$\sum_{l=-\infty}^{\infty} e^{2\pi i y(lk+j)^2} = k^{-1}2^{-\frac{1}{2}}(-iy)^{-\frac{1}{2}} \left\{ 1 + 2\sum_{v=1}^{\infty} \cos(2\pi jv/k)\, e^{-\pi i v^2/(2k^2 y)} \right\}. \tag{12}$$

Let

$$S_{h,k} = \sum_{j=0}^{k-1} e^{2\pi i h j^2/k}. \tag{13}$$

Then, by (11), (12), and (13),

$$f(h/k+y) = F + R, \tag{14}$$

where

$$F = k^{-1}2^{-\frac{1}{2}}(-iy)^{-\frac{1}{2}} S_{h,k}, \tag{15}$$

and

$$R = k^{-1}2^{-\frac{1}{2}}(-iy)^{-\frac{1}{2}} \sum_{v=1}^{\infty} e^{-\pi i v^2/(2k^2 y)} \sum_{j=0}^{k-1} 2\cos(2\pi jv/k)\, e^{2\pi i h j^2/k}. \tag{16}$$

Henceforth, let $A_1, A_2, \ldots$ denote sufficiently large constants which may be absolute or may depend only on r and s. We now estimate the error term R in the expression for $f(h/k+y)$. First it is clear that

$$\begin{aligned} &\left| \sum_{j=0}^{k-1} 2\cos(2\pi jv/k) e^{2\pi i h j^2/k} \right| \\ &\qquad \leqslant \left| \sum_{j=0}^{k-1} e^{2\pi i(hj^2+vj)/k} \right| + \left| \sum_{j=0}^{k-1} e^{2\pi i(hj^2-vj)/k} \right| < A_1 k^{\frac{1}{2}}, \end{aligned} \tag{17}$$

by an application of the well-known inequalities for the Gaussian sums.

* Whittaker and Watson, *A course of modern analysis* (1927), 475-476.

Then, from (16), (17), and (10), it follows that

$$|R| < A_1(k|y|)^{-\frac{1}{2}} \sum_{v=1}^{\infty} |e^{-\pi i v^2/(2k^2 y)}| \tag{18}$$

$$= A_1(k|y|)^{-\frac{1}{2}} \sum_{v=1}^{\infty} e^{-\pi v^2/(2k^2|y|^2 n)}.$$

Now

$$\sum_{v=1}^{\infty} e^{-\pi v^2/(2k^2|y|^2 n)} < \sum_{v=1}^{\infty} e^{-\pi v/(2k^2|y|^2 n)} \tag{19}$$

$$= e^{-\pi/(2k^2|y|^2 n)} \sum_{v=0}^{\infty} e^{-\pi v/(2k^2|y|^2 n)}.$$

Also, by (10), $\pi/(2k^2|y|^2 n) \geqslant \frac{1}{4}\pi$,

and hence

$$\sum_{v=0}^{\infty} e^{-\pi v/(2k^2|y|^2 n)} \leqslant \sum_{v=0}^{\infty} e^{-\pi v/4} = A_2. \tag{20}$$

Combining (18), (19), and (20), we have

$$|R| < A_3(k|y|)^{-\frac{1}{2}} e^{-\pi/(2k^2|y|^2 n)}, \tag{21}$$

from which it is obvious that

$$|R| < A_3(k|y|)^{-\frac{1}{2}}. \tag{22}$$

Using the fact that $e^{-x} < x^{-1}\,(x > 0)$, we obtain from (21) a second inequality for R, viz.

$$|R| < A_3 n(k|y|)^{\frac{3}{2}}. \tag{23}$$

Finally, on using the result $S_{h,k} = O(k^{\frac{1}{2}})$, we obtain from (15)

$$|F| < A_4(k|y|)^{-\frac{1}{2}}. \tag{24}$$

3.2. The function $\phi(h/k+y)$ defined by (2) has been investigated by Estermann*. We have the following results.

$$\phi(h/k+y) = Z+U, \tag{25}$$

where

$$Z = (-2\pi iky)^{-1}\{\gamma - 2\log k - \log(-2\pi iy)\}, \tag{26}$$

* *Loc. cit.*, equations (11), (12), (16), and (17).

γ denoting Euler's constant, and

$$|U| < A_5 k|y| n \log n < A_6 k^{-1}|y|^{-1} \log n. \tag{27}$$

Also, by (10) and (26),

$$|Z| < A_7 k^{-1}|y|^{-1} \log n. \tag{28}$$

4. In this paragraph we find an inequality for the error term introduced by replacing $f^r(h/k+y)\phi^s(h/k+y)$ by $F^r Z^s$ in equation (9).

We have

$$\begin{aligned} |(F+R)^r(Z+U)^s - F^r Z^s| \\ \leqslant |[(F+R)^r - F^r](Z+U)^s| + |F^r[(Z+U)^s - Z^s]|. \end{aligned} \tag{29}$$

Now it is obvious that

$$|(F+R)^r - F^r| < A_8 |R|(|F|^{r-1} + |R|^{r-1}). \tag{30}$$

By (22) and (24),

$$|F|^{r-1} + |R|^{r-1} < A_9(k|y|)^{-\frac{1}{2}r+\frac{1}{2}},$$

and hence, by (23) and (30),

$$|(F+R)^r - F^r| < A_{10} n(k|y|)^{-\frac{1}{2}r+2}. \tag{31}$$

Also, by (27) and (28),

$$|Z+U|^s < A_{11}(k|y|)^{-s} \log^s n,$$

and, on combining this inequality with (31), we have the following result:

$$|[(F+R)^r - F^r](Z+U)^s| < A_{12} n(k|y|)^{-\frac{1}{2}r-s+2} \log^s n. \tag{32}$$

In a similar manner we can find an inequality for the second term on the right-hand side of (29). It is obvious that

$$|(Z+U)^s - Z^s| < A_{13} |U|(|Z|^{s-1} + |U|^{s-1}),$$

and this inequality gives, by using (27) and (28),

$$\begin{aligned} |(Z+U)^s - Z^s| &< A_{14} k|y| n \log n (k|y|)^{-s+1} \log^{s-1} n \\ &= A_{14} n(k|y|)^{-s+2} \log^s n. \end{aligned} \tag{33}$$

By (24),

$$|F|^r < A_{15}(k|y|)^{-\frac{1}{2}r},$$

and hence, by (33),

$$|F^r[(Z+U)^s-Z^s]|<A_{16}n(k|y|)^{-\frac{1}{2}r-s+2}\log^s n. \tag{34}$$

Equations (29), (32), and (34) lead to the inequality

$$|(F+R)^r(Z+U)^s-F^rZ^s|<A_{17}n(k|y|)^{-\frac{1}{2}r-s+2}\log^s n. \tag{35}$$

Now let

$$m=\tfrac{1}{2}r+s-2. \tag{36}$$

Then $m\geqslant\frac{1}{2}$. Also, on putting $y=i/n+\vartheta$ and using (6) and (4), we have

$$\int_{i/n-\vartheta_1}^{i/n+\vartheta_2}|y|^{-\frac{1}{2}}dy<2\int_0^{1/(k\sqrt{n})}\vartheta^{-\frac{1}{2}}d\vartheta=4k^{-\frac{1}{2}}n^{-\frac{1}{4}}, \tag{37}$$

$$\begin{aligned}\int_{i/n-\vartheta_1}^{i/n+\vartheta_2}|y|^{-1}dy&<4\int_0^{1/(k\sqrt{n})}\left(\frac{1}{n}+\vartheta\right)^{-1}d\vartheta\\&=4\log(1+\sqrt{n}/k)<4[1+\log(\sqrt{n}/k)],\end{aligned} \tag{38}$$

and, if $m\geqslant\frac{3}{2}$,

$$\begin{aligned}\int_{i/n-\vartheta_1}^{i/n+\vartheta_2}|y|^{-m}dy&<2\int_0^{\infty}|y|^{-m}d\vartheta=2n^m\int_0^{\infty}|1+ni\vartheta|^{-m}d\vartheta\\&=2n^{m-1}\int_0^{\infty}|1+ix|^{-m}dx=A_{18}n^{m-1}.\end{aligned} \tag{39}$$

From formulae (35), (36), (37), (38), and (39), we obtain

$$\begin{aligned}&\int_{i/n-\vartheta_1}^{i/n+\vartheta_2}|(F+R)^r(Z+U)^s-F^rZ^s|dy\\&\qquad<\begin{cases}A_{19}n^{\frac{1}{4}}k^{-1}\log^s n & \text{if } m=\frac{1}{2};\\ A_{19}nk^{-1}\log^s n\,[1+\log(\sqrt{n}/k)] & \text{if } m=1;\\ A_{20}n^mk^{-m}\log^s n & \text{if } m\geqslant\frac{3}{2}.\end{cases}\end{aligned}$$

Hence, by (9), (10), (14), and (25),

$$\begin{aligned}&\left|a_{k,h}-\int_{i/n-\vartheta_1}^{i/n+\vartheta_2}F^rZ^se^{-2\pi inv}dy\right|\\&\qquad<\begin{cases}A_{21}n^{\frac{1}{4}}k^{-1}\log^s n & \text{if } m=\frac{1}{2};\\ A_{21}nk^{-1}\log^s n\,[1+\log(\sqrt{n}/k)] & \text{if } m=1;\\ A_{22}n^mk^{-m}\log^s n & \text{if } m\geqslant\frac{3}{2}.\end{cases}\end{aligned} \tag{40}$$

5. We now abandon the condition $-k^{-1}n^{-\frac{1}{2}} < \vartheta < k^{-1}n^{-\frac{1}{2}}$, but still assume that $k \leqslant \sqrt{n}$.

Let

$$b_k = \int_{i/n-\infty}^{i/n+\infty} F^r Z^s e^{-2\pi i n y}\, dy, \tag{41}$$

and consider the error introduced by replacing the integral occurring in (40) by b_k.

By (15) and (26) we have, on putting $y = i/n+\vartheta$ and using $|y| > |\vartheta|$,

$$|F| < A_{23}(k|\vartheta|)^{-\frac{1}{2}},$$

$$|Z| < A_{24}(k|\vartheta|)^{-1} \log \max(n, |\vartheta|).$$

Hence, by (4), (6), and (36),

$$\left| b_k - \int_{i/n-\vartheta_1}^{i/n+\vartheta_2} F^r Z^s e^{-2\pi i n y}\, dy \right| \tag{42}$$

$$< A_{25} \int_{1/(2k\sqrt{n})}^{\infty} (k\vartheta)^{-\frac{1}{2}r-s} \log^s \max(n, \vartheta)\, d\vartheta$$

$$< A_{26} k^{-\frac{1}{2}r-s} (k\sqrt{n})^{\frac{1}{2}r+s-1} \log^s n = A_{26} k^{-1} n^{\frac{1}{4}m+\frac{1}{2}} \log^s n.$$

Formulae (40) and (42), together with $k \leqslant \sqrt{n}$, lead to the required inequality for $|a_{k,h} - b_k|$, viz.

$$|a_{k,h} - b_k| < \begin{cases} A_{27} n^{\frac{1}{2}} k^{-1} \log^s n & \text{if } m = \frac{1}{2}; \\ A_{27} n k^{-1} \log^s n\, [1+\log(\sqrt{n}/k)] & \text{if } m = 1; \\ A_{28} n^m k^{-m} \log^s n & \text{if } m \geqslant \frac{3}{2}. \end{cases}$$

Hence, by (8) and (4), on using $\sum_h' |\xi_k^{-nh}| \leqslant k$, we have

$$\left| \nu(n) - \sum_{k=1}^{N} \sum_h' \xi_k^{-nh} b_k \right| \tag{43}$$

$$< \begin{cases} A_{27} n^{\frac{1}{2}} \log^s n & \text{if } m = \frac{1}{2}; \\ A_{27} n \log^s n \left[n^{\frac{1}{2}} + \sum_{k=1}^{N} \log(\sqrt{n}/k) \right] & \text{if } m = 1; \\ A_{28} n^m \log^s n \sum_{k=1}^{N} k^{-m+1} & \text{if } m \geqslant \frac{3}{2}. \end{cases}$$

To estimate the sum $\sum_{k=1}^{N} \log(\sqrt{n}/k)$, we first consider $\sum_{k=1}^{N} \log k$. The function $\log k$ steadily increases with k, from which it follows, by a

simple calculation, that

$$\sum_{k=1}^{N} \log k = \int_1^N \log x\, dx + O(\log N)$$
$$= N \log N + O(N) = \tfrac{1}{2}\sqrt{n} \log n + O(\sqrt{n}).$$

Hence

$$\sum_{k=1}^{N} \log(\sqrt{n}/k) = O(n^{\frac{1}{4}}). \tag{44}$$

Also

$$\sum_{k=1}^{N} k^{-m+1} < \begin{cases} A_{29} n^{\frac{1}{4}} & \text{if } m = \frac{3}{2}; \\ A_{30} \log n & \text{if } m = 2; \\ A_{31} & \text{if } m > 2. \end{cases} \tag{45}$$

It follows from (43), (44), (45), and (36) that

$$\left| \nu(n) - \sum_{k=1}^{N} \sum_{h}{}' \xi_k^{-nh} b_k \right| < A_{32} n^{r_1} \log^{s_1} n \tag{46}$$

(r_1 and s_1 have been defined in the introduction).

Now consider b_k defined by (41), where F and Z are given by equations (15) and (26), but without the restriction $k \leqslant \sqrt{n}$. We have (putting $-2\pi i n y = \omega$)

$$b_k = \pi^{\frac{1}{2}r} k^{-r-s} n^{\frac{1}{2}r+s-1} (S_{h,k})^r \tag{47}$$
$$\times \frac{1}{2\pi i} \int_{2\pi-i\infty}^{2\pi+i\infty} e^{\omega} \omega^{-s-\frac{1}{2}r} (\gamma - 2\log k - \log\omega + \log n)^s\, d\omega.$$

It is known that

$$\frac{1}{2\pi i} \int_{2\pi-i\infty}^{2\pi+i\infty} e^{\omega} \omega^{-t}\, d\omega = \frac{1}{\Gamma(t)} \quad (t > 1).$$

Let $a^{(0)}(t) = 1/\Gamma(t)$, and let $a^{(q)}(t)$ denote the q-th derivative of $a^{(0)}(t)$. Then

$$\frac{1}{2\pi i} \int_{2\pi-i\infty}^{2\pi+i\infty} e^{\omega} \omega^{-t} (-\log\omega)^q\, d\omega = a^{(q)}(t).$$

Hence, by (47), with the multinomial theorem, we have

$$b_k = \pi^{\frac{1}{2}r} n^{\frac{1}{2}r+s-1} k^{-r-s} (S_{h,k})^r \tag{48}$$
$$\times \sum_{\substack{l,m,q,t \\ l,m,q,t \geqslant 0 \\ l+m+q+t=s}} \frac{s!}{l!\ m!\ q!\ t!} \gamma^l (-2\log k)^m a^{(q)}(s+\tfrac{1}{2}r) \log^t n.$$

It follows that

$$|b_k| < A_{33} k^{-\frac{1}{2}r-s} n^{\frac{1}{2}r+s-1} \log^s k \quad \text{if} \quad k > N,$$

and hence that

$$\left| \sum_{k>N} \sum_{h}{}' \xi_k^{-nh} b_k \right| < A_{33} n^{\frac{1}{2}r+s-1} \sum_{k>N} k^{-\frac{1}{2}r-s+1} \log^s k$$
$$< A_{34} n^{\frac{1}{2}r+\frac{1}{2}s} \log^s n$$
$$\leqslant A_{34} n^{r_1} \log^{s_1} n.$$

Thus in (46) we can replace the finite series by the infinite series and write

$$\left| \nu(n) - \sum_{k=1}^{\infty} \sum_{h}{}' \xi_k^{-nh} b_k \right| < A_{35} n^{r_1} \log^{s_1} n. \tag{49}$$

6. *The singular series.*

Let

$$f(u) = \sum_{k=1}^{\infty} \sum_{h}{}' k^{-r-u} (S_{h,k})^r \xi_k^{-nh}, \tag{50}$$

where the infinite series clearly converges absolutely if $\frac{1}{2}r+u > 2$, since $S_{h,k} = O(k^{\frac{1}{2}})$; further, let $f^{(m)}(u)$ denote the m-th derivative of $f(u)$, the 0-th derivative meaning the function itself. Then

$$f^{(m)}(s) = \sum_{k=1}^{\infty} \sum_{h}{}' k^{-r-s} (S_{h,k})^r \xi_k^{-nh} (-\log k)^m,$$

and hence, by (48),

$$\sum_{k=1}^{\infty} \sum_{h}{}' \xi_k^{-nh} b_k = \pi^{\frac{1}{2}r} n^{\frac{1}{2}r+s-1} \sum_{t=0}^{s} \log^t n \sum_{m=0}^{s-t} c_{m,t} f^{(m)}(s), \tag{51}$$

where the c's are constants given by

$$c_{m,t} = \frac{2^m s!}{m!\ t!} \sum_{\substack{l,q \\ l,q \geqslant 0 \\ l+q=s-m-t}} \frac{1}{l!\ q!} \gamma^l a^{(q)}(s+\tfrac{1}{2}r). \tag{52}$$

The formulae (49) and (51) lead to the required asymptotic formula for $\nu(n)$, viz.

$$\nu(n) = \pi^{\frac{1}{2}r} n^{\frac{1}{2}r+s-1} \sum_{t=0}^{s} \log^t n \sum_{m=0}^{s-t} c_{m,t} f^{(m)}(s) + O(n^{r_1} \log^{s_1} n),$$

where the c's are defined by (52).

7. It only remains to investigate the function $f(u)$ defined in (50). A method of evaluating $f(0)$ has been given by Hardy in his paper, "On the representation of a number as the sum of any number of squares, and

in particular of five"*, and I shall follow this method in the subsequent discussion.

Write $$f(u) = \sum_{k=1}^{\infty} k^{-u} A_k,$$

where $$k^r A_k = \Sigma'_h (S_{h,k})^r \xi_k^{-nh}.$$

Hardy has proved that $k^{-u}A_k$ is multiplicative, *i.e.* that, if $(k_1, k_2) = 1$, then

$$(k_1^{-u} A_{k_1})(k_2^{-u} A_{k_2}) = (k_1 k_2)^{-u} A_{k_1 k_2}.$$

It follows that, if the series for $f(u)$ is absolutely convergent, $f(u)$ is expressible as the infinite product

$$f(u) = \prod_p \chi(p), \tag{53}$$

where

$$\chi(p) = 1 + p^{-u} A_p + p^{-2u} A_{p^2} + \ldots, \tag{54}$$

and p runs through all prime values. I quote the values of A_{p^λ} given in Hardy's paper.

It is necessary to consider separately the three cases $r \equiv 0$, $r \equiv 2$, $r \equiv 1, 3 \pmod 4$. Suppose that $n = 2^\alpha N$, where N is odd ($\alpha \geqslant 0$), and let p^t ($t > 0$) denote a general factor of the canonical decomposition of N.

7.1. *Case I.* $r \equiv 0 \pmod 4$.

We have the following results.

$$A_2 = 0.$$

$$A_{2^\lambda} = \begin{cases} (-1)^{\frac{1}{2}r} 2^{(1-\frac{1}{2}r)(\lambda-1)} & \text{if } 1 < \lambda < \alpha+1; \\ -(-1)^{\frac{1}{2}r} 2^{(1-\frac{1}{2}r)\alpha} & \text{if } \lambda = 1+\alpha > 1; \\ 0 & \text{if } \lambda > 1+\alpha. \end{cases}$$

If p is not a divisor of n,

$$A_p = -p^{-\frac{1}{2}r};$$

$$A_{p^\lambda} = 0 \quad \text{if} \quad \lambda > 1.$$

If p divides n,

$$A_{p^\lambda} = \begin{cases} (p-1)p^{\lambda-1-\frac{1}{2}r\lambda} & \text{if } 1 \leqslant \lambda < 1+t; \\ -p^{t-\frac{1}{2}r(1+t)} & \text{if } \lambda = 1+t; \\ 0 & \text{if } \lambda > 1+t. \end{cases}$$

---

* G. H. Hardy, *Trans. Amer. Math. Soc.*, 21 (1920), 255–284.

Hence by (54), putting

$$\tfrac{1}{2}r+u=y, \tag{55}$$

we have

$$\chi(2)=1+(-1)^{\frac{1}{2}r}2^{-u}\{2^{1-y}+2^{2(1-y)}+\ldots+2^{(\alpha-1)(1-y)}-2^{\alpha(1-y)}\}, \tag{56}$$

it being understood that $\chi(2)=1$ if $\alpha=0$;

if $p\nmid n$, $\qquad \chi(p)=1-p^{-y}$;

if $p\mid n$, $\qquad \chi(p)=1+\sum_{\lambda=1}^{t}(p-1)p^{\lambda-1-\lambda y}-p^{t-yt}$

$$=(1-p^{-y})(1+p^{1-y}+\ldots+p^{t(1-y)}).$$

Then, by (53),

$$\begin{aligned} f(u) &= \chi(2)\prod_{\substack{p\nmid n\\ p>2}}(1-p^{-y})\prod_{\substack{p\mid n\\ p>2}}\{(1-p^{-y})(1+p^{1-y}+\ldots+p^{t(1-y)})\} \\ &= \chi(2)\prod_{p>2}(1-p^{-y})\prod_{\substack{p\mid n\\ p>2}}(1+p^{1-y}+\ldots+p^{t(1-y)}) \\ &= \chi(2)(1-2^{-y})^{-1}[\zeta(y)]^{-1}\sum_{\substack{d\mid n\\ 2\nmid d}}d^{1-y}. \end{aligned} \tag{57}$$

If $\alpha=0$, $\chi(2)=1$, and (57) gives $f(u)$ in terms of the divisors of n. If $\alpha>0$, we have, by (56),

$$\chi(2)=1+(-1)^{\frac{1}{2}r}2^{-u}\{(2^y-1)(1+2^{1-y}+\ldots+2^{\alpha(1-y)})-1-2^y\},$$

and hence

$$\begin{aligned} \chi(2)\sum_{\substack{d\mid n\\ 2\nmid d}}d^{1-y} &= \{1-(-1)^{\frac{1}{2}r}2^{-u}(1+2^y)\}\sum_{\substack{d\mid n\\ 2\nmid d}}d^{1-y} \\ &\quad +(-1)^{\frac{1}{2}r}2^{-u}(2^y-1)\sum_{d\mid n}d^{1-y}, \end{aligned}$$

which, with (57), gives $f(u)$ as a simple function of the divisors of n.

7.2. *Case II.* $r\equiv 2 \pmod 4$.

We have the following results.

$$A_{2^\lambda}=0 \quad \text{unless} \quad \lambda=\alpha+2;$$

when $\lambda=\alpha+2$, $\qquad A_{2^\lambda}=(-1)^{\frac{1}{2}(2N-r)}2^{(1-\frac{1}{2}r)(\alpha+1)}$.

If p is not a divisor of n,

$$A_p=-(-1)^{\frac{1}{2}(p-1)}p^{-\frac{1}{2}r};$$

$$A_{p^\lambda}=0 \quad \text{if} \quad \lambda>1.$$

If p divides n,

$$A_{p^\lambda} = \begin{cases} (-1)^{\frac{1}{2}(p-1)\lambda}(p-1)p^{\lambda-1-\frac{1}{2}r\lambda} & \text{if } \lambda < t+1; \\ -(-1)^{\frac{1}{2}(p-1)(t+1)}p^{t-\frac{1}{2}r(1+t)} & \text{if } \lambda = t+1; \\ 0 & \text{if } \lambda > t+1. \end{cases}$$

Hence, by (54) and (55),

$$\chi(2) = 1+(-1)^{\frac{1}{4}(2N-r)}2^{(\alpha+1)(1-y)-u}; \tag{58}$$

if $p \nmid n$, $$\chi(p) = 1-(-1)^{\frac{1}{2}(p-1)}p^{-y};$$

if $p \mid n$, $$\chi(p) = 1+\sum_{\lambda=1}^{t}(-1)^{\frac{1}{2}(p-1)\lambda}(p-1)p^{\lambda(1-y)-1} - (-1)^{\frac{1}{2}(p-1)(t+1)}p^{(1+t)(1-y)-1}$$

$$= \{1-(-1)^{\frac{1}{2}(p-1)}p^{-y}\}\{1+(-1)^{\frac{1}{2}(p-1)}p^{1-y}+p^{2(1-y)} +\ldots+(-1)^{\frac{1}{2}(p-1)t}p^{t(1-y)}\}.$$

It follows from (53) that

$$f(u) = \chi(2)\prod_{\substack{p\nmid n\\ p>2}}\{1-(-1)^{\frac{1}{2}(p-1)}p^{-y}\}\prod_{\substack{p\mid n\\ p>2}}[\{1-(-1)^{\frac{1}{2}(p-1)}p^{-y}\} \times\{1+(-1)^{\frac{1}{2}(p-1)}p^{1-y}+\ldots+(-1)^{\frac{1}{2}(p-1)t}p^{t(1-y)}\}] \tag{59}$$

$$= \chi(2)\prod_{p>2}\{1-(-1)^{\frac{1}{2}(p-1)}p^{-y}\}\prod_{\substack{p\mid n\\ p>2}}\{1+(-1)^{\frac{1}{2}(p-1)}p^{1-y} +\ldots+(-1)^{\frac{1}{2}(p-1)t}p^{t(1-y)}\}.$$

Now it is obvious that

$$\prod_{\substack{p\mid n\\ p>2}}\{1+(-1)^{\frac{1}{2}(p-1)}p^{1-y}+\ldots+(-1)^{\frac{1}{2}(p-1)t}p^{t(1-y)}\} = \sum_{\substack{d\mid n\\ d\equiv 1 \pmod 4}} d^{1-y} - \sum_{\substack{d\mid n\\ d\equiv 3 \pmod 4}} d^{1-y},$$

and this equation, with (58) and (59), gives $f(u)$ as a simple function of the divisors of n.

7.3. *Case III. r odd.*

In this case the A's are given by the following rules:

$$A_2 = 0.$$

If λ is odd, $A_{2^\lambda} = 0$ unless $\lambda = \alpha+3$ and $N \equiv r \pmod 4$; when λ and N have such values,

$$A_{2^\lambda} = (-1)^{\frac{1}{4}(N-r)}2^{(1-\frac{1}{2}r)(\alpha+2)}.$$

If λ is even,

$$A_{2^\lambda} = \begin{cases} \cos(\frac{1}{4}\pi r)\, 2^{(1-\frac{1}{2}r)(\lambda-1)} & \text{if } \lambda < 1+\alpha; \\ -\cos(\frac{1}{4}\pi r)\, 2^{(1-\frac{1}{2}r)\alpha} & \text{if } \lambda = 1+\alpha; \\ (-1)^{\frac{1}{4}(N-1)} \sin(\frac{1}{4}\pi r)\, 2^{(1-\frac{1}{2}r)(\alpha+1)} & \text{if } \lambda = 2+\alpha; \\ 0 & \text{if } \lambda > 2+\alpha. \end{cases}$$

If p is not a divisor of n, $A_p = (n'/p)\, p^{\frac{1}{2}(1-r)}$, where $n' = (-1)^{\frac{1}{2}(r-1)} n$, and (n'/p) is the well-known symbol of Legendre and Jacobi; $A_{p^\lambda} = 0$ if $\lambda > 1$. If p divides n, $A_p = 0$; if λ is odd, $A_{p^\lambda} = 0$ except when $\lambda = t+1$, and then

$$A_{p^\lambda} = \left(\frac{\nu'}{p}\right) p^{t+\frac{1}{2}-\frac{1}{2}r(1+t)},$$

where $\nu = n/p^t$, and $\nu' = (-1)^{\frac{1}{2}(r-1)}\nu$; if λ is even,

$$A_{p^\lambda} = \begin{cases} (p-1)\, p^{\lambda-1-\frac{1}{2}r\lambda} & \text{if } \lambda < t+1; \\ -p^{t-\frac{1}{2}r(1+t)} & \text{if } \lambda = t+1; \\ 0 & \text{if } \lambda > t+1. \end{cases}$$

Let

$$\frac{1}{2}(r-1)+u = z, \tag{60}$$

and put

$$a = \begin{cases} 2l & \text{if } 2 \mid \alpha; \\ 2l+1 & \text{if } 2 \nmid \alpha. \end{cases}$$

Then

$$\chi(2) = 1 + \sum_{\lambda=1}^{l} \cos\left(\frac{\pi r}{4}\right) 2^{\lambda(1-2z)+\frac{1}{2}r-1} + a + b, \tag{61}$$

where

$$a = \begin{cases} \sin(\frac{1}{4}\pi r)(-1)^{\frac{1}{4}(N-1)}\, 2^{(\alpha+1)(\frac{1}{2}-z)-u} & \text{if } 2 \mid \alpha; \\ -\cos(\frac{1}{4}\pi r)\, 2^{\alpha(\frac{1}{2}-z)-u} & \text{if } 2 \nmid \alpha, \end{cases}$$

and $b = 0$ unless α is even and $N \equiv r \pmod 4$; when α and N have such values,

$$b = (-1)^{\frac{1}{4}(N-r)}\, 2^{(\alpha+2)(\frac{1}{2}-z)-u}.$$

If $p \nmid n$, $$\chi(p) = 1 + \left(\frac{n'}{p}\right) p^{-z}. \tag{62}$$

If $p \mid n$, $$\chi(p) = 1 + \sum_{\substack{2 \leqslant \lambda \leqslant t \\ 2 \mid \lambda}} (p-1)\, p^{\lambda(\frac{1}{2}-z)-1} + c, \tag{63}$$

where
$$c = \begin{cases} \left(\frac{\nu'}{p}\right) p^{\frac{1}{2}t - z(1+t)} & \text{if } 2|t; \\ -p^{\frac{1}{2}(t-1) - z(1+t)} & \text{if } 2\nmid t. \end{cases}$$

7.31. *Sum of the singular series when n contains no odd square factor.*

If n has no odd square factor, (63) gives:

(64) If $p|n$, $$\chi(p) = 1 - p^{-2z}.$$

With Hardy, we write

(65) $$\gamma_p(k) = \left\{1 - \left(\frac{k}{p}\right) p^{-z}\right\}^{-1}.$$

Then we may write (62) in the form:

(66) If $p \nmid n$, $$\chi(p) = (1 - p^{-2z})\,\gamma_p(n').$$

It follows from (53), (64), and (66) that

$$f(u) = \chi(2) \prod_{\substack{p \nmid n \\ p > 2}} \{(1 - p^{-2z})\gamma_p(n')\} \prod_{\substack{p|n \\ p>2}} (1 - p^{-2z})$$

$$= \chi(2) \prod_{p>2} (1 - p^{-2z}) \prod_{\substack{p \nmid n \\ p>2}} \gamma_p(n')$$

$$= \chi(2)(1 - 2^{-2z})^{-1} [\zeta(2z)]^{-1} \prod_{p \nmid 2n} \gamma_p(n').$$

Also, by (65),

(67) $$\prod_{p \nmid 2n} \gamma_p(n') = \sum_{\substack{m \\ (m, 2n)=1}} \left(\frac{n'}{m}\right) m^{-z},$$

and hence

$$f(u) = \chi(2)(1 - 2^{-2z})^{-1} [\zeta(2z)]^{-1} \sum_{\substack{m \\ (m, 2n)=1}} \left(\frac{n'}{m}\right) m^{-z}.$$

Now let us define $\psi(n)$ by the equation

(68) $$\psi(n) = \sum_{\substack{m \\ (m, 2n)=1}} \left(\frac{n'}{m}\right) m^{-z}.$$

7.32. Proceeding to the general case, write $n = \Omega^2 D$, where Ω^2 is the largest square dividing n. Then the odd primes fall into four classes

characterized as follows:

(i) Primes q for which $q \nmid n$. In this case we have, by (66),

$$\chi(q) = (1-q^{-2z})\,\gamma_q(n').$$

(ii) Primes ω_1 for which $\omega_1 | D$, $\omega_1 \nmid \Omega$. In this case we have, by (63),

$$\chi(\omega_1) = (1-\omega_1^{-2z}).$$

(iii) Primes ω_2 for which $\omega_2 \nmid D$, $\omega_2 | \Omega$. In this case t is even and we write $t = 2b_2$. By (63),

$$\chi(\omega_2) = 1 + \sum_{\lambda=1}^{b_2} (\omega_2 - 1)\,\omega_2^{\lambda(1-2z)-1} + \left(\frac{\nu'}{\omega_2}\right) \omega_2^{b_2 - z(2b_2+1)}.$$

On factorizing this series and using (65), we have

$$\chi(\omega_2) = (1-\omega_2^{-2z})\{1+\omega_2^{1-2z}+\ldots+\omega_2^{(b_2-1)(1-2z)}+\gamma_{\omega_2}(\nu')\,\omega_2^{b_2(1-2z)}\}.$$

(iv) Primes ω_3 for which $\omega_3 | D$, $\omega_3 | \Omega$. In this case t is odd and we write $t = 2b_3+1$. By (63),

$$\chi(\omega_3) = 1 + \sum_{\lambda=1}^{b_3} (\omega_3 - 1)\,\omega_3^{\lambda(1-2z)-1} - \omega_3^{b_3 - 2z(b_3+1)}$$

$$= (1-\omega_3^{-2z})(1+\omega_3^{1-2z}+\ldots+\omega_3^{b_3(1-2z)}).$$

It follows from (53) that

$$f(u) = \chi(2) \prod_{p>2} (1-p^{-2z})\,\phi(u) \tag{69}$$

$$= \chi(2)(1-2^{-2z})^{-1}\,[\zeta(2z)]^{-1}\,\phi(u),$$

with

$$\phi(u) = \prod_q \gamma_q(n') \prod_{\omega_2} \sum \lambda_l(\omega_2)\,\omega_2^{l(1-2z)} \prod_{\omega_3} \sum \omega_3^{k(1-2z)},$$

where, in the summations, $0 \leqslant l \leqslant b_2$, $0 \leqslant k \leqslant b_3$.

Also $\lambda_l(\omega_2) = 1$ if $l < b_2$; $\lambda_{b_2}(\omega_2) = \gamma_{\omega_2}(\nu')$, where $\nu' = (-1)^{\frac{1}{2}(r-1)}\,\omega_2^{-2b_2}\,n$.

Let $p_1, \ldots, p_h$ be the distinct primes of type ω_2, and let $l_1, \ldots, l_h$ be the corresponding l's; let $\pi_1, \ldots, \pi_j$ be the distinct primes of type ω_3, and let $k_1, \ldots, k_j$ be the corresponding k's. Then the general term of $\phi(u)$ is

$$\prod_q \gamma_q(n') \cdot \lambda_{l_1}(p_1) \ldots \lambda_{l_h}(p_h)\,[p_1^{l_1} \ldots p_h^{l_h}\,\pi_1^{k_1} \ldots \pi_j^{k_j}]^{(1-2z)}. \tag{70}$$

Suppose, first, that l does not assume the value b_2 for any ω_2; then it is clear that we may write the term (70) in the form

$$d^{1-2z} \prod_{\substack{p \\ p \nmid n/d^2}} \gamma_p\left(\frac{n'}{d^2}\right),$$

where d^2 is a typical odd square divisor of n. The result also holds if $l = b_2$ for any of the primes ω_2. For then the additional factor in

$$\prod_{\substack{p \\ p \nmid n/d^2}} \gamma_p \left(\frac{n'}{d^2}\right)$$

is allowed for by the factor $\lambda_{b_2}(\omega_2)$ in (70).

Hence
$$\phi(u) = \sum_{\substack{d \\ d^2 \mid n \\ 2 \nmid d}} d^{1-2z} \prod_{\substack{p \\ p \nmid n/d^2}} \gamma_p \left(\frac{n'}{d^2}\right),$$

and hence, by (67) and (68),

$$\phi(u) = \sum_{\substack{d \\ d^2 \mid n \\ 2 \nmid d}} d^{1-2z} \psi \left(\frac{n}{d^2}\right).$$

Finally, by (69), we have the required series for $f(u)$ in the case when r is odd, viz.:

$$f(u) = \chi(2)(1-2^{-2z})^{-1} [\zeta(2z)]^{-1} \sum_{\substack{d \\ d^2 \mid n \\ 2 \nmid d}} d^{1-2z} \psi \left(\frac{n}{d^2}\right),$$

where $\psi(n)$ is defined in equation (68).

I wish to thank Dr. T. Estermann for his valuable suggestions and criticisms.

A DIRICHLET PROBLEM FOR HARMONIC FUNCTIONALS, WITH APPLICATIONS TO ANALYTIC VARIETIES

By W. V. D. Hodge.

[Received and read 28 April, 1932.]

In his recent thesis† M. de Rham has established the existence of p-ple integrals of total differentials attached to a certain class of real analytic varieties of n-dimensions, which have arbitrary periods on the p-cycles of the variety, for all values of p from 0 to n. If we apply this important result to the simple case of a Riemann surface, we find that de Rham's work does not say anything about the harmonic integrals of the first kind which exist. The question then arises, can we impose conditions on the integrals considered by de Rham which will reduce to the conditions for a harmonic integral on a Riemann surface when $n = 2p = 2$? This is the question which I have sought to solve in this paper.

The conditions which I have imposed on the integrals are essentially those imposed by Volterra in his work on harmonic functionals. The integrals in question are indeed harmonic functionals of the boundary of the field of integration. It has been necessary to consider some problems which arise in connection with harmonic functionals in a Euclidean space of n-dimensions before proceeding to consider the problem proposed for analytic varieties, and these investigations are the subject of Sections I, II, III. Section I is devoted to definitions and simple properties of harmonic functionals. In it I find that most of the elementary properties of harmonic functions can be extended to harmonic functionals. I have also obtained an invariant form for the condition that a function of the $(p-1)$-bounding cycles of a space should be harmonic, which is valid in any system of coordinates. In Section II I have proposed a Dirichlet problem for harmonic functionals, and I have shown that there is just one harmonic functional

† G. de Rham, *Journal de Math.* (9), 10 (1931), 115-200.

in a Euclidean cell which has given values on the boundary, when this boundary, and the boundary values, are sufficiently smooth. In Section III I have considered the case $n = 2p$ and obtained for this a Poisson integral, and have used it to obtain certain results which are of importance in the sequel.

Section IV deals with the applications to analytic varieties. I first lay down the conditions to be satisfied by the variety. These conditions are that it should be conformally flat, and orientable. I then show that it is possible to solve the Dirichlet problem for cells on the variety, when $n = 2p$. I then proceed to establish my main theorem, that there exist p-ple integrals which are harmonic everywhere on the variety, and that these are all linearly dependent on R_p particular integrals of this type, where R_p is the p-th Betti number of the variety. The integrals are the generalizations of the harmonic integrals of the first kind on a Riemann surface. I also establish the existence of harmonic integrals of the second and third kinds.

In establishing the results of this paper, I have based my arguments very largely on standard works on harmonic functions. In most places small alterations have been necessary, and I have therefore had to repeat a certain amount of well-known analysis. For the sake of compactness, my plan has been to summarise as much as possible the argument on which I have based my extensions, and only where it is necessary have I given details. I have given references to works used, as far as possible, but I wish to state here the names of the four treatises to which I am chiefly indebted.

The topology is based on the work of Lefschetz†. In Section II the quotations from ordinary potential theory are from Kellogg's‡ treatise, and the discussion of the integral equation follows the account given in the tract of Heywood and Frechet§. The last part of Section III, and all Section IV, is modelled on the work of Weyl||, which I have followed as closely as possible, even to the extent of notation.

I. *Definitions and simple properties of harmonic functionals.*

1. In a real Euclidean space of n-dimensions take a system of rectangular Cartesian axes $(x_1, x_2, \ldots, x_n)$. Then we shall speak of the

† S. Lefschetz, "Topology", *American Math. Soc. Colloquium Publications* (1930).

‡ O. D. Kellogg, *Foundations of potential theory* (Springer, 1929).

§ H. B. Heywood et M. Frechet, *L'equation de Fredholm* (Hermann, 1912).

|| H. Weyl, *Die Idee der Riemannschen Fläche* (Teubner, 1923).

differential form

$$\phi = \sum_{i_1 \dots i_{p-1}} A_{i_1 \dots i_{p-1}} dx_i \dots dx_{i_{p-1}}$$

as a $(p-1)$-form, where the coefficients $A_{i_1 \dots i_{p-1}}$ are functions of the x's. In most of our applications we shall assume that these coefficients have second partial derivatives which are continuous. The form is then said to be regular. The elementary properties of these forms have been discussed by Cartan†, and in particular we know that a permutation of the order of the differentials in any term merely results, at most, in a change of sign in this term, and therefore that there is no loss of generality in supposing that each set of $p-1$ of the differentials $dx_1, dx_2, \dots, dx_n$ appears in only one term of ϕ. Again, the meaning to be attached to the integral

$$\int_{C_{p-1}} \phi,$$

taken over a given field C_{p-1} in our space, has been discussed fully by Poincaré‡. If C_{p-1} is a closed cycle γ in the space, we shall say that

$$\int_\gamma \phi$$

is a functional, or a function of γ.

We recall a few properties of $(p-1)$-forms. The covariant derivative of ϕ is defined as

$$\phi_x = \sum_{i_1 \dots i_p} B_{i_1 \dots i_p} dx_{i_1} \dots dx_{i_p},$$

where

$$B_{i_1 \dots i_p} = \sum_{k=1}^{p} (-1)^{k-1} \frac{\partial A_{i_1 \dots i_{k-1} i_{k+1} \dots i_p}}{\partial x_{i_k}}. \tag{1.1}$$

If γ is a cycle bounding a chain δ of p-dimensions,

$$\int_\gamma \phi = \int_\delta \phi_x.$$

This is the theorem of Stokes. We have, further, the identity

$$\phi_{xx} = 0.$$

If

$$\phi' = \sum_{j_1 \dots j_q} A'_{j_1 \dots j_q} dx_{j_1} \dots dx_{j_q}$$

† E. Cartan, *Leçons sur les invariants intégraux* (Hermann, 1922), Chap. 7.

‡ *Cf.* (for example) H. Poincaré, *Acta Math.*, 9 (1887), 187.

is a q-form, product of the ϕ by ϕ' is defined as

$$\phi \,.\, \phi' = \sum_{k_1 \dots k_{p+q-1}} C_{k_1 \dots k_{p+q-1}} dx_{k_1} \dots dx_{k_{p+q-1}},$$

where

$$C_{k_1 \dots k_{p+q-1}} = \Sigma \pm A_{i_1 \dots i_{p-1}} A'_{j_1 \dots j_q},$$

summed for all $i_1, \dots, i_{p-1}, j_1, \dots, j_q$ which are derangements of $k_1, \dots, k_{p+q-1}$, the positive sign being taken for the even derangements, the negative sign for the odd derangements. With this definition we have the following formulae:

$$\phi \,.\, \phi' = (-1)^{(p-1)q} \phi' \,.\, \phi,$$

$$(\phi \,.\, \phi')_x = \phi_x \,.\, \phi' + (-1)^{p-1} \phi \,.\, \phi_x'.$$

We verify immediately that products and covariant derivatives are unaltered by a change of coordinates.

2. Let us now associate with the p-form ϕ_x the $(n-p)$-form

$$\phi^x = \Sigma \pm B_{i_1 \dots i_p} dx_{i_{p+1}} \dots dx_{i_n},$$

where $i_1, \dots, i_n$ is a derangement of the first n integers, and we take the positive sign when the derangement is even, the negative sign when it is odd. We then write

$$(\phi^x)_x = \Delta\phi.$$

Following Volterra†, we shall say that $\int_\gamma \phi$ is a harmonic functional, or a *harmonic function* of γ, if ϕ_x is regular, and

$$\Delta\phi = 0.$$

Also, in this case, we shall speak of ϕ as a *harmonic form*. We observe that

(1) If $p = 1$, the condition is

$$\frac{\partial^2 \phi}{\partial x_1^2} + \dots + \frac{\partial^2 \phi}{\partial x_n^2} = 0.$$

(2) The condition is satisfied by the real or imaginary parts of the integral

$$\int_\delta \sum_i P_{i_1 \dots i_p} dz_{i_1} \dots dz_{i_p}$$

† V. Volterra, *Theory of functionals* (Blackie, 1930), Chap. 3.

when this is regarded as a function of the boundary of δ, and the form under the integral sign is a total differential in the complex variables $z_1, z_2, \ldots, z_m$, where $2m = n$, and

$$z_r = x_r + ix_{m+r}.$$

3. We can put the definition which we have just given in an invariant form. Let $(x_1, \ldots, x_n)$ be any parameters in an analytic cell, having the fundamental (definite and positive) quadratic form

$$ds^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} g_{ij} dx_i dx_j.$$

If ϕ is a $(p-1)$-form in this cell,

$$\phi = \sum_i A_{i_1 \ldots i_{p-1}} dx_{i_1} \ldots dx_{i_{p-1}},$$

the covariant derivative is defined as before,

$$\phi_x = \sum_i B_{i_1 \ldots i_p} dx_{i_1} \ldots dx_{i_p},$$

where $B_{i_1 \ldots i_p}$ has the value given by (1.1). We shall write

$$\begin{vmatrix} g_{i_1 j_1} \cdots g_{i_1 j_p} \\ g_{i_2 j_1} \cdots g_{i_2 j_p} \\ \cdots \quad \cdots \quad \cdots \\ g_{i_p j_1} \cdots g_{i_p j_p} \end{vmatrix} = \begin{pmatrix} i_1 \ldots i_p \\ j_1 \ldots j_p \end{pmatrix},$$

$$\begin{pmatrix} 1 \ldots n \\ 1 \ldots n \end{pmatrix} = \Omega^2,$$

and let Ω be the positive root of this equation. We define ϕ^x by the equation

$$\phi^x = \sum_i \sum \pm \frac{B_{i_1 \ldots i_p}}{\Omega} \begin{pmatrix} i_{p+1} \ldots i_n \\ j_{p+1} \ldots j_n \end{pmatrix} dx_{j_{p+1}} \ldots dx_{j_n}$$

with the signs determined as before. We can verify immediately that ϕ^x is invariant under a change of parameters. ϕ^x can therefore be called the contravariant derivative†. A harmonic form is one which satisfies the equation

$$\Delta\phi = (\phi^x)_x = 0.$$

† In the remainder of this paper, when we speak of the derivative of a form without an adjective attached, we shall mean the covariant derivative.

The generalized Beltrami parameter of the second order is

$$\Delta_2(\phi) = (\phi^x)^x,$$

and the condition can also be written in the form

$$\Delta_2(\phi) = 0.$$

In this paper we shall deal only with the case

$$g_{ij} = \delta_{ij}\dagger,$$

with one exception, which we shall now explain. We shall be concerned chiefly with the Dirichlet problem for harmonic functionals, and we shall find that we can solve this problem in a Euclidean n-space referred to rectangular Cartesian coordinates, in a large number of cases. If we can deduce from this the solution of the corresponding problem in a cell with a more general fundamental form we shall be able to extend the scope of our results accordingly. One case in which this is possible is obvious. The condition that a $(p-1)$-form ϕ should be harmonic when the fundamental form is

$$ds^2 = \lambda^2(dx_1^2 + \ldots + dx_n^2)$$

is the same as the condition that ϕ should be harmonic when the fundamental form is Euclidean, in the case $n = 2p$. In other words, if we have a conformal correspondence between the points of an analytic cell of $2p$ dimensions and a Euclidean cell, a $(p-1)$-form which is harmonic in the one corresponds to a harmonic form in the other.

The difference between the invariance of the $(p-1)$-forms in this special case and the invariance implied by the definition can be illustrated in this way. Suppose that ϕ is a $(p-1)$-form which is harmonic in a Euclidean space of n dimensions. Consider the form which is obtained by inverting in a sphere of radius a with centre at the origin. This is obtained by means of the equations

$$x_i' = \frac{a^2}{r^2}\, x_i, \quad r^2 = x_1^2 + \ldots + x_n^2.$$

ϕ becomes a form ϕ' which is harmonic with respect to the fundamental form

$$\left(\frac{r'}{a}\right)^4 (dx_1'^2 + \ldots + dx_n'^2).$$

† We shall always denote the Kronecker delta in this manner.

But when $n = 2p$ our result states that ϕ' is also harmonic with respect to the form

$$dx_1'^2 + \ldots + dx_n'^2.$$

The reasoning of Section III will show the importance of this fact.

We also notice in the case $n = 2p$, that, if ϕ is a $(p-1)$-form in a Euclidean n-space, and ϕ' is the form obtained by a conformal correspondence,

$$\phi^x = \phi'^{x'} \quad \text{or} \quad \phi^x = -\phi'^{x'},$$

according as the correspondence is direct or inverse, all the contravariant derivatives being calculated for the fundamental forms

$$dx_1^2 + \ldots + dx_n^2, \quad dx_1'^2 + \ldots + dx_n'^2.$$

A correspondence is said to be *direct* when the Jacobian of the transformation is positive, and *inverse* when it is negative.

4. We now return to a Euclidean n-space, with which we shall deal exclusively until Section IV.

Let us expand the $(n-p+1)$-form $\Delta\phi$, where

$$\phi = \sum_i A_{i_1 \ldots i_{p-1}} dx_{i_1} \ldots dx_{i_{p-1}}.$$

Let

$$\phi^x = \sum_i B_{i_1 \ldots i_p} dx_{i_{p+1}} \ldots dx_{i_n},$$

where we suppose the differentials to be always arranged so that $(i_1, \ldots, i_n)$ is an even derangement of the natural order. This is always possible when $n > 2$. No difficulty will arise in amending the proof in the case $n = 2$. Then

$$\Delta\phi = \sum_i C_{i_1 \ldots i_{p-1}} dx_{i_p} \ldots dx_{i_n},$$

where

$$C_{i_1 \ldots i_{p-1}} = \sum_{k=p}^{n} \frac{\partial B_{i_1 \ldots i_{p-1} i_k}}{\partial x_{i_k}}.$$

But

$$B_{i_1 \ldots i_{p-1} i_k} = \sum_{r=1}^{p-1} (-1)^{r-1} \frac{\partial A_{i_1 \ldots i_{r-1} i_{r+1} \ldots i_{p-1} i_k}}{\partial x_{i_r}} + (-1)^{p-1} \frac{\partial A_{i_1 \ldots i_{p-1}}}{\partial x_{i_k}}.$$

Therefore

$$C_{i_1 \ldots i_{p-1}} = \sum_{r=1}^{p-1} (-1)^{r-1} \frac{\partial}{\partial x_{i_r}} \sum_{k=p}^{n} \frac{\partial A_{i_1 \ldots i_{r-1} i_{r+1} \ldots i_{p-1} i_k}}{\partial x_{i_k}} + (-1)^{p-1} \sum_{k=p}^{n} \frac{\partial^2 A_{i_1 \ldots i_{p-1}}}{\partial x_{i_k}^2}$$

$$= \sum_{r=1}^{p-1} (-1)^{r-1} \frac{\partial}{\partial x_{i_r}} \sum_{k=1}^{n} \frac{\partial A_{i_1 \ldots i_{r-1} i_{r+1} \ldots i_{p-1} i_k}}{\partial x_{i_k}} + (-1)^{p-1} \sum_{k=1}^{n} \frac{\partial^2 A_{i_1 \ldots i_{p-1}}}{\partial x_{i_k}^2},$$

if we assume that $A_{i_1 \dots i_{p-1}} = 0$ when the suffixes are not all distinct. Again, if $\Delta\phi = 0$, we have

$$\sum_{r=1}^{p+1} (-1)^{r-1} \frac{\partial B_{i_1 \dots i_{r-1} i_{r+1} \dots i_{p+1}}}{\partial x_{i_r}} = 0,$$

and

$$\sum_{k=p}^{n} \frac{\partial B_{i_1 \dots i_{p-1} i_k}}{\partial x_{i_k}} = 0.$$

Hence

$$(-1)^{p-1} \frac{\partial B_{i_1 \dots i_p}}{\partial x_{i_k}} = \sum_{r=1}^{p} (-1)^{r-1} \frac{\partial}{\partial x_{i_r}} B_{i_1 \dots i_{r-1} i_{r+1} \dots i_p i_k} \quad (k > p).$$

Therefore

$$(-1)^{p-1} \sum_{k=p+1}^{n} \frac{\partial^2 B_{i_1 \dots i_p}}{\partial x_{i_k}^2} = \sum_{r=1}^{p} (-1)^{r-1} \frac{\partial}{\partial x_{i_r}} \sum_{k=p+1}^{n} \frac{\partial}{\partial x_{i_k}} B_{i_1 \dots i_{r-1} i_{r+1} \dots i_p i_k}$$

$$= (-1)^p \sum_{r=1}^{p} \frac{\partial^2}{\partial x_{i_r}^2} B_{i_1 \dots i_p}.$$

Hence

$$\sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2} B_{i_1 \dots i_p} = 0,$$

so that the coefficients of ϕ_x are harmonic functions of $(x_1, \dots, x_n)$.

5. Let us now consider a particular functional which corresponds to the logarithmic function in the case $p = 1$, $n = 2$. Let γ_1, γ_2 be any two $(p-1)$-cycles in our space. Consider the form

$$\omega_{p-1} = \sum_i \frac{d\xi_{i_1} \dots d\xi_{i_{p-1}} dx_{i_1} \dots dx_{i_{p-1}}}{r^{n-2}}$$

in two sets of variables $(\xi_1, \dots, \xi_n)$ and $(x_1, \dots, x_n)$, where

$$r^2 = \sum_{i=1}^{n} (x_i - \xi_i)^2.$$

When $n = 2$, we replace $1/r^{n-2}$ by $\log r$. The functional in question is

$$\int_{\gamma_1} \int_{\gamma_2} \omega_{p-1},$$

where the integration over γ_1 takes place with respect to the ξ-variables, and the integration over γ_2 is performed with respect to the x-variables. This can be regarded as a function either of γ_1 or of γ_2. We shall regard it as a function of γ_2, and prove that it is harmonic except when γ_1 meets γ_2.

It is first to be noted that, if

$$\phi = \sum_{i,j} A_{i_1 \dots i_p, j_1 \dots j_q} dx_{i_1} \dots dx_{i_p} d\xi_{j_1} \dots d\xi_{j_q}$$

is any form which is regular in the two sets of variables $(x_1, \dots, x_n)$ and $(\xi_1, \dots, \xi_n)$, and if the coefficients satisfy the conditions for the interchange of the order of differentiation in the second derivatives, we find by direct calculation that

$$\phi_{x\xi} = \phi_{\xi x}, \quad \phi^{x\xi} = \phi^{\xi x},$$

$$(\phi^x)_\xi = \phi_\xi{}^x = (\phi_\xi)^x.$$

Apply these results to ω_{p-1} and we have

$$\omega_{p-1,x} = -(n-2) \sum_i \sum_{k=i_p}^{i_n} \frac{x_k - \xi_k}{r^n} d\xi_{i_1} \dots d\xi_{i_{p-1}} dx_k dx_{i_1} \dots dx_{i_{p-1}},$$

$$\omega^x_{p-1} = -(n-2) \sum \pm \frac{x_k - \xi_k}{r^n} d\xi_{i_1} \dots d\xi_{i_{p-1}} dx_{j_1} \dots dx_{j_{n-p}},$$

where $k, i_1, \dots, i_{p-1}, j_1, \dots, j_{n-p}$ is a derangement of the first n integers, the positive sign being taken for an even derangement and the negative sign for an odd derangement. If $n \geqslant 4$, we shall find that it is convenient to write this

$$\omega^x_{p-1} = -(n-2) \sum_i \frac{x_{i_1} - \xi_{i_1}}{r^n} d\xi_{i_2} \dots d\xi_{i_p} dx_{i_{p+1}} \dots dx_{i_n} = (-1)^{n(p-1)-1} \omega^\xi_{n-p},$$

the summation being extended to all *even* derangements $(i_1, \dots, i_n)$ of the natural order, which give distinct terms. Then

$$\begin{aligned}
\omega^x_{p-1,\xi} &= p(n-2) \sum_i \frac{d\xi_{i_1} \dots d\xi_{i_p} dx_{i_{p+1}} \dots dx_{i_n}}{r^n} \\
&\qquad - n(n-2) \sum_{j=1}^{n} (x_j - \xi_j) d\xi_j \sum_i \frac{x_{i_1} - \xi_{i_1}}{r^{n+2}} d\xi_{i_2} \dots d\xi_{i_p} dx_{i_{p+1}} \dots dx_{i_n} \\
&= -(n-p)(n-2) \sum_i \frac{d\xi_{i_1} \dots d\xi_{i_p} dx_{i_{p+1}} \dots dx_{i_n}}{r^n} \\
&\qquad + n(n-2) \sum_{k=1}^{n} (x_k - \xi_k) dx_k \sum_i \frac{d\xi_{i_1} \dots d\xi_{i_p} (x_{i_{p+1}} - \xi_{i_{p+1}}) dx_{i_{p+2}} \dots dx_{i_n}}{r^{n+2}} \\
&= (-1)^{p(n-p)-1} \omega^\xi_{n-p-1,x}, \quad \text{or zero if } n = p.
\end{aligned}$$

Hence $\omega^{x\xi}_{p-1} = -\omega_{n-p-1,x\xi}$, or zero. Hence $\Delta_x \omega^\xi = 0$ and $\Delta_x \omega_\xi = 0$. Similarly $\Delta_\xi \omega_x = 0 = \Delta_\xi \omega^x$, except when $r = 0$. Hence $\int_{\gamma_1} \int_{\gamma_2} \omega_{p-1}$ is a

harmonic function of γ_2, provided that γ_2 does not meet γ_1. The cases $n < 4$ can easily be dealt with separately, and yield similar results.

Let Γ be any $(n-p)$-cycle not meeting γ_1 and let integration over this be with respect to x. Then our analysis shows that

$$\int_{\gamma_1}\int_{\Gamma}\omega_{p-1}^{x}$$

is unaltered however we deform γ_1 and Γ provided that they do not meet during the deformation. If γ_1 and Γ are "linked" k times, we can deform them into k $(p-1)$-spheres $\gamma_1', \ldots, \gamma_k'$, and k $(n-p)$-spheres $\Gamma_1', \ldots, \Gamma_k'$ respectively, γ_i', Γ_i' being linked, while γ_i', Γ_j' are not linked if $i \neq j$. Now

$$\int_{\gamma_{i'}}\int_{\Gamma_{i'}}\omega_{p-1}^{x} = a, \qquad \int_{\gamma_{i'}}\int_{\Gamma_{j'}}\omega_{p-1}^{x} = 0,$$

where a is a pure number. Hence

$$\frac{1}{a}\int_{\gamma_1}\int_{\Gamma}\omega_{p-1}^{x} = k,$$

the number of times that γ_1 and Γ are linked.

6. It is possible to establish the theorem of Green for our functionals. Let ϕ and ψ be two $(p-1)$-forms. If

$$\phi_x = \sum_i A_{i_1 \ldots i_p} dx_{i_1} \ldots dx_{i_p},$$

$$\psi_x = \sum_i B_{i_1 \ldots i_p} dx_{i_1} \ldots dx_{i_p},$$

we have $$\phi_x \cdot \psi^x = \sum_i A_{i_1 \ldots i_p} B_{i_1 \ldots i_p} dx_1 \ldots dx_n = \psi_x \cdot \phi^x.$$

Also $$(\phi^x \cdot \psi)_x = \Delta\phi \cdot \psi + (-1)^{n(p-1)} \psi_x \cdot \phi^x,$$

$$(\psi^x \cdot \phi)_x = \Delta\psi \cdot \phi + (-1)^{n(p-1)} \phi_x \cdot \psi^x,$$

so that $$(\phi^x \cdot \psi)_x - (\psi^x \cdot \phi)_x = \Delta\phi \cdot \psi - \Delta\psi \cdot \phi.$$

If D is any domain in our space, bounded by a contour Γ, we have

$$\int_{\Gamma} \phi^x \cdot \psi - \int_{\Gamma} \psi^x \cdot \phi = \int_{\Delta} \Delta\phi \cdot \psi - \int_{D} \Delta\psi \cdot \phi,$$

and, in particular, if $\int_{\gamma} \phi$ and $\int_{\gamma} \psi$ are harmonic functions of γ for all

$(p-1)$-cycles in D,

$$\int_\Gamma \phi^x \cdot \psi = \int_\Gamma \psi^x \cdot \phi.$$

As a special case, let $\psi = dx_{i_1} \dots dx_{i_{p-1}}$. Then

$$\int_\Gamma \phi^x \cdot dx_{i_1} \dots dx_{i_{p-1}} = 0.$$

7. Let us consider any domain Δ of our space having a boundary Γ and let ϕ be a $(p-1)$-form which is harmonic in Δ. If $(\xi_1, \dots, \xi_n)$ is any point A of Δ, the coefficient of $d\xi_{i_1} \dots d\xi_{i_p}$ in $\omega^x_{p-1,\xi}$ is harmonic in Δ except at A. If Γ' is a sphere of radius ρ about A, enclosing a region Δ', this coefficient is harmonic in $\Delta - \Delta'$. Hence, applying Green's theorem, we have

$$\int_\Gamma \omega^x_{p-1,\xi} \cdot \phi - \int_\Gamma \phi^x \cdot \omega_{p-1,\xi} = \int_{\Gamma'} \omega^x_{p-1,\xi} \phi - \int_{\Gamma'} \phi^x \cdot \omega_{p-1,\xi}.$$

Now

$$\int_{\Gamma'} \omega^x_{p-1,\xi} \cdot \phi = -\frac{(n-2)(n-p)}{\rho^n} \int_{\Gamma'} \sum_i d\xi_{i_1} \dots d\xi_{i_p} dx_{i_{p+1}} \dots dx_{i_n} \cdot \phi \quad (n > 2)$$

$$\left(\text{since } \sum_{k=1}^n (x_k - \xi_k) dx_k = 0 \text{ on } \Gamma'\right)$$

$$= (-1)^{n-p+1} \frac{(n-2)(n-p)}{\rho^n} \int_{\Delta'} \sum_i d\xi_{i_1} \dots d\xi_{i_p} dx_{i_{p+1}} \dots dx_{i_n} \cdot \phi_x$$

$$= (-1)^{n(p+1)-1} \frac{(n-2)(n-p)}{\rho^n} \int_{\Delta'} \sum_i B_{i_1 \dots i_p} d\xi_{i_1} \dots d\xi_{i_p} \cdot dx_1 \dots dx_n,$$

where

$$\phi_x = \sum_i B_{i_1 \dots i_p} dx_{i_1} \dots dx_{i_p}.$$

Now $B_{i_1 \dots i_p}$ is harmonic in Δ'. Hence

$$\int_{\Delta'} B_{i_1 \dots i_p} dV = \rho^n \alpha_n B_{i_1 \dots i_p} \quad (\text{at } A),$$

where $\rho^n \alpha_n$ is the content of Δ'. Similarly

$$\int_{\Gamma'} \phi^x \cdot \omega_{p-1,\xi} = \frac{n-2}{\rho^n} \int_{\Gamma'} \phi^x \cdot \sum_1^n (x_k - \xi_k) d\xi_k \sum_i d\xi_{i_1} \dots d\xi_{i_{p-1}} dx_{i_1} \dots dx_{i_{p-1}}$$

$$= (-1)^{n(p+1)} \frac{(n-2)p}{\rho^n} \int_{\Delta'} \sum_i B_{i_1 \dots i_p} d\xi_{i_1} \dots d\xi_{i_p} dx_1 \dots dx_n$$

$$= (-1)^{n(p+1)} (n-2) p \alpha_n \sum_i B_{i_1 \dots i_p} d\xi_{i_1} \dots d\xi_{i_p} \quad (\text{at } A).$$

Hence

$$\int_\Gamma \omega_{p-1,\xi} \cdot \phi^x - \int_\Gamma \phi \cdot \omega^x_{p-1,\xi} = n(n-2)\, a_n \sum_i B_{i_1 \dots i_p} d\xi_{i_1} \dots d\xi_{i_p},$$

where the coefficients have their values at A. It follows that, if γ is any $(p-1)$-cycle lying in Δ,

$$k_n \int_\gamma \phi = \int_\gamma \int_\Gamma \omega_{p-1} \cdot \phi^x - \int_\gamma \int_\Gamma \phi \cdot \omega^x_{p-1}, \quad \text{where} \quad k_n = n(n-2)\, a_n \quad (n > 2),$$
$$k_2 = -2\pi,$$

and thus we can express the functional in terms of the values of ϕ and ϕ^x on the boundary of the domain.

8. We can define an integral associated with any form whose derivative has continuous coefficients which is the natural extension of the Dirichlet integral which is so important in the theory of harmonic functions. Consider any such $(p-1)$-form ϕ. Let

$$\phi_x = \sum_i B_{i_1 \dots i_p} dx_{i_1} \dots dx_{i_p}.$$

Then

$$\phi_x \cdot \phi^x = \sum_i B^2_{i_1 \dots i_p} dx_{i_1} \dots dx_{i_n}.$$

If Δ is any domain in our space with the boundary Γ,

$$D_\Delta(\phi) = \int_\Delta \phi_x \cdot \phi^x$$

is called the Dirichlet integral of ϕ in Δ. The integral is greater than or equal to 0, and only vanishes if $B_{i_1 \dots i_p} = 0$ for all $i_1, \dots, i_p$, and hence if $\phi_x = 0$. Consider all the $(p-1)$-forms which have continuous derivatives in Δ and which define on Γ the same $(p-1)$-forms. Let their Dirichlet integrals in Δ have a lower limit d. If there exists a form ϕ satisfying these conditions, such that $D_\Delta(\phi) = d$, we can show that ϕ is harmonic everywhere within Δ. Consider any region Δ' lying within Δ and let u be any $(p-1)$-form having continuous derivatives in Δ and vanishing outside Δ'. Let λ be any constant. Then

$$D_\Delta(\phi + \lambda u) = d + \lambda^2 D_{\Delta'}(u) + 2\lambda \int_{\Delta'} u_x \phi^x$$

$$= d + \lambda^2 D_{\Delta'}(u) + 2\lambda \left[\int_{\Gamma'} u \cdot \phi^x + (-1)^p \int_{\Delta'} u \cdot \Delta\phi \right].$$

On Γ', $u = 0$ and ϕ^x is finite, hence $\int_\Gamma u \,.\, \phi^x = 0$. Hence $D_\Delta(\phi)$ cannot be a minimum unless

$$\int_{\Delta'} u \,.\, \Delta\phi = 0.$$

But we can easily choose u so that this integral is positive, except in the case $\Delta\phi = 0$. Hence, if there exists a form ϕ satisfying the given conditions and such that $D_\Delta(\phi) = d$, it must be harmonic everywhere inside Δ. We shall see later that when Γ satisfies some general conditions, and the values of the $(p-1)$-form on Γ are suitably restricted, such a form does exist.

9. So far we have considered only points at a finite distance. It will be necessary for the next section to consider harmonic forms in an unbounded region when $n > 2$. On such forms ϕ we impose the additional restriction that the coefficients of $r^{n-2}\phi$ and $r^{n-1}\phi_x$ remain finite as x tends to infinity, r being the distance from x to a fixed point. In these circumstances we easily verify that the Dirichlet integral of ϕ throughout an unbounded region has a finite value, and only vanishes when $\phi_x = 0$.

II. *The Dirichlet problem for harmonic functionals.*

In what follows we shall suppose that $n > 2$, and we shall always suppose that the differentials are arranged so that the n suffixes $i_1, \ldots, i_n$ which appear in calculations form an even derangement of the first n integers.

10. In a Euclidean space of n-dimensions, let us consider a cell Δ which has as boundary the $(n-1)$-cycle Γ. We subject Γ to the condition of being regular, *i.e.*

(1) At each point P of Γ there is a unique tangent hyperplane.

(2) If we make a change of axes so that P is the new origin and the tangent hyperplane at P is the hyperplane $x_n = 0$, the equation of Γ can be written as

$$x_n = f(x_1, x_2, \ldots, x_{n-1}) \qquad (2.1)$$

in the neighbourhood of P, where $f(x_1, x_2, \ldots, x_{n-1})$ has continuous partial derivatives of the third order near P, so that the expansion by means of Taylor's theorem in the form

$$x_n = \tfrac{1}{2}\left[x_1 \frac{\partial}{\partial x_i^0} + \ldots x_{n-1} \frac{\partial}{\partial x_{n-1}^0}\right]^2 f + \tfrac{1}{6}\left[\Sigma x_i \frac{\partial}{\partial (x_i^0 + \theta x_i)}\right]^3 f,$$

where $0 < \theta < 1$, and $\partial f/\partial x_i^0$ denotes the value of $\partial f/\partial x_i$ at the origin, is valid whenever

$$x_1^2 + \ldots + x_{n-1}^2 < c^2,$$

where c is a constant greater than zero, which is independent of the point P on Γ. We consider a $(p-1)$-form ϕ defined on Γ,

$$\phi = \sum_i A_{i_1 \ldots i_{p-1}} dx_{i_1} \ldots dx_{i_{p-1}}.$$

We shall say that this is continuous on Γ if, when at any point P we use the axes defined above and express x_n in ϕ by means of (2.1), the coefficients of the resulting form are continuous in the open domain

$$x_1^2 + \ldots + x_{n-1}^2 < c^2.$$

We say that ϕ satisfies uniform Hölder conditions† on Γ if each coefficient of the $(p-1)$-form defined in this open domain satisfies the condition that its value at the point $(x_1, \ldots, x_{n-1})$ differs from its value at the origin by less than

$$M(x_1^2 + \ldots + x_{n-1}^2)^\alpha,$$

where M and α are constants greater than zero which are independent of the point P on Γ. We notice that ϕ is only defined as a form on Γ, and that two forms are to be regarded as equivalent on Γ when they differ by a form $\chi \sum_i \lambda_i dx_i$, where χ is any $(p-2)$-form, and $(\lambda_1, \ldots, \lambda_n)$ are the directions cosines of the normal to Γ at $(x_1, \ldots, x_n)$.

11. We first establish some lemmas concerning $(p-1)$-forms defined on the regular boundary Γ of a cell Δ.

Definition. If a $(p-1)$-form defined on Γ has a covariant derivative on Γ which is equal to zero at every point it is said to be *improper* on Γ. A similar definition applies to improper forms in Δ.

If ϕ is an improper $(p-1)$-form in Δ we can construct a one-valued continuous $(p-2)$-form ψ in Δ such that $\psi_x = \phi$‡. Similarly, if ϕ is an improper $(p-1)$-form on Γ, we can construct a one-valued continuous $(p-2)$-form ψ on Γ such that $\psi_x = \phi$, provided that $p < n$.

We cannot say the same if $p = n$, unless

$$\int_\Gamma \phi = 0.$$

† *Cf.* Kellogg, *Foundations of potential theory* (1929), 152.

‡ de Rham, *loc. cit.*, Chap. 3, Lemmas 2, 3.

LEMMA I. *If ϕ is a $(p-1)$-form on Γ whose coefficients have continuous derivatives, we can construct a $(p-1)$-form ψ in Δ whose coefficients have continuous derivatives in Δ, and which reduces to ϕ on Γ. Moreover, if ϕ $(p<n)$ is improper on Γ, we can find a ψ satisfying the conditions which is improper in Δ.*

Since Γ is regular, no normals to Γ at adjacent points of Γ will intersect within a distance d of Γ, d being some positive distance. Along each inward normal to Γ mark off a distance a $(0<a<d)$. We thus obtain a locus Γ_1, which has a unique tangent hyperplane at each point which varies continuously. Let Δ_1 be the region lying between Γ_1 and Γ. Consider any point Q of this region. It lies on a unique normal to Γ, say at P; suppose that $PQ=r$. If ψ_Q denotes the value of ψ at Q and ϕ_P that of ϕ at P, we define ψ by the relations:

(1) In Δ_1, $$\psi_Q=\left(\frac{r-a}{a}\right)^2\phi_P.$$

(2) In $\Delta-\Delta_1$, $$\psi=0.$$

If $\phi=\phi_x'$, where ϕ' is a $(p-2)$-form, we define ψ by the relations:

(1) In Δ_1, $$\psi_Q=\frac{(r-a)^3}{a^3}\phi_P+\frac{3(r-a)^2}{a^3}dr\,\phi'_P.$$

(2) In $\Delta-\Delta_1$, $$\psi=0.$$

We verify immediately that these forms satisfy the requirements of the lemma.

COROLLARY. *If ϕ and ψ are improper forms of orders p and $n-p-1$ respectively $(0<p<n-1)$ on Γ, then*

$$\int_\Gamma \phi\,.\,\psi=0.$$

LEMMA II. *If the $(p-1)$-form ϕ is improper on Γ, then*

$$\int_\Gamma \omega_{n-p}\,.\,\phi$$

is a harmonic form in ξ, where ξ is any point of space not on Γ.

If $p=n$, the theorem is well known. Suppose then that $p<n$. By Lemma I we can suppose that ϕ is defined as an improper form in Δ.

Suppose that ξ is within Δ, and let Γ' be a sphere whose centre is ξ and whose radius is ρ, which is so small that Γ' lies entirely within Δ.

We have

$$\Delta_\xi \int_\Gamma \omega_{n-p} \cdot \phi = (-1)^{n(p-1)-1} \int_\Gamma \omega^x_{p-1,\xi} \phi$$

$$= (-1)^{n(p-1)-1} \int_{\Gamma'} \omega^x_{p-1,\xi} \phi,$$

since $\omega^x_{p-1,\xi}$ and ϕ are both improper in the region between Γ and Γ'. Hence

$$\Delta_\xi \int_\Gamma \omega_{n-p} \cdot \phi = (-1)^{n(p-1)} \frac{(n-2)(n-p)}{\rho^n} \int_{\Gamma'} \sum_i d\xi_{i_1} \dots d\xi_{i_p} dx_{i_{p+1}} \dots dx_{i_n} \cdot \phi = 0.$$

If ξ is outside Δ, we first observe that

$$\int_\Gamma \omega_{n-p} \cdot \phi$$

satisfies the condition of § 9 with regard to its behaviour at infinity, and the result follows as before.

12. We now formulate the Dirichlet problem for harmonic functionals. We consider a cell Δ having a regular boundary Γ as in § 10, and ask whether we can find a function of the $(p-1)$-cycles which is harmonic in Δ and takes assigned values on the $(p-1)$-cycles of Γ. The value on any cycle of Γ will be given as the integral over the cycle of a form θ which we shall suppose satisfies uniform Hölder conditions on Γ, and in actual fact we shall show that we can find a particular form which is harmonic in Δ and reduces to θ on Γ. θ is determined on Γ by the values of the functional, and hence is known save for an improper form on Γ, and hence, if we can show that a harmonic form in Δ which is improper on Γ is improper everywhere in Δ, we shall be able to assert that there is a unique solution to the problem which we have proposed. This result will appear in the ensuing analysis. We can also propose the problem for the space outside Δ, and thus we have the Dirichlet problem for internal and external regions. We naturally ask at the same time whether we can extend the Neumann problem to our functionals. It is readily seen that this corresponds to the problem of finding a harmonic $(p-1)$-functional $\int_\gamma \phi$ when we are given the form to which ϕ^x reduces on Γ. Now ϕ^x is an improper $(n-p)$-form, and hence, if $p > 1$, we can find an $(n-p-1)$-form θ on Γ such that $\theta_x = \phi^x$ on Γ. We therefore consider the Dirichlet problem of

finding an $(n-p-1)$-form reducing to θ on Γ, and the $(p-1)$-form which is the conjugate of this gives the required functional. If $p=1$, we require

$$\int_\Gamma \phi^x = 0 = \int_\Gamma \frac{\partial \phi}{\partial n}\, d\Gamma,$$

and the problem is then soluble in the same way. In the solution of the Dirichlet problem which we shall give, certain extreme values of p require special consideration. These are $p=1$, $p=n-1$, and $p=n$. The first corresponds to the Dirichlet problem for harmonic functions, and the second is a way of expressing the Neumann problem for functions, in which the condition for a solution is satisfied. The equation which we obtain below can easily be seen to be, in the case $p=n$, the usual integral equation for the Neumann problem, and the resulting equation is only soluble under certain conditions. For the sake of brevity, we regard these cases as well known, and therefore we shall content ourselves with the consideration of the Dirichlet problem for the cases $1<p<n-1$.

The process of solution which we shall adopt is as follows. Let ϕ be a $(p-1)$-form which we shall suppose to satisfy uniform Hölder conditions on Γ. Then

$$\int_\Gamma \omega^\xi_{n-p} \cdot \phi = (-1)^{n(p-1)-1} \int_\Gamma \omega^x_{p-1} \cdot \phi$$

is clearly a $(p-1)$-form which is harmonic at all points of space other than those on Γ. We seek to find ϕ so that, as ξ tends to a point of Γ from the inside (or the outside), this form takes the assigned value in the interior (or exterior) Dirichlet problem. We shall see that we have to solve an integral equation for ϕ.

13. Let $$\phi = \sum_i A_{i_1 \dots i_{p-1}}\, dx_{i_1} \dots dx_{i_{p-1}}$$

be any $(p-1)$-form on Γ satisfying uniform Hölder conditions on Γ. Suppose that $(\xi_1, \dots, \xi_n)$ is any point of Δ, and consider

$$\int_\Gamma \omega_{n-p} \cdot \phi = \sum_i \int_\Gamma \omega_{n-p} \cdot A_{i_1 \dots i_{p-1}}\, dx_{i_1} \dots dx_{i_{p-1}},$$

$$\int_\Gamma \omega_{n-p} \cdot A_{1 \dots p-1}\, dx_1 \dots dx_{p-1}$$

$$= (-1)^{(n-p)(p-1)} \sum_{k=p}^{n} (-1)^{k-1} B_k\, d\xi_p \dots d\xi_{k-1}\, d\xi_{k+1} \dots d\xi_n,$$

where $$B_k = \int_\Gamma A_{1 \dots p-1} \lambda_k \frac{d\Gamma}{r^{n-2}}.$$

Then

$$\int_\Gamma \omega_{n-p,\,\xi} \cdot A_{1\dots p-1}\, dx_1 \dots dx_{p-1}$$

$$= (-1)^{(p-1)(n-p+1)} \sum_{k=p}^{n} \frac{\partial B_k}{\partial \xi_k}\, d\xi_p \dots d\xi_n$$

$$+ (-1)^{(p-1)(n-p)} \sum_{k=p}^{n} \sum_{r=1}^{p-1} (-1)^{k-1} \frac{\partial B_k}{\partial \xi_r}\, d\xi_r d\xi_p \dots d\xi_{k-1} d\xi_{k+1} \dots d\xi_n.$$

Hence

$$\int_\Gamma \omega^{\xi}_{n-p} \cdot A_{1\dots p-1}\, dx_1 \dots dx_{p-1}$$

$$= \sum_{k=p}^{n} \frac{\partial B_k}{\partial \xi_k}\, d\xi_1 \dots d\xi_{p-1} - \sum_{k=p}^{n} \sum_{r=1}^{p-1} (-1)^{r-1} \frac{\partial B_k}{\partial \xi_r}\, d\xi_k d\xi_1 \dots d\xi_{r-1} d\xi_{r+1} \dots d\xi_{p-1}$$

$$= \sum_{k=1}^{n} \frac{\partial B_k}{\partial \xi_k}\, d\xi_1 \dots d\xi_{p-1} - \sum_{k=1}^{n} \sum_{r=1}^{p-1} (-1)^{r-1} \frac{\partial B_k}{\partial \xi_r}\, d\xi_k d\xi_1 \dots d\xi_{r-1} d\xi_{r+1} \dots d\xi_{p-1}.$$

Now
$$\frac{\partial B_k}{\partial \xi_j} = (n-2) \int_\Gamma A_{1\dots p-1} \frac{(x_j - \xi_j)\lambda_k}{r^n}\, d\Gamma.$$

If ψ is the angle between the internal normal at x and the line joining x to ξ, we have

$$r \cos\psi = \sum_{k=1}^{n} \lambda_k (x_k - \xi_k).$$

Hence

$$\int_\Gamma \omega^{\xi}_{n-p} \cdot A_{1\dots p-1}\, dx_1 \dots dx_{p-1}$$

$$= (n-2) \int_\Gamma A_{1\dots p-1}\, d\xi_1 \dots d\xi_{p-1} \frac{\cos\psi}{r^{n-1}}\, d\Gamma - (n-2)$$

$$\times \int_\Gamma A_{1\dots p-1} \sum_{k=1}^{n} \lambda_k\, d\xi_k \sum_{r=1}^{p-1} (-1)^{r-1} (x_r - \xi_r)\, d\xi_1 \dots d\xi_{r-1} d\xi_{r+1} \dots d\xi_{p-1} \frac{d\Gamma}{r^n}.$$

Hence
$$\int_\Gamma \omega^{\xi}_{n-p} \cdot \phi = (n-2)(A-B),$$

where

$$A = \int_\Gamma \sum_i A_{i_1 \dots i_{p-1}}\, d\xi_{i_1} \dots d\xi_{i_{p-1}} \frac{\cos\psi}{r^{n-1}}\, d\Gamma,$$

$$B = \int_\Gamma \sum_i \sum_j A_{j i_1 \dots i_{p-2}} (x_j - \xi_j) \sum_{k=1}^{n} \lambda_k\, d\xi_k \cdot d\xi_{i_1} \dots d\xi_{i_{p-2}} \frac{d\Gamma}{r^n}.$$

14. We now wish to study the behaviour of these expressions as ξ tends to a point P of Γ. The discussion of A is classical, and we have

$$\lim_{\xi\to P} A - n\alpha_n \sum_i A_{i_1\dots i_{p-1}} d\xi_{i_1}\dots d\xi_{i_{p-1}} = A - \tfrac{1}{2}n\alpha_n \sum_i A_{i_1\dots i_{p-1}} d\xi_{i_1}\dots d\xi_{i_{p-1}},$$

i.e.

$$\lim_{\xi\to P} A = A + \tfrac{1}{2}n\alpha_n \sum_i A_{i_1\dots i_{p-1}} d\xi_{i_1}\dots d\xi_{i_{p-1}},$$

where in the right-hand side of this equation A is calculated for ξ at P and $A_{i_1\dots i_{p-1}}$ has its value at P. Now consider B. Let λ_i^0 be the value of λ_i at P. Then

$$B = B_1 + B_2,$$

where

$$B_1 = \int_\Gamma \sum_i \sum_j A_{ji_1\dots i_{p-2}}(x_j - \xi_j) \sum_{k=1}^{n} (\lambda_k - \lambda_k^0)\, d\xi_k . d\xi_{i_1}\dots d\xi_{i_{p-2}} \frac{d\Gamma}{r^n},$$

$$B_2 = \sum_1^n \lambda_k^0 d\xi_k \int_\Gamma \sum_i \sum_j A_{ji_1\dots i_{p-2}}(x_j - \xi_j)\, d\xi_{i_1}\dots d\xi_{i_{p-2}} \frac{d\Gamma}{r^n}.$$

First let us consider B_1. Take as axes those of § 10, with P as origin. In $x_n = 0$, describe a sphere with centre the origin and radius $\sigma < c$, enclosing a region Σ. Let the cylinder which has this sphere as base and the x_n-axis as its axis cut out a portion Γ_1 of Γ near P, and denote the remaining portion by Γ_2. We can divide B_1 into the sum of two integrals, one over Γ_1 and one over Γ_2. Clearly the latter is continuous as ξ moves on to Γ at P. Consider the integral over Γ_1. Expressing x_n in terms of $(x_1, \dots, x_{n-1})$ by means of (2.1), we obtain this as an integral over Σ. Now

$$\left| A_{ji_1\dots i_{p-2}} \frac{x_j - \xi_j}{r} (\lambda_k - \lambda_k^0) \right| \leqslant |A_{ji_1\dots j_{p-2}}(\lambda_k - \lambda_k^0)|,$$

and the right-hand side of this inequality vanishes at P, and satisfies a uniform Hölder condition on Γ_1. Hence

$$\left| \int_{\Gamma_1} A_{ji_1\dots i_{p-2}} \frac{(x_j - \xi_j)(\lambda_k - \lambda_k^0)}{r^n}\, d\Gamma \right| < M' \int_\Sigma \frac{\rho^\alpha d\Sigma}{r^{n-1}} \quad (\alpha > 0),$$

where M' is a constant and $\rho^2 = x_1^2 + \dots + x_{n-1}^2$. Suppose that ξ is very near P and let the foot of the perpendicular from P to $x_n = 0$ meet this hyperplane in P_1. Let Σ_1 be the sphere in $x_n = 0$ whose centre is P_1 and whose radius σ' is such that it is the maximum sphere with P_1 as centre lying entirely within Σ. Then $\int_\Sigma \frac{\rho^\alpha d\Sigma}{r^{n-1}}$ differs from $\int_{\Sigma_1} \frac{\rho^\alpha d\Sigma_1}{r^{n-1}}$ by a quantity η

which tends to zero as P_1 tends to P. Now, if c is the n-th coordinate of ξ,

$$\left|\int_{\Sigma_1} \frac{\rho^\alpha d\Sigma_1}{r^{n-1}}\right| < a_{n-1}\int_0^{\sigma'} \frac{\rho^{n-2+\alpha}d\rho}{(\rho^2+c^2)^{\frac{1}{2}(n-1)}} < K\sigma^\beta,$$

where K is a constant independent of c, and $\beta > 0$. It therefore follows that B_1 remains continuous as ξ moves up to P.

To discuss B_2 we consider

$$\int_\Gamma A_{ji_1\ldots i_{p-2}}\frac{d\Gamma}{r^{n-2}}.$$

Since $A_{ji_1\ldots i_{p-2}}$ satisfies uniform Hölder conditions on Γ, the argument given by Kellogg† shows that

$$\frac{\partial}{\partial\xi_j}\int_\Gamma A_{ji_1\ldots i_{p-2}}\frac{d\Gamma}{r^{n-2}} = (n-2)\int_\Gamma A_{ji_1\ldots i_{p-2}}\frac{x_j-\xi_j}{r^n}\,d\Gamma$$

remains finite as ξ tends to P, and is also finite when $\xi = P$. Hence the discontinuity in B as ξ passes to P is of the form

$$\chi\Sigma\lambda_k{}^0 d\xi_k,$$

where χ is a finite $(p-2)$-form. Therefore

$$\lim_{\xi\to P}\int_\Gamma \omega^\xi_{n-p}\cdot\phi = \Big[\int_\Gamma \omega^\xi_{n-p}\cdot\phi\Big]_{\xi=P} + \tfrac{1}{2}k_n\phi + \chi\Sigma\lambda_k{}^0 d\xi_k,$$

so that, as ξ tends to a point of Γ, $\int_\Gamma \omega^\xi_{n-p}\cdot\phi$ defines a form on Γ which is equal to

$$\int_\Gamma \omega^\xi_{n-p}\cdot\phi + \tfrac{1}{2}k_n\phi.$$

A similar argument shows that, when ξ is outside Δ, $\int_\Gamma \omega^\xi_{n-p}\cdot\phi$ defines a form on Γ which is equal to

$$\int_\Gamma \omega^\xi_{n-p}\cdot\phi - \tfrac{1}{2}k_n\phi.$$

Hence, to solve the Dirichlet problem for the functions of the $(p-1)$-cycles of Δ, we must solve the integral equation

$$\theta = \int_\Gamma \omega^\xi_{n-p}\cdot\phi + \tfrac{1}{2}k_n\phi, \qquad (2.2)$$

† *Loc. cit.*, 165.

and to solve the exterior problem we have the equation

$$\theta = \int_\Gamma \omega^{\xi}_{n-p} \cdot \phi - \tfrac{1}{2} k_n \phi. \tag{2.3}$$

We therefore consider the problem of solving such equations.

15. We first consider the equation

$$\phi(\xi) = \lambda \int_\Gamma K(\xi, x) \cdot \phi(x) + \theta(\xi),$$

where $\phi(\xi)$ and $\theta(\xi)$ are $(p-1)$-forms on Γ, and $K(\xi, x)$ is a $(p-1)$-form in ξ, and a $(n-p)$-form in x, which is continuous on Γ. The solution of this equation is achieved by the method usually used to solve the ordinary integral equation of Fredholm's type. We need only summarise the process, basing our account on that given by Heywood and Frechet†.

We write

$$K\begin{pmatrix} z_1 \dots z_r \\ z_1 \dots z_r \end{pmatrix} = \begin{vmatrix} K(z_1, z_1) & K(z_1, z_2) & \dots & K(z_1, z_r) \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ K(z_r, z_1) & & \dots & K(z_r, z_r) \end{vmatrix}.$$

The meaning of this determinant requires some explanation, owing to the fact that the order in which the factors in any term of the expansion are written is important. Consider the term which corresponds to

$$\eta a_{1i_1} \dots a_{ri_r} \quad (\eta = \pm 1)$$

in the expansion of the determinant of order r whose elements a_{ij} are numbers. Our first factor is $\eta K(z_1, z_{i_1})$, our second that which is a $(p-1)$-form in z_{1_1}, and so on until we reach a factor which is an $(n-p)$-form in z_1. If we have not exhausted all the factors we begin again with any of the remaining factors and proceed in the same way, and so on until all the factors are exhausted. We thus arrange the factors into groups, and we see that, except for a change in the order of the groups and a cyclic permutation of the order of the factors in a group, the term is uniquely determined. But these changes are equivalent to a change in the order of the sets of parameters $z_1, z_2, \dots, z_r$, and, since we are going to integrate over Γ with respect to each of these sets of parameters, these

† Heywood and Frechet, *L'équation de Fredholm* (1912), 49 *et seq.*

sets are really dummies, and the arbitrariness in question does not affect the final result. In this way we give a definite meaning to the determinant. In a similar manner we give a definite meaning to

$$K\begin{pmatrix}\xi z_1 \dots z_r\\ x z_1 \dots z_r\end{pmatrix} = \begin{vmatrix} K(\xi, x) & K(\xi, z_1) & \dots & K(\xi, z_r)\\ \dots & \dots & \dots & \dots & \dots\\ \dots & \dots & \dots & \dots & \dots\\ K(z_r, x) & K(z_r, z_1) & \dots & K(z_r, z_r)\end{vmatrix}.$$

We prove the convergence of the series

$$D\begin{pmatrix}\xi\\ x\end{pmatrix}, \lambda\Big) = \sum_0^\infty (-1)^r \frac{\lambda^r}{r!} \int_\Gamma \dots \int_\Gamma K\begin{pmatrix}\xi z_1 \dots z_r\\ x z_1 \dots z_r\end{pmatrix},$$

$$D(\lambda) = \sum_0^\infty (-1)^r \frac{\lambda^r}{r!} \int_\Gamma \dots \int_\Gamma K\begin{pmatrix}z_1 \dots z_r\\ z_1 \dots z_r\end{pmatrix}$$

in the usual manner, and deduce that, if $D(\lambda) \neq 0$,

$$k(\xi, x) + \lambda K(\xi, x) = \lambda \int_\Gamma K(\xi, z) . k(z, x) = \lambda \int_\Gamma k(\xi, z) . K(z, x),$$

where

$$k(\xi, x) = -\lambda \frac{D\begin{pmatrix}\xi\\ x\end{pmatrix}, \lambda\Big)}{D(\lambda)}.$$

Hence, if

$$\phi(\xi) = \theta(\xi) - \int_\Gamma k(\xi, x) . \theta(x), \tag{2.4}$$

we have

$$\lambda \int_\Gamma K(\xi, x) . \phi(x) = \lambda \int K(\xi, x) . \theta(x) - \lambda \int_\Gamma \int_\Gamma K(\xi, x) . k(x, z) . \theta(z)$$

$$= -\int_\Gamma k(\xi, x) . \theta(x),$$

and it follows that (2.4) is a solution of our integral equation. It can further be shown in the usual way that it is the only solution.

16. We have now to consider the case in which $K(\xi, x)$ is discontinuous on Γ. This is the case when we consider the equation (2.2). Let us write

$$K_1(\xi, x) = K(\xi, x),$$

$$K_s(\xi, x) = \int_\Gamma K(\xi, z) . K_{s-1}(z, x).$$

$K_2(\xi, x)$, etc., are the "iterated kernels". We show, just as in the case

of ordinary integral equations, that if, for s large enough, $K_s(\xi, x)$ is continuous, we can solve the equation. We now show that this is the case when

$$K(\xi, x) = \omega_{n-p}^{\xi}.$$

In this case the kernel is continuous except when $x = \xi$. We shall first show that, if x and ξ tend to coincidence at a point P, the coefficients of $K(\xi, x)$ become infinite like $1/r^{n-2}$. Let $(u_1, \ldots, u_n)$ be the parameters of Γ valid in the neighbourhood of P. The equations which express $(x_1, \ldots, x_n)$ in terms of $(u_1, \ldots, u_{n-1})$ have continuous partial derivatives of the third order, since Γ is regular. Let $(\nu_1, \ldots, \nu_{n-1})$ be the parameters corresponding to $(\xi_1, \ldots, \xi_n)$. Then we have

$$\omega_{n-p}^{\xi} = (n-2) \sum_{ij} K_{k_1 \ldots k_{p-1}}^{j_1 \ldots j_{n-p}} d\nu_{k_1} \ldots d\nu_{k_{p-1}} du_{j_1} \ldots du_{j_{n-p}},$$

where

$$K_{k_1 \ldots k_{p-1}}^{j_1 \ldots j_{n-p}} = \pm \frac{1}{r^n} \begin{vmatrix} x_1 - \xi_1 & \cdots & x_n - \xi_n \\ \dfrac{\partial \xi_1}{\partial \nu_{k_1}} & \cdots & \dfrac{\partial \xi_n}{\partial \nu_{k_1}} \\ \cdots & \cdots & \cdots \\ \dfrac{\partial \xi_1}{\partial \nu_{k_{p-1}}} & \cdots & \dfrac{\partial \xi_n}{\partial \nu_{k_{p-1}}} \\ \dfrac{\partial x_1}{\partial u_{j_1}} & \cdots & \dfrac{\partial x_n}{\partial u_{j_1}} \\ \cdots & \cdots & \cdots \\ \dfrac{\partial x_1}{\partial u_{j_{n-p}}} & \cdots & \dfrac{\partial x_n}{\partial u_{j_{n-p}}} \end{vmatrix}.$$

When x is near to ξ,

$$x_i - \xi_i = \sum_j \frac{\partial \xi_i}{\partial \nu_j} (u_j - \nu_j) + \text{terms of higher order in } u - \nu,$$

and

$$\frac{\partial x_i}{\partial u_j} = \frac{\partial \xi_i}{\partial \nu_j} + \sum_k \frac{\partial^2 \xi_i}{\partial \nu_k \partial \nu_j} (u_k - \nu_k) + \text{terms of higher order in } u - \nu.$$

Hence

$$K_{k_1 \ldots k_{p-1}}^{j_1 \ldots j_{n-p}} = \frac{1}{r^n} \left[\sum_{ij} P_{ij} (u_i - \nu_i)(u_j - \nu_j) + \text{terms of higher order} \right],$$

the functions P_{ij} being continuous in the neighbourhood of P. But $(u - \nu)/r$ remains finite as $\nu \to u$, hence $K_{k_1 \ldots k_{p-1}}^{j_1 \ldots j_{n-p}}$ becomes infinite like $1/r^{n-2}$.

We now apply the reasoning given by Heywood and Frechet† and show that we can find s so large that $K_s(\xi, x)$ is continuous on Γ. Indeed, we can show that this satisfies uniform Hölder conditions on Γ.

17. It only remains to consider what happens if $D(\lambda) = 0$, *i.e.* if λ is a characteristic value of the parameter. It is easy to show (in the usual manner) that, when this is the case, there is a continuous form ϕ, not identically zero, such that

$$\phi(\xi) = \lambda \int_\Gamma K(\xi, x) \cdot \phi(x).$$

Hence, if we can show that there is no form ϕ other than zero satisfying the equation

$$\phi = -\frac{2}{k_n} \int_\Gamma \omega^\xi_{n-p} \cdot \phi, \tag{2.5}$$

we shall know that equation (2.2) has a unique solution. We therefore suppose that (2.5) has a non-vanishing solution, and examine the consequences. Let this solution be

$$\phi(x) = \sum_i A_{i_1 \dots i_{p-1}} dx_{i_1} \dots dx_{i_{p-1}}.$$

Then

$$\phi(\xi) = -\frac{2(n-2)}{k_n} \left\{ \begin{array}{l} \displaystyle\int_\Gamma \sum_i A_{i_1 \dots i_{p-1}} d\xi_{i_1} \dots d\xi_{i_{p-1}} \frac{\cos\psi}{r^{n-1}} d\Gamma \\ \displaystyle -\int_\Gamma \sum_1^n \lambda_k d\xi_k \sum_i \sum_j A_{j i_1 \dots i_{p-2}} (x_j - \xi_i) d\xi_{i_1} \dots d\xi_{i_{p-2}} \frac{d\Gamma}{r^n} \end{array} \right\}.$$

Now

$$\sum_1^n \lambda_k d\xi_k = \cos\psi_1 \sum_1^n k_i d\xi_i,$$

where $(k_1, \dots, k_n)$ are the direction cosines of the projection of the line joining ξ to x on the tangent hyperplane at ξ, and ψ_1 is the angle between this line and the normal at x. Now we can find a constant N such that everywhere on Γ we have

$$\left| \frac{\cos\psi}{r^{n-1}} \right| < \frac{N}{r^{n-2}}, \qquad \left| \cos\psi_1 \frac{(x_j - \xi_j) k_i}{r^n} \right| < \frac{N}{r^{n-2}},$$

$$\left| \frac{\partial}{\partial s} \left(\frac{\cos\psi}{r^{n-1}} \right) \right| < \frac{N}{r^{n-1}}, \qquad \left| \frac{\partial}{\partial s} \left\{ \cos\psi_1 \frac{(x_j - \xi_j) k_i}{r^n} \right\} \right| < \frac{N}{r^{n-1}},$$

† *Loc. cit.*, 142.

where $\partial/\partial s$ means differentiation with respect to ξ in any direction on Γ. Hence, if we apply Kellogg's argument†, we find that ϕ satisfies uniform Hölder conditions on Γ.

Now let us consider

$$\int_\Gamma \Delta_\xi \omega_{n-p} \cdot \phi,$$

where ξ is a point of Γ. We take as axes those defined in § 10 with P as origin, so that (2.1) is the equation of Γ. For brevity write

$$\frac{\partial x_n}{\partial x_i} = p_i, \qquad \frac{\partial \xi_n}{\partial \xi_i} = q_i.$$

With centre the origin and radius $\sigma < c$, we construct a sphere lying in $x_n = 0$, as in § 13, and use the notation of that paragraph. We have

$$\int_\Gamma \Delta_\xi \omega_{n-p} \cdot \phi = \int_{\Gamma_1} \Delta_\xi \omega_{n-p} \cdot \phi + \int_{\Gamma_2} \Delta_\xi \omega_{n-p} \cdot \phi.$$

The second integral is clearly finite. We examine the former integral. In Γ_1 we can express x_n, ξ_n in terms of $(x_1, \ldots, x_{n-1})$, $(\xi_1, \ldots, \xi_{n-1})$ respectively. Now

$$\frac{\cos\psi}{r^{n-1}} = \frac{\sum_1^{n-1} p_i(x_i - \xi_i) - (x_n - \xi_n)}{r^n (1+\Sigma p_i^2)^{\frac{1}{2}}},$$

hence

$$\frac{\partial}{\partial \xi_i}\left(\frac{\cos\psi}{r^{n-1}}\right) = \frac{q_i - p_i}{r^n (1+\Sigma p_i^2)^{\frac{1}{2}}} + n\,\frac{\cos\psi}{r^{n-1}}\left[\frac{x_i - \xi_i}{r^2} + \frac{(x_n - \xi_n)\,q_i}{r^2}\right].$$

Now

$$\int_{\Gamma_1} A_{i_1 \ldots i_{p-1}} \frac{\partial}{\partial \xi_i}\left(\frac{\cos\psi}{r^{n-1}}\right) d\Gamma$$

$$= \int_{\Gamma_1} (A_{i_1 \ldots i_{p-1}} - A^0_{i_1 \ldots i_{p-1}}) \frac{\partial}{d\xi_i}\left(\frac{\cos\psi}{r^{n-1}}\right) d\Gamma + A^0_{i_1 \ldots i_{p-1}} \int_{\Gamma_1} \frac{\partial}{\partial \xi_i}\left(\frac{\cos\psi}{r^{n-1}}\right) d\Gamma,$$

$A^0_{i_1 \ldots i_{p-1}}$ denoting the value of $A_{i_1 \ldots i_{p-1}}$ at the origin. Now

$$\left| \int_{\Gamma_1} (A_{i_1 \ldots i_{p-1}} - A^0_{i_1 \ldots i_{p-1}}) \frac{\partial}{\partial \xi_i}\left(\frac{\cos\psi}{r^{n-1}}\right) d\Gamma \right| < MN \int_\Sigma \frac{\rho^\alpha d\Gamma}{r^{n-1}} < K\sigma^\alpha \quad (\alpha > 0),$$

† *Loc. cit.*, 300.

where K is a number independent of σ. Again,

$$\int_{\Gamma_1} \frac{\partial}{\partial \xi_i}\left(\frac{\cos\psi}{r^{n-1}}\right) d\Gamma = -\int_\Sigma \frac{P_i d\Sigma}{r^n} + n\int_\Sigma \frac{x_i d\Sigma}{r^n} + \int_\Sigma \chi d\Sigma,$$

when $\xi = 0$, where χ is a function which becomes infinite like $1/r^{n-2}$ at the origin, and P_i stands for the terms of lowest order in p_i. P_i is therefore a linear homogeneous expression in $(x_1, \ldots, x_{n-1})$. It follows from the fact that Σ is a sphere that the first two terms in the above expression vanish. Hence

$$\int_{\Gamma_1} \frac{\partial}{\partial \xi_i}\left(\frac{\cos\psi}{r^{n-1}}\right) d\Gamma$$

has a finite value. Hence we deduce that

$$\int_\Gamma \sum_i A_{i_1 \ldots i_{p-1}} d\xi_{i_1} \ldots d\xi_{i_{p-1}} \frac{\partial}{\partial \xi_i}\left(\frac{\cos\psi}{r^{n-1}}\right) d\Gamma$$

has a finite value. An exactly similar argument shows that

$$\int_\Gamma \Sigma\, d\xi_k d\xi_{i_1} \ldots d\xi_{i_{p-2}} A_{j i_1 \ldots i_{p-2}} \frac{\partial}{\partial \xi_i}\left\{\frac{\cos\psi_1 (x_j - \xi_j) k_k}{r^n}\right\} d\Gamma$$

has a definite finite value, and hence that

$$\int_\Gamma \Delta_\xi \omega_{n-p} \cdot \phi$$

is a finite p-form on Γ.

We next show that this p-form is continuous. Let ξ, ξ' be two points of Γ within a distance δ of each other. If δ is sufficiently small we can find a region Γ' of Γ containing both ξ and ξ' such that

$$(1) \qquad \left\{\int_{\Gamma-\Gamma'} \Delta_\xi \omega_{n-p} \cdot \phi\right\}_{\text{at } \xi} - \left\{\int_{\Gamma-\Gamma'} \Delta_\xi \omega_{n-p} \cdot \phi\right\}_{\text{at } \xi'}$$

is a p-form whose coefficients are less than $\frac{1}{3}\epsilon$ in absolute magnitude, ϵ being any given positive number, and such that

$$(2) \qquad \left\{\int_{\Gamma'} \Delta_\xi \omega_{n-p} \cdot \phi\right\}_{\text{at } \xi} \quad \text{and} \quad \left\{\int_{\Gamma'} \Delta_\xi \omega_{n-p} \cdot \phi\right\}_{\text{at } \xi'}$$

are forms whose coefficients are less than $\frac{1}{3}\epsilon$ in absolute magnitude.

Hence the continuity of $\int_\Gamma \Delta_\xi \omega_{n-p} \cdot \phi$ is established. From this we deduce the validity of differentiation under the sign of integration, and hence that ϕ_x exists and is continuous on $\mathbb{F}$.

Now suppose that s is so large that $K_s(\xi, x)$ satisfies uniform Hölder conditions on Γ. Then

$$\int_\Gamma \Delta_\xi \omega_{n-p}(\xi, z) \,.\, K_s(z, x) = K_{s+1,\xi}(\xi, x)$$

is continuous on Γ. Now

$$\phi = \left(-\frac{2}{k_n}\right)^{s+1} \int_\Gamma K_{s+1}(\xi, x) \,.\, \phi.$$

Hence

$$\phi_\xi = \left(-\frac{2}{k_n}\right)^{s+1} \int_\Gamma K_{s+1,\xi}(\xi, x) \,.\, \phi$$

satisfies uniform Hölder conditions on Γ.

18. Let ϕ be any form in Δ which has a continuous derivative and reduces to the solution of (2.5) on Γ. Let ξ be any point of Δ and consider

$$\psi = (-1)^{(n-p)(p-1)} \int_\Gamma \omega^\xi_{n-p} \,.\, \phi = -\int_\Gamma \omega^x_{p-1} \,.\, \phi.$$

This is a harmonic form in ξ which reduces to zero on Γ. Following an argument similar to that of Lemma II of § 11, we have

$$\psi^\xi = (-1)^{n-p} \int_\Gamma \omega_{n-p-1,\xi} \,.\, \phi_x.$$

Now ϕ_x satisfies uniform Hölder conditions on Γ. Hence† ψ^ξ remains finite as ξ tends to a point of Γ. Now the Dirichlet integral

$$D_\Delta(\psi) = \int_\Delta \psi_\xi \,.\, \psi^\xi = \int_\Gamma \psi \,.\, \psi^\xi = 0,$$

since $\psi = 0$ on Γ, and ψ^ξ is finite. Hence ψ is improper. It follows that

$$\psi_1 = \int_\Gamma \omega_{n-p} \,.\, \phi$$

is a harmonic $(n-p)$-form in Δ, such that $\psi_1{}^\xi$ reduces to zero on Γ.

But

$$D_\Delta(\psi_1) = \int_\Gamma \psi_1 \,.\, \psi_1{}^\xi = 0.$$

† *Cf.* Kellogg, *loc. cit.*, 165.

Hence ψ_1 is also improper, *i.e.* $\int_\Gamma \omega_{n-p}^{\xi} \cdot \phi = 0$ when ξ is in Δ. Next consider

$$\psi_2 = -\frac{1}{k_n}\int_\Gamma \omega_{n-p}^{\xi} \cdot \phi,$$

where ξ is a point outside Γ. This is a harmonic form in ξ which behaves at infinity in the manner laid down in § 9. As ξ tends to a point of Γ we find that this defines a form equal to ϕ. Hence

$$-\frac{1}{k_n}\int_\Gamma \Delta_\xi\, \omega_{n-p} \cdot \phi$$

defines a form equal to ϕ_ξ. But

$$-\frac{1}{k_n}\int_\Gamma \Delta_\xi\, \omega_{n-p} \cdot \phi = -\frac{1}{k_n}\int_\Gamma \omega_{n-p-1}^{\xi} \cdot \phi_x,$$

and hence defines on Γ a form equal to

$$-\frac{1}{k_n}\int_\Gamma \omega_{n-p-1}^{\xi} \cdot \phi_x + \tfrac{1}{2}\phi_\xi.$$

Hence

$$\phi_\xi = -\frac{2}{k_n}\int_\Gamma \omega_{n-p-1}^{\xi} \cdot \phi_x.$$

As before, we show that, if ξ is inside Δ,

$$\psi_3 = \int_\Gamma \omega_{n-p-1} \cdot \phi_x$$

is improper in Δ. If ξ is outside Δ, ψ_3 is harmonic, and, since ψ_3 is continuous everywhere, ψ_3 is improper on Γ. Evaluating the Dirichlet integral throughout the space exterior to Γ, we have

$$D(\psi_3) = -\int_\Gamma \psi_3 \cdot \psi_3{}^{\xi} = 0 \quad (\text{since } 1 < p < n-1),$$

since ψ_3 and $\psi_3{}^{\xi}$ are both improper on Γ. Hence ψ_3 is improper everywhere. But the discontinuity in $\psi_3{}^{\xi}$ as ξ crosses Γ is $k_n \phi_x$, which is hence zero. Thus ϕ is improper on Γ. Repeating the argument, we find that $\phi = 0$, and therefore we can solve the equation (2.2), and the solution is unique.

19. Exactly similar considerations apply to the exterior Dirichlet problem, and hence we can say that the Dirichlet problem is solvable for

the region interior or exterior to a regular boundary for all values of p except $p = n$. Further, we shall see that the solution is unique if we can show that, if θ is improper at every point of Γ, the solution of (2.2) or (2.3) is improper. For then

$$\int_\Gamma \omega^\xi_{n-p} \cdot \phi$$

is improper, inside or outside Γ. Consider the interior problem. Suppose that $p > 1$, the case $p = 1$ being well known. If $p < n$, there exists a $(p-2)$-form θ' on Γ such that $\theta_x' = \theta$, if θ is improper. We can find a harmonic form in Δ which reduces to θ' on Γ, which is given in the form

$$\int_\Gamma \omega^\xi_{n-p-1} \cdot \phi'.$$

Now
$$\int_\Gamma \Delta_\xi \omega_{n-p-1} \cdot \phi' = \int_\Gamma \omega^\xi_{n-p} \cdot \phi_x'.$$

Hence the solution of

$$\theta = \int_\Gamma \omega^\xi_{n-p} \cdot \phi + \tfrac{1}{2} k_n \phi$$

is $\phi = \phi_x'$, from which the result follows. A similar argument applies to the external problem, and we have therefore established the uniqueness of the solution of our problem.

III. *The generalized Poisson integral.*

Up to the present we have considered general values of p and n. We saw, however, in §3 that special interest attaches to the case $n = 2p$. The rest of this paper deals with this case.

20. In § 7 we obtained the formula

$$k_n \int_\gamma \phi = \int_\gamma \int_\Gamma \omega_{p-1} \cdot \phi^x - \int_\gamma \int_\Gamma \phi \cdot \omega^x_{p-1}.$$

In the case in which Γ is a sphere of radius a and centre the origin we deduce from this an expression for $\int_\gamma \phi$ which depends only on the value of ϕ on Γ,

just as we obtain the Poisson integral for harmonic functions. Let

$$\sigma^2 = \sum_1^n \xi_i^2,$$

and write $$\xi_i' = \frac{a^2}{\sigma^2}\,\xi_i \quad (i = 1, \ldots, n).$$

This is a conformal transformation of coordinates, and hence

$$\int_{\gamma_1}\int_{\gamma_2} \omega'_{p-1},$$

where $$\omega'_{p-1} = \sum_i \frac{d\xi'_{i_1} \ldots d\xi'_{i_{n-1}}\, dx_{i_1} \ldots dx_{i_{p-1}}}{r'^{n-2}}$$

$\left(\text{where } r'^2 = \sum_1^n (x_i - \xi_i')^2\right)$ is expressed as a form in $(\xi_1, \ldots, \xi_n)$ before integrating over γ_1, is a harmonic functional of γ_1 or γ_2. If γ is a $(p-1)$-cycle within the sphere we find, as in § 7, that

$$0 = \int_\gamma \int_\Gamma \omega'_{p-1} \cdot \phi^x - \int_\gamma \int_\Gamma \phi \cdot \omega'^x_{p-1}.$$

Hence

$$k_n \int_\gamma \phi = \int_\gamma \int_\Gamma (\omega_{p-1} - \omega'_{p-1}) \cdot \phi^x - \int_\gamma \int_\Gamma \phi \cdot (\omega^x_{p-1} - \omega'^x_{p-1}).$$

We now show that the first term of the right-hand side vanishes. We have

$$d\xi'_{i_1} \ldots d\xi'_{i_{p-1}} = \left(\frac{a}{\sigma}\right)^{2p-2} d\xi_{i_1} \ldots d\xi_{i_{p-1}}$$

$$-2\frac{d\sigma}{\sigma}\left\{\sum_{r=1}^{p-1} (-1)^{r-1}\, \xi_{i_r}\, d\xi_{i_1} \ldots d\xi_{i_{r-1}}\, d\xi_{i_{r+1}} \ldots d\xi_{i_{p-1}}\right\}\left(\frac{a}{\sigma}\right)^{2p-2}.$$

Also, since x is on Γ,

$$r' = \frac{a}{\sigma}\, r.$$

Hence

$$\omega'_{p-1} = \omega_{p-1} - 2\frac{d\sigma}{\sigma}\sum_i \left\{\sum_{r=1}^{p-1} (-1)^{r-1}\, \xi_{i_r}\, d\xi_{i_1} \ldots d\xi_{i_{r-1}}\, d\xi_{i_{r+1}} \ldots d\xi_{i_{p-1}}\right\} \frac{dx_{i_1} \ldots dx_{i_{p-1}}}{r^{n-2}}$$

$$= \omega_{p-1} - 2\frac{d\sigma}{\sigma}\sum_1^n \xi_k\, dx_k \sum_i \frac{d\xi_{i_1} \ldots d\xi_{i_{n-2}}\, dx_{i_1} \ldots dx_{i_{p-2}}}{r^{n-2}}.$$

But on Γ we have $$\sum_1^n x_k dx_k = 0.$$

Hence

$$\int_\Gamma (\omega_{p-1} - \omega'_{p-1}) \cdot \phi^x = -2 \frac{d\sigma}{\sigma} \int_\Gamma \sum_1^n (x_k - \xi_k) dx_k \cdot \omega_{p-2} \cdot \phi^x.$$

Let $$\nu = \frac{1}{n-4} \sum_i \frac{d\xi_{i_1} \dots d\xi_{i_{p-2}} dx_{i_1} \dots dx_{i_{p-2}}}{r^{n-4}} \qquad (n > 4),$$

and $$\nu = -\log r \qquad (n = 4).$$

Then $$\int_\Gamma (\omega_{p-1} - \omega'_{p-1}) \cdot \phi^x = 2 \frac{d\sigma}{\sigma} \int_\Gamma \nu_x \cdot \phi^x.$$

Let Γ' and Δ' be defined as in § 7. Then

$$\int_\Gamma \nu_x \cdot \phi^x = \int_{\Gamma'} \nu_x \cdot \phi^x$$

$$= \frac{1}{\rho^{n-2}} \int_{\Gamma'} \sum_1^n (x_k - \xi_k) dx_k \sum_i d\xi_{i_1} \dots d\xi_{i_{p-2}} dx_{i_1} \dots dx_{i_{p-2}} \cdot \phi^x = 0,$$

since, on Γ', $$\sum_1^n (x_k - \xi_k) dx_k = 0.$$

Hence $$k_n \int_\gamma \phi = \int_\gamma \int_\Gamma \phi \cdot (\omega'^x_{p-1} - \omega^x_{p-1}). \qquad (3.1)$$

21. We thus see that if there exists a harmonic functional in Δ, taking the values of $\int_\gamma \phi$ on Γ, its value is given inside Δ by (3.1). We have seen in § 2 that, if ϕ satisfies uniform Hölder conditions on Γ, such a functional exists, but we can easily verify directly that in these circumstances (3.1) gives a harmonic functional which takes the assigned values on Γ. In the first place, if $(\xi_1, \dots, \xi_n)$ is a point of Δ, we have

$$\Delta_\xi \int_\Gamma \phi \cdot (\omega'^x_{p-1} - \omega^x_{p-1}) = -\int_\Gamma \phi \cdot \Delta_{\xi'} \omega'^x_{p-1} - \int_\Gamma \phi \cdot \Delta_\xi \omega^x_{p-1} = 0,$$

and hence $$\frac{1}{k_n} \int_\gamma \int_\Gamma \phi \cdot (\omega'^x_{p-1} - \omega^x_{p-1})$$

is a harmonic functional of γ in Δ.

Let $$\phi = \sum_i A_{i_1 \dots i_{p-1}} dx_{i_1} \dots dx_{i_{p-1}}.$$

From the analysis of § 13 we have

$$\int_\Gamma \phi \,.\, \omega^x_{p-1} = -(n-2)\int_\Gamma \sum_i A_{i_1 \dots i_{p-1}} d\xi_{i_1} \dots d\xi_{i_{p-1}} \frac{\cos\psi}{r^{n-1}}\, d\Gamma$$

$$+(n-2)\int_\Gamma \sum_i \sum_{j=1}^{n} A_{j i_1 \dots i_{p-2}} (x_j - \xi_j) \sum_{k=1}^{n} \lambda_k d\xi_k d\xi_{i_1} \dots d\xi_{i_{p-2}} \frac{d\Gamma}{r^n}.$$

We obtain a similar expression for $\int_\Gamma \phi \,.\, \omega'^x_{p-1}$. From these we find that

$$\frac{1}{k_n}\int_\Gamma \phi \,.\, (\omega'^x_{p-1} - \omega^x_{p-1})$$

$$= \frac{n-2}{k_n}\int_\Gamma \sum_i A_{i_1 \dots i_{p-1}} d\xi_{i_1} \dots d\xi_{i_{p-1}} \frac{a^2-\sigma^2}{a r^n}\, d\Gamma$$

$$+ \frac{2(n-2)}{k_n}\frac{d\sigma}{a}\int_\Gamma \frac{\cos\psi'}{r^{n-1}} \sum_i \sum_j A_{j i_1 \dots i_{p-2}} \xi_j d\xi_{i_1} \dots d\xi_{i_{p-2}} d\Gamma$$

$$- \frac{n-2}{k_n}\int_\Gamma \left(1 - \frac{\sigma^2}{a^2}\right) \sum_i \sum_j A_{j i_1 \dots i_{p-2}} x_j \sum_1^n \lambda_k d\xi_k d\xi_{i_1} \dots d\xi_{i_{p-2}} \frac{d\Gamma}{r^n}$$

$$+ \frac{2(n-2)}{k_n}\frac{d\sigma}{\sigma}\int_\Gamma \sum_i \sum_j A_{j i_1 \dots i_{p-2}} \left(x_j \frac{\sigma^2}{a^2} - \xi_j\right) \sum_1^n \lambda_k \xi_k d\xi_{i_1} \dots d\xi_{i_{p-2}} \frac{d\Gamma}{r^n}$$

$$- \frac{2(n-2)}{k_n}\frac{d\sigma}{\sigma}\int_\Gamma \Bigg[\sum_i \sum_j A_{j i_1 \dots i_{p-2}} \left(x_j \frac{\sigma^2}{a^2} - \xi_j\right)$$

$$\times \sum_1^n \lambda_k d\xi_k \sum_1^{p-2} (-1)^{r-1} \xi_{i_r} d\xi_{i_1} \dots d\xi_{i_{r-1}} d\xi_{i_{r+1}} \dots d\xi_{i_{p-2}} \Bigg] \frac{d\Gamma}{r^n}$$

and a calculation on the lines of § 14 shows that, as ξ tends to a point of Γ, this tends to the value of

$$\phi + \chi \Sigma x_i dx_i$$

at this point, where χ is some $(p-2)$-form in x, when the x's are replaced by ξ's.

22. There is another form of the Poisson integral which is also useful. Following the argument of § 18, we see that

$$\int_\Gamma \phi \,.\, \omega^x_{p-1\,\xi} = -\int_\Gamma \phi_x \,.\, \omega^\xi_{p-1},$$

$$\int_\Gamma \phi \,.\, \omega'^x_{p-1\,\xi} = -\int_\Gamma \phi_x \omega'^\xi_{p-1}.$$

Hence $$\phi_\xi = \frac{1}{k_n}\int_\Gamma \phi_x(\omega^\xi_{p-1} - \omega'^\xi_{p-1}) = \frac{1}{k_n}\int_\Gamma \phi_x(\omega'^x_p - \omega_p{}^x),$$

and we deduce from this that, if ϕ_x satisfies uniform Hölder conditions on Γ, ϕ_ξ and ϕ^ξ remain finite as ξ tends to a point of Γ. The importance of this formula lies in the fact that it is sometimes more convenient to consider the functional as the integral of its covariant derivative (which is unique) over an open domain than as the integral of a $(p-1)$-form (which is not unique) over the boundary of the domain.

23. By means of the results which we have obtained for the Poisson integral we have a new method of solving the Dirichlet problem in the case $n = 2p$, which applies to domains of more generality than those considered in Section II. The method is the extension of the calculus of variations method used in the case of harmonic functions of two variables. As, however, we are going to use this method in Section IV to solve a particular problem in connection with analytic varieties, when it will be fully exemplified, we do not give it here.

Some preliminary work must be done, however, and this is now given. Let Δ be any domain in a space of $2p$ dimensions, with boundary Γ, and suppose that the values of a continuous $(p-1)$-form are given on this. We consider all the $(p-1)$-forms in Δ which have derivatives which satisfy uniform Hölder conditions inside Δ and which reduce to the given form on Γ. We have seen that, if there exists a form among these for which the Dirichlet integral over Δ attains its minimum d, that form is harmonic. We now show that, if ϕ is any of our forms which is not harmonic at every point of Δ, we can deduce from it another form $\tilde{\phi}$, possessing a derivative which satisfies uniform Hölder conditions in Δ, which reduces to the given form on Γ, and is such that

$$D_\Delta(\tilde{\phi}) < D_\Delta(\phi).$$

Let Γ' be a sphere of radius a in Δ, enclosing a region Δ', in which ϕ is not harmonic. ϕ_x satisfies uniform Hölder conditions on Γ'. When we take the centre of Γ' as origin,

$$\bar{\phi} = \frac{1}{k_n}\int_{\Gamma'} \phi \,.\, (\omega'^x_{p\ 1} - \omega^x_{p-1})$$

is a harmonic form which reduces to ϕ on Γ', and which is such that $\bar{\phi}^x$ is finite in Δ' and on Γ'. Let

$$v = \phi - \bar{\phi};$$

we then see that $$\int_{\Delta'} v_x \cdot \bar{\phi}^x = \int_{\Gamma'} v \cdot \bar{\phi}^x = 0,$$

since v reduces to zero on Γ'. Now

$$\begin{aligned} D_{\Delta'}(\phi) &= D_{\Delta'}(v) + 2\int_{\Delta'} v_x \cdot \bar{\phi}^x + D_{\Delta'}(\bar{\phi}) \\ &> D_{\Delta'}(\bar{\phi}), \end{aligned}$$

unless $D_{\Delta'}(v) = 0$. This can happen only if ϕ is harmonic in Δ', however, and this is contrary to hypothesis.

Consider a sphere Γ'', concentric with Γ', of radius $b < a$. If r denotes distance from the centre of this sphere, we define $\tilde{\phi}$ as follows:

$$\tilde{\phi} = \bar{\phi} \qquad (r \leqslant b), \tag{1}$$

$$\tilde{\phi} = \bar{\phi} + \left(\frac{r-b}{a-b}\right)^2 (\phi - \bar{\phi}) \qquad (b \leqslant r \leqslant a), \tag{2}$$

$$\tilde{\phi} = \phi \qquad (r \geqslant a). \tag{3}$$

Clearly $\tilde{\phi}$ is continuous in Δ. Also $\tilde{\phi}_x$ is continuous everywhere except perhaps when $r = a$. When $b < r < a$,

$$\tilde{\phi}_x = \bar{\phi}_x + \left(\frac{r-b}{a-b}\right)^2 (\phi_x - \bar{\phi}_x) + 2\,\frac{r-b}{(a-b)^2}\, dr(\phi - \bar{\phi}) \to \phi_x,$$

when $r \to a$, since $\phi - \bar{\phi} \to \chi\, dr$, where χ is a $(p-2)$-form; hence $\tilde{\phi}_x$ is also continuous in Δ. It is also clear that $\tilde{\phi}_x$ satisfies uniform Hölder conditions in Δ. We now have

$$\begin{aligned} D_{\Delta}(\phi) - D_{\Delta}(\tilde{\phi}) &= D_{\Delta'}(\phi) - D_{\Delta'}(\tilde{\phi}) \\ &= D_{\Delta'}(\phi - \bar{\phi}) - D_{\Delta'}(\tilde{\phi} - \bar{\phi}). \end{aligned}$$

Consider

$$D_{\Delta'}(\tilde{\phi} - \bar{\phi}) = D_{\Delta' - \Delta''}(\tilde{\phi} - \bar{\phi}) = D_{\Delta' - \Delta''}\left\{\left(\frac{r-b}{a-b}\right)^2 (\phi - \bar{\phi})\right\}$$

$$= \int_{\Delta' - \Delta''} \left(\frac{r-b}{a-b}\right)^4 \sum_i B^2_{i_1 \ldots i_p}\, dx_i \ldots dx_n$$

$$+ \frac{4}{(a-b)^4} \int_{\Delta' - \Delta''} \frac{(r-b)^2}{r^2} \sum_i C^2_{i_1 \ldots i_p}\, dx_1 \ldots dx_n,$$

where $$v = \phi - \bar{\phi} = \sum_i A_{i_1 \dots i_{p-1}} dx_{i_1} \dots dx_{i_{p-1}},$$

$$v_x = \phi_x - \bar{\phi}_x = \sum_i B_{i_1 \dots i_p} dx_{i_1} \dots dx_{i_p},$$

and $$C_{i_1 \dots i_p} = \sum_{r=1}^{p} (-1)^{r-1} x_{i_r} A_{i_1 \dots i_{r-1} i_{r+1} \dots i_p}.$$

Let $$\psi = \sum_i C_{i_1 \dots i_p} dx_{i_{p+1}} \dots dx_{i_n}.$$

Then $$v \,.\, \psi = \sum_{k=1}^{n} (-1)^{k-1} D_k dx_1 \dots dx_{k-1} dx_{k+1} \dots dx_n,$$

where $$D_k = (-1)^{p-1} \sum A_{i_1 \dots i_{p-1}} C_{i_1 \dots i_{p-1} k}.$$

Hence $$\sum_1^n x_k dx_k \,.\, v \,.\, \psi = \sum_1^n x_k D_k dx_1 \dots dx_n = \sum C^2_{i_1 \dots i_p} dx_1 \dots dx_n.$$

Hence

$$\int_{\Delta' - \Delta''} \frac{(r-b)^2}{r^2} \sum C^2_{i_1 \dots i_p} dx_1 \dots dx_n = \int_{\rho=b}^{\rho=a} \frac{(\rho-b)^2}{\rho} d\rho \int_{r=\rho} v \,.\, \psi.$$

Now $$\int_{r=\rho} v \,.\, \psi = \frac{1}{\rho} \int_{r=\rho} \sum x_k D_k d\Gamma = \frac{1}{\rho} \int_{r=\rho} \sum C^2_{i_1 \dots i_p} d\Gamma.$$

Now, when $r = a$, $v = \chi dr$, hence $\sum_1^n x_k D_k = 0$, and therefore

$$C_{i_1 \dots i_p} = 0 \qquad (\text{all } i_1, \dots, i_p).$$

Now let $C_{i_1 \dots i_p}(r_1)$, $C_{i_1 \dots i_p}(r_2)$ be the values of $C_{i_1 \dots i_p}$ at two points on the same radius ($r_2 > r_1$). Then

$$[C_{i_1 \dots i_p}(r_2) - C_{i_1 \dots i_p}(r_1)]^2 = \left[\int_{r_1}^{r_2} \frac{\partial C_{i_1 \dots i_p}(r)}{\partial r} dr \right]^2$$

$$\leqslant \int_{r_1}^{r_2} \left[\frac{\partial C_{i_1 \dots i_p}}{\partial r} \right]^2 r^{n-1} dr \int_{r_1}^{r_2} \frac{dr}{r^{n-1}}.$$

If $d\omega$ is an element of solid angle at the origin, we have, therefore,

$$\int_{r=\rho} C^2_{i_1 \dots i_p} d\omega \leqslant \int_{\Delta' - \Delta''} \sum_{k=1}^{n} \left(\frac{\partial C_{i_1 \dots i_p}}{\partial x_k} \right)^2 dx_1 \dots dx_n \left\{ \frac{1}{n-2} \left(\frac{1}{\rho^{n-2}} - \frac{1}{a^{n-2}} \right) \right\}.$$

Therefore

$$\int_{r=\rho} C^2_{i_1 \dots i_p} d\Gamma \leqslant \frac{\rho}{n-2} \frac{a^{n-2} - \rho^{n-2}}{a^{n-2}} D^{r=a}_{r=\rho}(C_{i_1 \dots i_p}),$$

where $D^{r=a}_{r=\rho}(C_{i_1 \dots i_p})$ denotes the Dirichlet integral of $C_{i_1 \dots i_p}$ between

$r=\rho$ and $r=a$. Hence

$$\int_{\Delta'-\Delta''} \frac{(r-b)^2}{r^2} \sum_i C^2_{i_1 \dots i_p} dx_1 \dots dx_n$$

$$\leqslant \frac{1}{(n-2)\,a^{n-2}} \sum_i D^{r=a}_{r=b}(C_{i_1 \dots i_p}) \int_b^a (\rho-b)^2 (a^{n-2}-\rho^{n-2})\, d\rho$$

$$< \frac{1}{a} \sum_i D^{r=a}_{r=b}(C_{i_1 \dots i_p}) \int_b^a (\rho-b)^2 (a-\rho)\, d\rho$$

$$< \frac{(a-b)^4}{12a} \sum_i D^{r=a}_{r=b}(C_{i_1 \dots i_p}).$$

Hence $$D_\Delta(\phi) - D_\Delta(\tilde{\phi}) = D_{\Delta'}(\phi-\bar{\phi}) - \eta,$$

where $$0 < \eta < D_{\Delta'-\Delta''}(\phi-\bar{\phi}) + \frac{1}{3a} \sum_i D^{r=a}_{r=b}(C_{i_1 \dots i_p}).$$

By taking $a-b$ sufficiently small we can therefore make η as small as we please, and hence

$$D_\Delta(\tilde{\phi}) < D_\Delta(\phi).$$

The importance of this result is that it enables us to construct a sequence of forms

$$\phi_1, \phi_2, \dots, \phi_n, \dots, \tag{3.2}$$

satisfying the boundary conditions, and having covariant derivatives which satisfy uniform Hölder conditions in Δ, such that the sequence

$$D_\Delta(\phi_1), D_\Delta(\phi_2), \dots, D_\Delta(\phi_n), \dots$$

has the limit d. If we can show that the sequence (3.2) converges, the Dirichlet problem is solved. The manner in which this is proved is illustrated in Section IV.

24. By means of the Poisson integral we can immediately extend Harnack's theorem for harmonic functions, and hence we can state that, if

$$\phi_1, \phi_2, \dots, \phi_n, \dots$$

is a sequence of $(p-1)$-forms which are harmonic within a sphere of radius a, and the sequence converges to a limit ϕ in this sphere, ϕ is also harmonic within the sphere. The proof is the same as that given by Weyl† for the case $n=2$.

† Weyl, *Die Idee der Riemannschen Fläche* (1923), 83.

IV. *Applications to analytic varieties.*

25. Consider a closed analytic variety V of n-dimensions, lying in a Euclidean space of r-dimensions. Let us assume

(1) That V is entirely covered by a finite number of overlapping domains, each having the property that it can be represented by equations

$$x_i = f_i(u_1, \ldots, u_n),$$

where the f's are analytic functions of the parameters $(u_1, \ldots, u_n)$ whose domain is an open Euclidean n-cell, $E(u)$, in which $(u_1, \ldots, u_n)$ are the Cartesian coordinates.

(2) In each cell

$$\sum_{i=1}^{r} dx_i^2 = \lambda^2 \sum_{i=1}^{n} du_i^2,$$

where λ is a function of $(u_1, \ldots, u_n)$.

(3) If a point P lies in two of these domains, the parameters of P in each system can be expressed as analytic functions of the parameters in the other. The transformation is conformal, *i.e.*

$$\lambda^2 \sum_{1}^{n} du_i^2 = \lambda'^2 \sum_{1}^{n} du'_i{}^2.$$

We assume further that it is direct.

Consider a sphere lying entirely in a cell $E(u)$. The image of this will be spoken of as a sphere on V. The intersections of the regions bounded by two spheres in the same parametric system is a cell. Consider now a sphere in the parametric system $(u_1', \ldots, u_n')$ which overlaps a sphere in the system $(u_1, \ldots, u_n)$. We can show that the part common to the regions which they bound may be broken up into a finite number of cells. Substituting for $(u_1', \ldots, u_n')$ in terms of $(u_1, \ldots, u_n)$, we have then to consider the portion of a spherical region which is bounded partly by the sphere, and partly by an analytic variety. But we know that this can be reduced to a finite number of cells†. A simple application of the Heine-Borel covering theorem enables us to see that we can find a finite system Σ of spheres on V having the property that each point of the variety V lies within a region bounded by at least one sphere of the system Σ. We can suppose further that no sphere of the system Σ lies in the region bounded by another sphere of the system.

† B. O. Koopman and A. B. Brown, *Trans. Amer. Math. Soc.*, 34 (1932), 231.

These spheres divide V up into a finite number of pieces, each of which can be divided into a finite number of cells. It is clear that V has by this process been reduced to a complex and, since the neighbourhood of each point is an analytic cell, V is an absolute manifold M.

26. In § 3 we gave a definition of a harmonic form on the variety V. We saw there that, if $n = 2p$,

$$\phi = \sum_i A_{i_1 - i_{p-1}} du_{i_1} \dots du_{i_{p-1}}$$

is a harmonic form in the domain of V which corresponds to the cell $E(u)$. ϕ is also a harmonic form regarded as a form in this cell. Hence we can solve the Dirichlet problem for any cell in this domain if we can solve it for the corresponding cell in $E(u)$. In particular, we can solve it for the image on V of the interior of a sphere in $E(u)$, by means of the Poisson integral, and this is the only case which we shall use in this section.

For the remainder of this section we shall suppose that V is of $2p$-dimensions, and concern ourselves solely with functions of the $(p-1)$-cycles. Now consider any form ϕ which has a continuous derivative in a certain region R of V. If R lies in the domain corresponding to $E(u)$, and is represented there by R_1, the Dirichlet integral $D_R(\phi)$ of ϕ over the region R is obtained by expressing ϕ in terms of $(u_1, \dots, u_n)$ and writing

$$D_R(\phi) = \int_{R_1} \phi_u \cdot \phi^u.$$

If R also lies in the domain corresponding to $(u_1', \dots, u_n')$, express $\phi = \phi'$ in terms of $(u_1', \dots, u_n')$. Then

$$\int_{R_1'} \phi'_{u'} \cdot \phi'^{u'} = \int_{R_1} \phi_u \cdot \phi^u = D_R(\phi),$$

since at any point of R_1' $\phi_u = \phi'_{u'}$ and $\phi^u = \phi'^{u'}$ at the corresponding point of R_1', since the transformation between the parameters is conformal and direct. Now consider any region R on V. Let

$$R = R_a + R_b + \dots + R_k,$$

where each region such as R_a lies entirely within a cell of M. Then the Dirichlet integral $D_{R_a}(\phi)$ of ϕ is uniquely defined. We define the Dirichlet integral $D_R(\phi)$ by the equation

$$D_R(\phi) = D_{R_a}(\phi) + D_{R_b}(\phi) + \dots + D_{R_k}(\phi).$$

In particular, if ϕ has a continuous derivative over the whole of V, we can

define in this way $$D_V(\phi) = D(\phi).$$

We shall have to consider a form ϕ which is defined everywhere on V, but which has a certain sphere of discontinuity. If the derivative is continuous everywhere else and the sphere is one of the spheres of Σ, the expression

$$\Sigma D_{E_i}(\phi),$$

summed over the cells of V, is uniquely defined, and is taken as the value of $D(\phi)$. More generally, we shall find that ϕ is not uniquely determined in our problem, but that ϕ_u is determined and has the discontinuity described above. If ϕ_u is continuous elsewhere, $D(\phi)$ is defined by the above formula.

27. Consider a point P of V. Let it lie in the domain corresponding to $E(u)$, and let the parameters of P be taken as $u_i = 0$ $(i = 1, \ldots, n)$. We shall establish the existence of a form ϕ which is harmonic everywhere on V except at P, and is such that, in the neighbourhood of P,

$$\phi - \sum_{i=1}^{p} (-1)^{i-1} \frac{u_i du_1 \ldots du_{i-1} du_{i+1} \ldots du_p}{r^n},$$

where $r^2 = u_1^2 + \ldots + u_n^2$, is harmonic. Consider a sphere about P of radius $r = a$. Write

$$\Phi = \sum_{i=1}^{p} (-1)^{i-1} \frac{u_i du_1 \ldots du_{i-1} du_{i+1} \ldots du_p}{r^n} + \sum_{i=1}^{p} (-1)^{i-1} \frac{u_i du_1 \ldots du_{i-1} du_{i+1} \ldots du_p}{a^n},$$

so that

$$\Phi_u = p\, du_1 \ldots du_p \left(\frac{1}{r^n} + \frac{1}{a^n}\right) - n \sum_{1}^{n} u_i du_i \sum_{i=1}^{p} (-1)^{i-1} \frac{u_i du_1 \ldots du_{i-1} du_{i+1} \ldots du_p}{r^{n+2}},$$

$$\Phi^u = p\, du_{p+1} \ldots du_n \left(\frac{1}{r^n} + \frac{1}{a^n}\right) - n \frac{du_{p+1} \ldots du_n}{r^n} + (-1)^p n \sum_{i=1}^{n} u_i du_i \sum_{k=p+1}^{n} (-1)^{k-1} u_k \frac{du_{p+1} \ldots du_{k-1} du_{k+1} \ldots du_n}{r^{n+2}},$$

$$\Delta_u \Phi = n(n-p) \sum_{i=1}^{n} u_i du_i \frac{du_{p+1} \ldots d_n}{r^{n+2}} - n(n-p) \sum_{i=1}^{n} \frac{u_i du_i du_{p+1} \ldots du_n}{r^{n+2}} = 0 \quad (\text{if } r \neq 0).$$

Also, when $r = a$, $\Phi^u = dr\chi$,

where χ is a $(p-1)$-form. Let κ_0 be the sphere whose centre is P, and whose radius is a, and let K_0 be the region enclosed by it. Let κ_1 be a concentric sphere (in the same coordinate system) of radius $b > a$, K_1 the region which it encloses. We construct a $(p-1)$-form ψ having a discontinuity on κ_0, but possessing elsewhere an exterior derivative which satisfies uniform Hölder conditions on V, such that associated with it there is a $(p-1)$-form ψ^* which is continuous in K_1, whose derivative satisfies uniform Hölder conditions in K_1, and which is equal to ψ in K_0 and to $\psi - \Phi$ in $K_1 - K_0$. Any form satisfying these conditions is said to satisfy conditions (α). We can find one such form ψ as follows. Let c be any number $0 < c < a$, and let Φ_1 be

$$\Phi_1 = 2\Sigma(-1)^{i-1}\frac{u_i du_1 \dots du_{i-1} du_{i+1} \dots du_p}{a^n},$$

then, if

(1) $\psi = -\Phi_1 \qquad (0 < r < c),$

(2) $\psi = -\Phi_1 - \left(\frac{r-c}{a-c}\right)^2 (\Phi - \Phi_1) \quad (c < r < a),$

(3) $\psi = 0$ outside K_0,

then ψ satisfies the required conditions.

28. We have defined the Dirichlet integral $D(\psi)$ of ψ over V. Suppose that ψ is not harmonic in the neighbourhood of a point Q of $V - K_0$. With centre Q and suitable radius we construct a spherical region K about Q. Following the reasoning of § 23, we can find a $(p-1)$-form $\tilde{\psi}$ satisfying the conditions (α) which is equal to ψ outside this sphere, and is such that

$$D(\tilde{\psi}) < D(\psi).$$

Suppose next that the point Q lies in K_0 or on κ_0, and that ψ^* is not harmonic in the neighbourhood of Q. We construct the spherical region as before. If K lies wholly within K_0, the argument we have just given can be applied. If K extends outside K_0, let K' be the part within K_0, K'' the remainder, which we suppose lies in K_1. We can construct $\tilde{\psi}^*$ such that

$$D_K(\tilde{\psi}^*) < D_K(\psi^*).$$

Let $\tilde{\psi}$ be the form satisfying conditions (α) which corresponds to $\tilde{\psi}^*$. Then

$$D_K(\psi^*)-D_K(\tilde{\psi}^*) = D_{K'}(\psi)-D_{K'}(\tilde{\psi})+D_{K''}(\psi^*)-D_{K''}(\tilde{\psi}^*).$$

But $$D_{K''}(\psi^*+\Phi) = D_{K''}(\psi^*)+D_{K''}(\Phi)+2\int_{\kappa''}\psi^*.\Phi^u,$$

where the last integral is evaluated over the boundary κ'' of K''. Similarly

$$D_{K''}(\tilde{\psi}^*+\Phi) = D_{K''}(\tilde{\psi}^*)+D_{K''}(\Phi)+2\int_{\kappa''}\tilde{\psi}^*.\Phi^u.$$

Now $$\int_{\kappa''}(\psi^*-\tilde{\psi}^*).\Phi^u = 0,$$

since, on the part of κ'' which belongs to the boundary of K, $\psi^*-\tilde{\psi}^*$ defines a zero form, and on the remainder, which belongs to κ_0, Φ^u defines a zero form. Hence

$$D_K(\tilde{\psi}) < D_K(\psi),$$

and therefore $$D(\tilde{\psi}) < D(\psi).$$

We can therefore construct a sequence of forms,

$$\psi_1,\ \psi_2,\ \ldots,\ \psi_n,\ \ldots,$$

satisfying conditions (α), such that

$$D(\psi_n) < D(\psi_m) \quad (\text{if } n > m).$$

Since $D(\psi_n) > 0$ for all values of n, the sequence

$$D(\psi_1),\ D(\psi_2),\ \ldots,\ D(\psi_n),\ \ldots$$

must tend to a limit d. Given any ϵ, we can find an n_0 such that

$$D(\psi_n) < d+\epsilon$$

for all $n > n_0$. In what follows we shall denote by

$$\bar{\psi}_1,\ \bar{\psi}_2,\ \ldots,\ \bar{\psi}_n,\ \ldots$$

the harmonic forms which define the same forms as

$$\psi_1,\ \psi_2,\ \ldots,\ \psi_n,\ \ldots$$

on a certain sphere κ on V. If this sequence has a limiting form, the remark made in § 24 shows that this limit is harmonic.

29. Let us write

$$(\psi_n)_u = \sum_i B^{(n)}_{i_1 \dots i_p} du_{i_1} \dots du_{i_p}$$

$$(\bar{\psi}_n)_u = \sum_i \bar{B}^{(n)}_{i_1 \dots i_p} du_{i_1} \dots du_{i_p}.$$

Consider the point Q of $V - K_0$, and the region K of radius a as in § 28.

$$\left| \int_K (B^{(n_1)}_{i_1 \dots i_p} - B^{(n_2)}_{i_1 \dots i_p}) du_1 \dots du_n \right| \leqslant c a^p \sqrt{\{D(\psi_{n_1} - \psi_{n_2})\}}$$

$$\leqslant c a^p [\sqrt{\{D(\psi_{n_1}) - d\}} + \sqrt{\{D(\psi_{n_2}) - d\}}]$$

by Levi's inequality†, where $c^2 a^n$ is the measure of a sphere of radius a in space of n-dimensions.

It follows that the sequence whose general term is

$$\int_K B^{(n)}_{i_1 \dots i_p} du_1 \dots du_n$$

converges to a limit as $n \to \infty$, and hence, if $(\mu_1, \dots, \mu_n)$ are the parameters of Q,

$$\frac{1}{c^2 a^n} \int_K (\psi_n)_u \Sigma d\mu_{i_1} \dots d\mu_{i_p} du_{i_{p+1}} \dots du_{i_n} \qquad (4.1)$$

defines a definite p-form at Q as $n \to \infty$.

If K' is a spherical region in K whose centre is at Q', and whose radius is a', a similar argument will show that

$$\frac{1}{c^2 a^n} \int_{K'} (\bar{\psi}_n)_u \Sigma d\mu_{i_1} \dots d\mu_{i_p} du_{i_{p+1}} \dots du_{i_n}$$

tends to a definite form at Q'. But, since the coefficients of $(\bar{\psi}_n)_u$ are harmonic functions of $(u_1, \dots, u_n)$,

$$\frac{1}{c^2 a^n} \int_{K'} (\bar{\psi}_n)_u \sum_i d\mu_{i_1} \dots d\mu_{i_p} du_{i_{p+1}} \dots du_{i_n}$$

is equal to the value of $(\bar{\psi}_n)_u$ at Q', when $(\mu_1, \dots, \mu_n)$ are written for $(u_1, \dots, u_n)$. Hence

$$\lim_{n \to \infty} (\bar{\psi}_n)_u = \bar{\psi}_u$$

exists, and it is the derivative of a harmonic form in K.

† *Cf.* Weyl, *Die Idee der Riemannschen Fläche*, 101.

We must now show that the limit of (4. 1) is independent of the sphere K. First consider a concentric sphere $\bar{K}$, in the same coordinate system, of radius $\bar{a} < a$. We can find a form $\tilde{\psi}_n$ as in § 28, such that

$$D_K(\tilde{\psi}_n - \bar{\psi}_n) = D_K(\tilde{\psi}_n) - D_K(\bar{\psi}_n)$$

is as small as we please. Now

$$\begin{aligned} D_K(\psi_n - \bar{\psi}_n) &= D_K(\psi_n) - D_K(\bar{\psi}_n) \\ &= D_K(\psi_n) - D_K(\tilde{\psi}_n) + D_K(\tilde{\psi}_n - \bar{\psi}_n) \\ &\leqslant [D(\psi_n) - d] + D_K(\tilde{\psi}_n - \bar{\psi}_n); \end{aligned}$$

hence, by taking n sufficiently large, we can make

$$D_K(\psi_n - \bar{\psi}_n)$$

as small as we please. But

$$\frac{1}{c^2 a^n} \left| \int_K (B^{(n)}_{i_1 \dots i_p} - \bar{B}^{(n)}_{i_1 \dots i_p}) \, du_1 \dots du_n \right| \leqslant \frac{1}{c a^p} \sqrt{\{D_K(\psi_n - \bar{\psi}_n)\}},$$

and hence

$$\lim \frac{1}{c^2 a^n} \int_K (\psi_n)_u \Sigma \, d\mu_{i_1} \dots d\mu_{i_p} du_{i_{p+1}} \dots du_{i_n} = \bar{\psi}_u \quad \text{at } Q.$$

An exactly similar argument shows that

$$\lim \frac{1}{c^2 \bar{a}^n} \int_{\bar{K}} (\psi_n)_u \Sigma \, d\mu_{i_1} \dots d\mu_{i_p} du_{i_{p+1}} \dots du_{i_n} = \bar{\psi}_u \quad \text{at } Q.$$

Hence this limit is independent of the radius a. Next suppose that $\bar{K}$ is a sphere of radius $\bar{a}$ in another system of coordinates $(u', \dots, u_n')$, valid in the neighbourhood of Q. We can suppose that $\bar{K}$ lies in K. We then prove in the same way that

$$\lim \frac{1}{c^2 \bar{a}^n} \int_{\bar{K}} (\psi_n')_{u'} \Sigma \, d\mu'_{i_1} \dots d\mu'_{i_p} du'_{i_{p+1}} \dots du'_{i_n} = \bar{\psi}'_{u'} = \bar{\psi}_u,$$

since the transformation of coordinates is conformal. A similar argument, with only minor variations, is valid when Q is a point in K_0 or on κ_0, if we consider $\psi_n{}^*$ in place of ψ_n.

30. We now show that the limit of (4 . 1) is the derivative of a harmonic form. Again, take Q in $V - K_0$ and consider the region K. Let R be any

point of K and with centre R and radius r describe a spherical region K_R, lying within K. Then as before we show that

$$\frac{1}{c^2 r^n}\left|\int_{K_R} B^{(n)}_{i_1 \dots i_p} du_1 \dots du_n - \int_{K_R} \bar{B}^{(n)}_{i_1 \dots i_p} du_1 \dots du_n\right| < \frac{4}{cr^p}\,[D(\psi_n)-d]^{\frac{1}{2}},$$

and hence that the limit of (4.1) at R is equal to the value of $\bar{\psi}_u$ at R. Hence this limit is the derivative of a harmonic form in the neighbourhood of Q, and hence everywhere on $V-K$. We deal similarly with points in K_0. We thus see that the limit of (4.1) defines at each point of V a form ϕ_u which is the covariant derivative of a harmonic form (except, of course, on κ_0), and which behaves on V like the covariant derivative of a form satisfying conditions (α). We could define from this a continuous $(p-1)$-form in each cell of M which has ϕ_u as its derivative, but this will not be necessary.

An argument similar to that of Weyl† shows that

$$D(\phi) = d.$$

If there existed another minimising sequence of forms

$$\psi_1', \psi_2', \dots, \psi_n',$$

satisfying the conditions (α), such that

$$\lim D(\psi_n') = d' \neq d,$$

we could find from this another form ϕ_u' having the same properties as ϕ_u such that

$$D(\phi') = d'.$$

Then $\chi = \phi_u - \phi_u'$ would be the covariant derivative of a form on V which was everywhere finite, and (as we shall see in § 31) was such that

$$\int \chi$$

over any cycle was zero. But we shall show in § 32 that this would imply

$$\chi = 0,$$

and hence $d = d'$, and therefore ϕ_u is unique.

31. Let us consider

$$\int_{\Gamma_p} \phi_u$$

† *Loc. cit.*, 107.

over any p-cycle of V. Let K', K'', ... be regions bounded by a finite set of spheres of the system Σ which cover a portion of V in which Γ_p is embedded. We can divide Γ_p up into portions Γ_p', Γ_p'', ... each lying wholly in one of these spheres. Let $\bar{\psi}_n'$, $\bar{\psi}_n''$, ... be the harmonic forms which correspond to ψ_n in each of these spherical regions. Since ϕ_u is the limit of $(\bar{\psi}_n)_u$ we can find n such that

$$\left| \int_{\Gamma_p} \phi_u - \Sigma \int_{\Gamma_{p'}} (\bar{\psi}_n')_u \right|$$

is as small as we please. Also, by taking n sufficiently large,

$$\int_{\Gamma_{p'}} (\bar{\psi}_n')_u \quad \text{differs from} \quad \int_{\Gamma_{p'}} (\psi_n)_u$$

by as little as we please. Hence

$$\int_{\Gamma_p} \phi_u \quad \text{differs from} \quad \int_{\Gamma_p} (\psi_n)_u$$

by as little as we please. But ψ_n is one valued on V. Hence the latter integral is zero, and it follows that

$$\int_{\Gamma_p} \phi_u = 0.$$

We therefore deduce that there exists a p-form θ which is the covariant derivative of a harmonic form everywhere on V, except at P, where it behaves like Φ_u. Also, we have

$$\int_{\Gamma_p} \theta = 0$$

over any p-cycle Γ_p of V. We shall say that

$$\int \theta$$

is an integral on V which is harmonic everywhere except at P.

More generally, we can find a harmonic integral on V which behaves in the neighbourhood of P like the coefficient of $dv_{i_1} \dots dv_{i_{p'}}$ in

$$(-1)^{p-1} \int \omega^{\nu}_{p-1,\,u} = \int \omega^{u}_{p-1,\,\nu},$$

ω_{p-1} having its usual meaning with ν written for ξ, u for x. P is the point $\nu = u$. Such integrals we shall call normal harmonic integrals of the second kind on V, having P for pole.

32. Let us write

$$\int \mu_{\nu u}$$

for the integral (with respect to u) which is harmonic on V except at P, and is such that

$$\int \mu_{\nu u} - \int \omega^u_{p-1,\nu}$$

is harmonic in the neighbourhood of P. Let $E_p{}^i$ be any p-cell of the manifold M, and let

$$E_p{}^i \to C_{p-1}.$$

Let $\int_{E_p{}^i} \mu_{\nu u} = \lambda_{i,u}$ (the integration being with respect to ν). Then $\int \lambda_{i,u}$ is a harmonic integral on V, except on C_{p-1}, where it behaves like $\iint_{C_{p-1}} \omega^u_{p-1}$. This is what we call a harmonic integral of the third kind on V. From the manner in which $\lambda_{i,u}$ is constructed we see that, if $\Gamma_p{}'$ is any p-cycle on V, then

$$\int_{\Gamma_p{}'} \lambda_{i,u} = 0$$

if $\Gamma_p{}'$ does not meet $E_p{}^i$, and if $\Gamma_p{}'$ meets $E_p{}^i$, we can deform it into a cycle not meeting $E_p{}^i$ + a cycle $\Gamma_p{}''$ in the region about $E_p{}^i$ in which the parameters $(u_1, \ldots, u_n)$ are valid, and deduce that

$$\int_{\Gamma_p{}'} \lambda_{i,u} = \int_{\Gamma_p{}''} \int_{C_{p-1}} \omega^u_{p-1} = K\alpha,$$

where α is the constant of § 5, and $K = (\Gamma_p{}' \,.\, E_p)$. Now let

$$\Gamma_p = \sum_i a_i E_p{}^i$$

be any cycle of M, and write

$$w = \frac{1}{\alpha} \sum_i a_i \lambda_{i,u}.$$

We see that w is harmonic everywhere on V, and that

$$\int_{\Gamma_p{}'} w = (\Gamma_p{}' \,.\, \Gamma_p).$$

We speak of this as the period of $\int w$ on $\Gamma_p{}'$. Hence, if Γ_p is a bounding cycle or a zero divisor, all its periods are zero. But, in this case, we can prove that $w = 0$. We construct the product of V by itself, and employ

the reasoning of my previous paper† to show that

$$\int_V w \,.\, \bar{w} = 0,$$

where

$$\bar{w} = \frac{1}{a} \Sigma a_i \lambda_i^u,$$

and this can only happen if $w = 0$. On the other hand, if Γ_p is a true cycle, w cannot be zero, since the integral has at least one non-zero period. If

$$\Gamma_p', \ldots, \Gamma_p^{R_p}$$

are the R_p p-cycles of V, we therefore have R_p integrals

$$\int w_1, \ldots, \int w_{R_p},$$

such that

$$\int_{\Gamma_p^i} w_i = \delta_{ij},$$

and which are harmonic everywhere on V, and any integral which is harmonic everywhere on V is linearly dependent on these integrals. The R_p integrals written above are called the normal harmonic integrals of the first kind on V.

These names have been introduced because the integrals reduce, in the case $p = 1$, to the real or imaginary parts of the Abelian integrals of the first, second, and third kinds on a Riemann surface.

† Hodge, *Journal London Math. Soc.*, 5 (1930), 283.

ON QUANTITATIVE SUBSTITUTIONAL ANALYSIS

(*Seventh Paper.*)

By ALFRED YOUNG.

[Received 23 May, 1932.—Read 19 May, 1932.]

The subject here is the application of quantitative substitutional analysis to the theory of invariants. In the second paper, Section V*, it was shown that a very convenient notation for the invariants of a single binary form could be obtained,

$$f(a_0, a_1, \ldots, a_n),$$

by means of the symbolical calculus and the tableaux. Equations were there obtained expressing the linear relations between invariants thus given. Two problems at once arise, the discovery of a linearly independent set of such forms in terms of which the rest can be expressed; and the expression of the product of two such forms in terms of members of the set. The notation may be used for covariants as well as for invariants, and for any concomitants of one or more quantics in any number of variables.

The first section deals with the application to concomitant types, *i.e.* concomitants linear in the coefficients of each of the ground forms concerned. And here a very simple answer is obtained in Theorem I to the first problem.

The second section develops the relations between the forms, more particularly in the binary case. The forms for this purpose are arranged according to some definite sequence, and here two sequences are introduced, (A) and (B), each of which has its own advantage.

In the third section the relations of forms given in the present tableau notation to seminvariants expressed otherwise are sought. It is shown in particular that these forms are identical with the seminvariants obtained by Elliott by means of Ω and O operators from the corresponding gradient. The

* *Proc. London Math. Soc.* (1), 34 (1902), 388.

expression for them is also obtained in terms of continued transvectants. Conversely a continued transvectant is expressed in terms of these forms. And further the actual expression of any gradient as a sum of terms of covariants is directly obtained. The fourth section gives a complete discussion of covariants of degree 4. In the fifth section the theory is illustrated by the binary cubic and quartic. In the sixth section the question of sequences is discussed. A pre-arranged sequence must underlie any enumeration of the forms. In addition to the sequences (A) and (B) already used, a sequence (C) is introduced, which is the reciprocal of (A) and (B). It is shown that this sequence (C) underlies the whole of the classical treatment of binary forms; but, that while it contains certain obvious advantages here, in that it can be said at once whether a form belongs to the linearly independent set or no, it has the disadvantage that a criterion needs to be obtained as to whether the form in question actually exists for a given finite order. A very intimate connection is here indicated between a form which belongs to the independent set according to one of the three sequences and a form which has the corresponding gradient for its leading gradient. A numerical diagram is introduced as a means by which it may be ascertained whether a seminvariant with such a leading gradient exists.

In Section VII, the invariants of degrees 5 and 6 for any binary form are obtained, with the corresponding invariants of the quintic and sextic.

In Section VIII, generating functions for types are obtained for forms in any number of variables, by means of the results of Section I.

In Section IX, these are illustrated by the discussion of the invariant types of degree 5 for binary forms.

In Section X, the independent set of seminvariants of degree 5 for a binary form is given, and exhibited as leading gradients.

In Section XI, the method is applied to obtain the irreducible invariants of a quaternary cubic.

Section XII contains some concluding remarks.

I. *Application of the analysis to algebraic forms.*

1. The general concomitant for forms with q variables is symbolically written as a product of factors of the form

$$(a_1 a_2 \dots a_r u_{r+1} \dots u_q),$$

where the original forms are $a_{1_x}^{n_1}$, $a_{2_x}^{n_2}$, ..., which may be the same or different forms:

$$a_{1_x} = a_{11}x_1 + a_{12}x_2 + \dots + a_{1q}x_q,$$

and $u_{r+1}, \ldots, u_q$ are cogredient sets of variables which are contragredient to x. For uniformity we replace x by $(u_2 u_3 \ldots u_q)$ so that the symbolical factors are all of the same type. By polarization this symbolical product can be represented as a sum of products which are linear in every symbolical letter which appears; this summation is best represented by a substitutional operator.

We take the *first term* of this concomitant, which corresponds to the seminvariant, as the coefficient of $u_{2,2}^{\lambda_2} u_{3,3}^{\lambda_3} \ldots u_{q,q}^{\lambda_q}$, where λ_2 is the number of factors which contain u_2 and is the order; λ_3 is the number of factors which contain u_3—for ternary forms the sum of the order and class—and so on.

We use a triple suffix notation where

$$a_{\lambda\mu_x} = \Sigma a_{\lambda\mu\tau} x_\tau,$$

and the symbolical product is in general

$$\Pi(a_{\lambda 1} a_{\lambda 2} \ldots a_{\lambda r_\lambda} u_{r_\lambda + 1} \ldots u_q).$$

We use as letters of permutation the letters $a_{\lambda\mu}$ with two suffixes, so that, for instance, the transposition $(a_{\kappa\lambda} a_{\mu\nu})$ changes the product $a_{\kappa\lambda\tau} a_{\mu\nu s}$ into $a_{\mu\nu\tau} a_{\kappa\lambda s}$. Then the seminvariant is

$$\Pi \{a_{\lambda 1} a_{\lambda 2} \ldots a_{\lambda r_\lambda}\}' a_{\lambda 11} a_{\lambda 22} \ldots a_{\lambda r_\lambda r_\lambda} = \Pi \{a_{\lambda 1} a_{\lambda 2} \ldots a_{\lambda r_\lambda}\}' K,$$

where

$$K = G_{a_1} G_{a_2} \ldots G_{a_q} K,$$

and G_{a_r} is a positive symmetric group of degree a_r divided by its order, which contains one letter from every symbolical factor of the seminvariant which has r letters at least; for K is unaltered by the interchange of a pair of letters whose third suffix in K is the same. That is, the symbolical product is of the nature

$$NPK,$$

where NP belongs to $T_{a_1 a_2 \ldots a_q}$.

If we multiply by ΣPN, we can express this in terms of

$$\Sigma PNK.$$

The forms thus defined will all be linearly independent when the tableaux for P are standard. When, however, as is the case here, there are implicitly understood substitutional operators on the left-hand side, a further investigation is needed to determine the standard forms, since these operators usually disturb the sequence on which the standard property depends.

2. Let us consider types, that is forms which are linear in the coefficients of every ground form concerned. To start with, the ground forms are arranged in definite sequence. Let $b_x{}^n$ be one of them. Then this quantic in the substitutional expressions must be represented by n letters $b_1, b_2, \ldots, b_n$; and the fact that they are all representative of the one quantic is denoted by an operator $B = \{b_1 b_2 \ldots b_n\}$ on the left of PN. Standard tableaux will include some in which two or more b's are in the same column; let us suppose that the following is part of such a tableau F:

$$\begin{matrix} \ldots & \ldots & \ldots & b_1 b_2 \ldots b_r & c_1 \ldots c_k \\ d_1 \ldots d_l & b_{r+1} \ldots b_{r+s} \ldots b_n & \ldots & \ldots, & \end{matrix}$$

where the letters $c_1, \ldots, c_k$ are later than b_n, and the letters $d_1, \ldots, d_l$ are earlier than b_1.

Then, if PN is given by the tableau F,

$$BPNK$$

is a sum of terms some of which are not standard. Thus one is obtained by interchange of b_1 and b_{r+s}.

Then

$$B\{d_1 \ldots d_l b_{r+1} \ldots b_{r+s} b_1 \ldots b_r c_1 \ldots c_k\} PNK = 0,$$

and hence

$$\lambda BPNK = -\Sigma \mu BP' NK,$$

where F', the tableau from which P' is formed, is obtained from F by exchanging some of the letters $d_1, \ldots, d_l, b_{r+1}, \ldots, b_{r+s}$ for some of the letters $b_1, \ldots, b_r, c_1, \ldots, c_k$, and since in virtue of the presence of B interchange of letters b among themselves has no effect, at least either one of the d's is moved up or one of the c's is moved down. Thus F' is an earlier tableau than F, and also it has fewer pairs of letters b in the same column. Thus, when the tableau of a concomitant has a pair of letters belonging to the same ground form in the same column, the concomitant can be expressed in terms of others whose tableaux are earlier. We may refer to the case as overlapping, and state the result as follows:

THEOREM I. *A complete set of linearly independent concomitants linear in the coefficient of each of the ground forms* $a_{r_x}^{n_r}$ $(r = 1, 2, \ldots, \delta)$ *is given by those forms* $GPNK$ *whose tableaux* F *are standard and do not overlap.*

Here G is that operator which expresses that the first n_1 symbolical letters refer to the first quantic, the next n_2 to the second quantic, and so on.

That these concomitants are all linearly independent follows at once from the fact that

$$GPNK = \Sigma(PNK)_\rho,$$

where every tableau F_ρ is standard. Unfortunately, the question of the reducibility of these forms is not so easily settled.

3. For the binary case the tableaux contain only two rows, and it is sufficient to define a seminvariant by giving a scheme which states how many letters belonging to each quantic lie in the second row. Thus, if the r-th quantic has λ_r letters in the second row, we may write the seminvariant as $(a_1{}^0 a_2^{\lambda_2} a_3^{\lambda_3} \dots a_\delta^{\lambda_\delta})$, or more shortly $(0\,\lambda_2 \dots \lambda_\delta)$.

The first quantic can have no letters in the second row, since that would mean overlapping. This condition further requires that

$$\lambda_2 \leqslant n_1, \quad \lambda_3 + 2\lambda_2 \leqslant n_1 + n_2, \quad \lambda_4 + 2\lambda_3 + 2\lambda_2 \leqslant n_1 + n_2 + n_3,$$

and so on. There is nothing novel about this, as a similar result is obtained at once by expressing the form as a continued transvectant

$$(\dots((f_1, f_2)^{\lambda_2} f_3)^{\lambda_3} \dots f_\delta)^{\lambda_\delta},$$

when the same inequalities must be satisfied. The similar result for forms with more variables is not so obvious.

II. *The relations between seminvariants.*

4. When two or more of the δ forms are the same we have to introduce a fresh operator on the left which disturbs the sequence of forms, and some of the seminvariants with tableaux in which there is no overlapping are expressible in terms of others with earlier tableaux.

It was proved* that all relations between such forms as PNK may be derived from relations

$$\Gamma PNK = 0,$$

where Γ is a positive symmetric group containing all the letters of one row of the tableau and one letter out of a lower row. Let PNK be the form $(a_1{}^0 a_2^{\lambda_2} \dots a_\delta^{\lambda_\delta})$ and let Γ contain all the letters of the first row and one

* Q.S.A., II, 370.

letter a_r out of the second row of the tableau. Then

$$\Gamma PNK = \sum_{s \neq r} (n_s - \lambda_s)(a_1^{\lambda_0} a_2^{\lambda_2} \dots a_s^{\lambda_s+1} \dots a_r^{\lambda_r-1} \dots a_\delta^{\lambda_\delta}) + (n_r - \lambda_r + 1)(a_1^{\lambda_0} a_2^{\lambda_2} \dots a_r^{\lambda_r} \dots a_\delta^{\lambda_\delta}) = 0.$$

This may be written

$$O(a_1^{\lambda_0} a_2^{\lambda_2} \dots a_r^{\lambda_r-1} \dots a_\delta^{\lambda_\delta}) = 0. \tag{I}$$

There is a close relation between our seminvariant $(a_1^{\lambda_0} a_2^{\lambda_2} \dots a_\delta^{\lambda_\delta})$ and the gradient

$$A_{1,0} A_{2,\lambda_2} \dots A_{\delta,\lambda_\delta},$$

where $A_{r,s}$ is defined by

$$a_{rx}^{n_r} = (A_{r,0}, A_{r,1}, \dots, A_{r,n_r})(x_1, x_2)^{n_r}.$$

In this respect the operator O of equation (I) appears as the well-known algebraic operator

$$O = \sum_r \sum_s (n_r - s) A_{r,s+1} \frac{\partial}{\partial A_{r,s}}.$$

In the case of seminvariants of a single form the notation may be changed with advantage to

$$(A_0^{\lambda_0} A_1^{\lambda_1} \dots A_n^{\lambda_n})$$

or simply

$$(\lambda_0 \lambda_1 \dots \lambda_n),$$

where λ_r is the number of sets of n letters which have r letters in the lower row of the tableau. Then

$$O(A_0^{\lambda_0} A_1^{\lambda_1} \dots A_n^{\lambda_n}) = 0, \tag{II}$$

where O is the ordinary algebraic operator.

5. The seminvariants of a single binary form will now be considered. Those of the same weight will be arranged in a definite sequence. For this purpose two different laws will be considered, since each possesses a certain advantage; they will be quoted as sequence (A) and sequence (B); but, since the former is that in general adopted, it will be assumed to be the one in use unless otherwise stated.

(A) The sequence from the last letter:

$$(A_0^{\lambda_0} A_1^{\lambda_1} \dots A_n^{\lambda_n}) \quad \text{precedes} \quad (A_0^{\mu_0} A_1^{\mu_1} \dots A_n^{\mu_n})$$

when the first of the differences

$$\lambda_n - \mu_n, \quad \lambda_{n-1} - \mu_{n-1}, \quad \dots, \quad \lambda_0 - \mu_0$$

which does not vanish is positive.

(B) The sequence from the first letter requires that the first of the differences

$$\lambda_0 - \mu_0, \quad \lambda_1 - \mu_1, \quad \dots, \quad \lambda_n - \mu_n$$

which does not vanish must be positive. A seminvariant which can be linearly expressed in terms of earlier seminvariants is called *reductible*, a seminvariant which cannot be so expressed is called *irreductible*.

From equation (II) it follows at once that all seminvariants for which $\lambda_1 > 0$ are reductible according to either sequence.

6. Consider now seminvariant types of degree 3. They are given by the forms

$$(a_1^{\lambda_0} a_2^{\lambda} a_3^{\mu}).$$

As in Q.S.A., II, § 28, we shall operate with the positive symmetric group of the n letters a_2; as in the place quoted this may be treated in two ways, either as a sum of permutations of the n letters a_2, in which case the effect is merely multiplication by $n!$, or as bringing up all the n letters a_2 into the first row and filling the λ places of the second row by letters a_1 and a_3 from the first row in all possible ways. By equating the two results we get

$$(-)^{\lambda} \binom{n}{\lambda} (a_1^{\lambda_0} a_2^{\lambda} a_3^{\mu}) = \sum_u \binom{n}{u} \binom{n-\mu}{\lambda-u} (a_1^{u} a_2^{\lambda_0} a_3^{\lambda-u+\mu}),$$

i.e.

$$(-)^{\lambda} \binom{n}{\lambda} \binom{n}{\mu} (a_1^{\lambda_0} a_2^{\lambda} a_3^{\mu}) = \Sigma \binom{\lambda-u+\mu}{\mu} \binom{n}{u} \binom{n}{\lambda-u+\mu} (a_1^{u} a_2^{\lambda_0} a_3^{\lambda-u+\mu}),$$

which suggests a change of notation with the definition

$$\{a_1^{\lambda} a_2^{\mu} a_3^{\nu}\} = \binom{n}{\lambda} \binom{n}{\mu} \binom{n}{\nu} (a_1^{\lambda} a_2^{\mu} a_3^{\nu}).$$

Our equation then becomes

$$(-)^{\lambda} \{a_1^{\lambda_0} a_2^{\lambda} a_3^{\mu}\} = \sum_v \binom{v}{\mu} \{a_1^{\lambda+\mu-v} a_2^{\lambda_0} a_3^{v}\}.$$

More generally, if the orders n_r are not necessarily equal, we obtain in the same way

$$\text{(III)} \qquad (-)^{\lambda_2}\{a_1{}^0 a_2{}^{\lambda_2} a_3{}^{\lambda_3} \dots a_\delta{}^{\lambda_\delta}\} = \Sigma\left[\prod_{r=3}^{\delta}\binom{v_r}{\lambda_r}\right]\{a_1{}^{w-\Sigma v} a_2{}^0 a_3{}^{v_3} \dots a_\delta{}^{v_\delta}\}.$$

7. THEOREM II. *The following relation connects seminvariant types of degree* 3:

$$(-)^{\gamma_3}\sum_{r=0}^{\gamma_1}\binom{\gamma_1+\gamma_2-r}{\gamma_2}\{a_1{}^0 a_2{}^r a_3{}^{w-r}\}+(-)^{\gamma_1}\sum_{r=0}^{\gamma_2}\binom{\gamma_2+\gamma_3-r}{\gamma_3}\{a_1{}^{w-r} a_2{}^0 a_3{}^r\}$$

$$+(-)^{\gamma_2}\sum_{r=0}^{\gamma_3}\binom{\gamma_3+\gamma_1-r}{\gamma_1}\{a_1{}^r a_2{}^{w-r} a_3{}^0\} = 0,$$

where $$\gamma_1+\gamma_2+\gamma_3 = w-1.$$

This is analogous to Stroh's equation*.

To prove this we use the equations

$$\{a_1{}^{w-r} a_2{}^0 a_3{}^r\} = (-)^{w-r}\Sigma\binom{w-s}{r}\{a_1{}^0 a_2{}^s a_3{}^{w-s}\},$$

$$\{a_1{}^r a_2{}^{w-r} a_3{}^0\} = (-)^r\Sigma\binom{s}{w-r}\{a_1{}^0 a_2{}^s a_3{}^{w-s}\}$$

to transform the last two sums, and we then consider the coefficient of $\{a_1{}^0 a_2{}^s a_3{}^{w-s}\}$.

When $s \leqslant \gamma_1$ the last sum provides no part of this coefficient. A sum is required of the form

$$\binom{\xi+\lambda}{\xi}-\eta\binom{\xi+\lambda-1}{\xi-1}+\binom{\eta}{2}\binom{\xi+\lambda-2}{\xi-2}-\dots+(-)^\xi\binom{\eta}{\xi}\binom{\lambda}{0}.$$

This may be regarded as a function of λ of order ξ; let us replace λ by $-\mu$, then $\binom{\xi+\lambda}{\xi}$ is

$$\frac{(\xi-\mu)(\xi-\mu-1)\dots(-\mu+1)}{\xi!} = (-)^\xi\binom{\mu-1}{\xi}.$$

The function becomes

$$(-)^\xi\left[\binom{\mu-1}{\xi}+\eta\binom{\mu-1}{\xi-1}+\dots+\binom{\eta}{\xi}\binom{\mu-1}{0}\right] = (-)^\xi\binom{\mu+\eta-1}{\xi},$$

---

* Grace and Young, *Algebra of invariants*, 64.

and the original sum is

$$(-)^{\xi}\binom{\eta-\lambda-1}{\xi}. \tag{IV}$$

Thus $$\sum_{r=0}^{\gamma_2}(-)^r\binom{w-s}{r}\binom{\gamma_2+\gamma_3-r}{\gamma_3}=(-)^{\gamma_2}\binom{w-s-\gamma_3-1}{\gamma_2},$$

and the total coefficients of the terms $s\leqslant\gamma_1$ are all zero.

When s lies between γ_1 and $\gamma_1+\gamma_2+1$, then

$$0\leqslant w-s-\gamma_3-1<\gamma_2,$$

and hence the coefficient obtained from the second sum is zero, while no such terms are obtained from the other two sums.

When $s>\gamma_1+\gamma_2$ it is necessary to consider the last sum; the coefficient of the s term is

$$\sum_{r=0}^{\gamma_3}(-)^{\gamma_2+r}\binom{s}{w-r}\binom{\gamma_3+\gamma_1-r}{\gamma_1},$$

the coefficient of x^{γ_1} in the expansion of

$$(-)^{w+\gamma_2}\Big[\{1-(1+x)\}^s(1+x)^{\gamma_3+\gamma_1-w}-(1+x)^{\gamma_1+\gamma_3-w}+s(1+x)^{\gamma_1+\gamma_3-w+1}$$
$$-\ldots+(-)^{\gamma_1+\gamma_3-w}\binom{s}{w-\gamma_1-\gamma_3-1}(1+x)^{-1}\Big],$$

and this is

$$(-)^{\gamma_3}\left[\binom{\gamma_2+\gamma_1}{\gamma_2}-\binom{s}{1}\binom{\gamma_2+\gamma_1-1}{\gamma_2-1}+\ldots+(-)^{\gamma_2}\binom{s}{\gamma_2}\binom{\gamma_1}{0}\right]$$
$$=(-)^{\gamma_2+\gamma_3}\binom{s-\gamma_1-1}{\gamma_2}.$$

In this case the coefficient provided by the second sum is the coefficient of x^{γ_3} in

$$(-)^{w+\gamma_1}[1-(1+x)^{-1}]^{w-s}(1+x)^{\gamma_2+\gamma_3}=(-)^{w+\gamma_1}x^{w-s}(1+x)^{\gamma_2+\gamma_3+s-w},$$

since the index of $1+x$ is always positive. This gives

$$(-)^{\gamma_2+\gamma_3+1}\binom{s-\gamma_1-1}{s-w+\gamma_3},$$

and once more the total coefficient is zero. Thus the identity is proved. A transformation by means of equation (III) gives the following variation of the result just proved, simply and directly; we give it as a corollary because it is sometimes useful.

COROLLARY.

$$(-)^{\gamma_3}\sum_{r=0}^{\gamma_1}\binom{\gamma_1+\gamma_2-r}{\gamma_2}\{a_1{}^0a_2{}^ra_3{}^{w-r}\}+(-)^{\gamma_1}\sum_{r=0}^{\gamma_2}\binom{\gamma_2+\gamma_3-r}{\gamma_3}\{a_1{}^{w-r}a_2{}^0a_3{}^r\}$$

$$+(-)^{\gamma_2+\gamma_3}\sum_{r=0}^{\gamma_3}\binom{\gamma_3+\gamma_2-r}{\gamma_2}\{a_1{}^0a_2{}^{w-r}a_3{}^r\}=0,$$

when $\gamma_1+\gamma_2+\gamma_3=w-1$.

8. It is to be observed that Theorem II may be applied to seminvariant types of any degree greater than 3. For by (III) we obtain in all cases

$$(-)^{\lambda_2}\{a_1{}^0a_2{}^{\lambda_2}a_3{}^{\lambda_3}\dots a_\delta{}^{\lambda_\delta}\}=\Sigma\binom{v}{\lambda_3}\{a_1{}^{\lambda_2+\lambda_3-v}a_2{}^0a_3{}^va_4{}^{\lambda_4}\dots a_\delta{}^{\lambda_\delta}\}+K,$$

where K represents terms in which some of the indices later than that of a_3 are increased but none are diminished. Thus, in particular, we may suppose Theorem II always applicable to the first three quantics, and the result is true except for forms in which the weight is shifted to later quantics. The theorem may be applied exactly as Stroh's series or Jordan's lemma to prove that for degree 3 we may express a seminvariant type in terms of the types

$$(a_1{}^0a_2{}^\lambda a_3{}^{w-\lambda}),\quad (a_1{}^{w-\lambda}a_2{}^0a_3{}^\lambda),\quad (a_1{}^\lambda a_2{}^{w-\lambda}a_3{}^0),$$

where $$w-\lambda\geqslant 2\lambda.$$

Further, when $w=3\lambda+1$,

$$(a_1{}^0a_2{}^\lambda a_3{}^{2\lambda+1})+(a_1{}^{2\lambda+1}a_2{}^0a_3{}^\lambda)+(a_1{}^\lambda a_2{}^{2\lambda+1}a_3{}^0)$$

is expressible in terms of forms in which the highest index is increased. This is true also of the differences

$$(a_1{}^0a_2{}^\lambda a_3{}^{2\lambda})-(a_1{}^{2\lambda}a_2{}^0a_3{}^\lambda),$$

and so on.

When we apply these results to the seminvariants of a single form, we see that the lowest index is, of course, zero; the second must be even or by (III) the seminvariant is reductible.

Let the second lowest index be 2λ and the third λ_3. Then the form is reductible according to either sequence when $\lambda_3<4\lambda$, or when $\lambda_3=4\lambda+1$.

We have seen (§ 2) that there is no relation between types whose tableaux have no overlapping, and in which the quantics are arranged according to the fixed sequence. There are then, in general, $w+1$ independent forms $\{a_1{}^0a_2{}^\lambda a_3{}^\mu\}$, and there are no relations other than those given by

Theorem II. And thus for seminvariants of degree 3 there are, in general, no others than those which we have obtained. In other words, they are really irreductible.

9. The equation (I) for seminvariant types is one in terms of which all other equations between types may be expressed. This leads to a remarkable result when the equation is expressed in terms of the new forms introduced in § 6, *i.e.*

$$\{a_1^{\lambda_1} a_2^{\lambda_2} \dots a_s^{\lambda_s}\} = \left[\prod_{r=1}^{s} \binom{n_r}{\lambda_r}\right] (a_1^{\lambda_1} a_2^{\lambda_2} \dots a_s^{\lambda_s}).$$

The equation (II) may now be written

$$\text{(V)} \qquad \Sigma \left(a_r \frac{\partial}{\partial a_r} a_r\right) \{a_1^{\lambda_1} a_2^{\lambda_2} \dots a_s^{\lambda_s}\} = 0;$$

the operator, of course, operates on the argument, and the numerical coefficients are brought outside. Thus the equation is

$$\Sigma(\lambda_r + 1)\{a_1^{\lambda_1} a_2^{\lambda_2} \dots a_r^{\lambda_r+1} \dots a_s^{\lambda_s}\} = 0.$$

It is to be observed that these equations are entirely independent of the order n_r of the forms. The only occurrence of the order is when the index $\lambda_r > n_r$, the coefficient $\binom{n_r}{\lambda_r}$ is zero, and the form $\{a_1^{\lambda_1} a_2^{\lambda_2} \dots\}$ is zero. Thus the relations between these types are independent of the order of the forms, so that we have

THEOREM III. *By the introduction of a numerical factor the relations between seminvariant forms may be made independent of the order of the forms and the same as for perpetuants. In particular, seminvariants of a single form are irreductible or reductible independently of its order, according to any sequence.*

10. Consider the seminvariant, of a single quantic,

$$\{a_1^0 a_2^{\lambda_2} \dots a_s^{\lambda_s}\},$$

where $\lambda_s \geqslant \lambda_{s-1} \geqslant \lambda_{s-2} \dots \geqslant \lambda_2$.

If the order n of this quantic is less than λ_s, this seminvariant vanishes owing to the factor $\binom{n}{\lambda_s}$; otherwise it is reductible or irreductible independently of the value of n. Let us take $n = \lambda_s$.

Then it may happen that

$$w = \lambda_2 + \lambda_3 + \ldots + \lambda_\delta > \tfrac{1}{2}\delta\lambda_\delta.$$

In this case, there is no such seminvariant for $n = \lambda_\delta$, so that the form is zero. This means that for any value of n this form can be expressed in terms of forms which have one index greater than λ_δ. Hence the form is reductible according to sequence (A), but not necessarily according to sequence (B).

It has been remarked that all *types* for which λ_1 is zero are linearly independent, and also that equations (III) enable us to express all other types in terms of these, then all our relations for reductibility may be derived from (III). In fact, they are the result of taking each letter in turn as the one with zero index and then using the fact that the letters are all equivalent.

It follows from this that, if it has been proved that

$$\{a_1^{\,0} a_2^{\lambda_2} \ldots a_r^{\lambda_r}\} \quad (r < \delta)$$

is reductible, then also must

$$\{a_1^{\,0} a_2^{\lambda_2} \ldots a_r^{\lambda_r} \ldots a_\delta^{\lambda_\delta}\}$$

be reductible according to sequence (A), the indices being in ascending order of magnitude. For the reduction of the form of degree r might have been obtained from equations (III) [or indeed from equations (V)], and, in either case, it is applicable to the form of degree δ, it being remembered that any increase in the later indices itself constitutes a reduction.

We may state then

THEOREM IV. *A seminvariant, of a single quantic,*

$$\{a_1^{\,0} a_2^{\lambda_2} \ldots a_\delta^{\lambda_\delta}\},$$

in which the indices are in ascending order of magnitude, is reductible according to sequence (A) *when*

$$\lambda_2 + \lambda_3 + \ldots + \lambda_r > \tfrac{1}{2} r \lambda_r,$$

where $r \leqslant \delta$.

It may be noticed that the consequence of putting $r = 3$ is the result already obtained, that for irreductibility $\lambda_3 \geqslant 2\lambda_2$.

11. The general identities proved in Q.S.A., II, 390–393, have already been referred to. They were proved for invariants, but the proof given is true also for seminvariants. The notation used differs from that used here in that the forms are defined by the numbers of letters of each set which lie in the upper row of the tableau instead of the lower row. To change the notation to that used here, each index λ must be changed to $n-\lambda$. Further, it is to be remembered that the result is given in terms of forms $(a_1^{\lambda_1} a_2^{\lambda_2} \dots a_\delta^{\lambda_\delta})$, or rather of the forms $(A_0^{\lambda_0} \dots A_n^{\lambda_n})$, and not of those denoted by $\{a_1^{\lambda_1} \dots a_\delta^{\lambda_\delta}\}$.

When the changes indicated are made we have an identity

$$\Pi \left\{x_\lambda + \mu x_{\lambda+1} + \binom{\mu}{2} x_{\lambda+2} + \dots + x_{\lambda+\mu}\right\}^{\beta_{\lambda,\mu}}$$

$$= (-)^{\varpi-\rho} \Pi \left\{x_\lambda + (n-\lambda-\mu) x_{\lambda+1} + \binom{n-\lambda-\mu}{2} x_{\lambda+2} + \dots + x_{n-\mu}\right\}^{\beta_{\lambda,\mu}},$$

where $\varpi - \rho = w - \Sigma \lambda \beta_{\lambda\mu}$, to which the following meaning is to be attached. Each side is expanded, but only those terms which are of weight w, when $x_0, x_1, \dots, x_n$ are regarded as the coefficients of a quantic, are retained. Then, if $\phi(\lambda_0, \lambda_1, \dots, \lambda_n)$ and $\psi(\lambda_0, \lambda_1, \dots, \lambda_n)$ are the coefficients of $x_0^{\lambda_0} x_1^{\lambda_1} \dots x_n^{\lambda_n}$ on the two sides, we have

$$\Sigma \phi(\lambda_0, \lambda_1, \dots, \lambda_n)(A_0^{\lambda_0} A_1^{\lambda_1} \dots A_n^{\lambda_n}) = \Sigma \psi(\lambda_0, \lambda_1, \dots, \lambda_n)(A_0^{\lambda_0} A_1^{\lambda_1} \dots A_n^{\lambda_n}).$$

The most useful results are obtained by choosing the indices such that $\lambda \beta_{\lambda,\mu}$ is always zero.

Let us write

$$X_\mu = x_0 + \mu x_1 m + \binom{\mu}{2} x_2 m^2 + \dots + x_\mu m^\mu.$$

The result states that the coefficient of m^w in the equation

$$\prod_\mu X_\mu^{\beta_{0,\mu}} = (-)^w \prod_\mu X_{n-\mu}^{\beta_{0,\mu}}$$

gives an identity between seminvariants.

Now Elliott* has proved that

$$\Pi (m^{-\mu} X_\mu)^{\beta_{0,\mu}} = e^{m^{-1}\Omega} \Pi x_\mu^{\beta_{0,\mu}},$$

where

$$\Omega = x_0 \frac{\partial}{\partial x_1} + 2x_1 \frac{\partial}{\partial x_2} + \dots.$$

* *Algebra of quantics*, 111.

Here we write m^{-1} for m in Elliott's result and use our special function for his general one.

Let $$\Sigma \mu\beta_{0,\mu} = \varpi, \quad \Sigma \beta_{0,\mu} = \delta,$$

$$n\delta - 2w = R.$$

Our equation becomes

$$m^{\varpi} e^{m^{-1}\Omega} \Pi x_{\mu}^{\beta_{0,\mu}} = (-)^{w} m^{n\delta-\varpi} e^{m^{-1}\Omega} \Pi x_{n-\mu}^{\beta_{0,\mu}}.$$

The result required is obtained from this by taking the coefficient of m^w on each side. On the left this is zero when $\varpi < w$, and it is $\Pi x_{\mu}^{\beta_{0,\mu}}$ when $\varpi = w$. Similarly it is zero on the right when $\varpi > n\delta - w$.

Let $\varpi = w + \rho$; then

$$n\delta - \varpi = w + R - \rho,$$

and the result of equating the coefficients of m^w gives

$$\text{(VI)} \quad \frac{1}{\rho!} \Omega^{\rho} (A_0^{\beta_{0,0}} A_1^{\beta_{0,1}} \dots A_n^{\beta_{0,n}}) = (-)^{w} \left(\frac{1}{(R-\rho)!}\right) \Omega^{R-\rho} (A_0^{\beta_{0,n}} A_1^{\beta_{0,n-1}} \dots A_n^{\beta_{0,0}}),$$

where R is the order of the seminvariant. Also we obtain

$$\text{(VII)} \quad \Omega^{\rho}(A_0^{\beta_{0,0}} A_1^{\beta_{0,1}} \dots A_n^{\beta_{0,n}}) = 0$$

when $\rho > R$.

12. For an invariant this gives

$$(A_0^{\lambda_0} A_1^{\lambda_1} \dots A_n^{\lambda_n}) = (-)^{w} (A_0^{\lambda_n} A_1^{\lambda_{n-1}} \dots A_n^{\lambda_0}),$$

a relation obtainable at once directly from the consideration that in the tableau for an invariant there is the same number w of letters in each of the two rows, and these rows may be interchanged, the result being multiplied by $(-)^w$.

It has been seen that, for an irreductible form, λ_1 is zero; it follows that, for an invariant, λ_{n-1} is also zero. Also, if the form is irreductible according to sequence (A), the first of the differences

$$\lambda_n - \lambda_0, \quad \lambda_{n-2} - \lambda_2, \quad \lambda_{n-3} - \lambda_3, \quad \dots$$

which does not vanish is positive; according to sequence (B) it will be negative.

Further, when all these differences vanish the form is zero unless w is even.

13. THEOREM V. *The seminvariant of order R,*

$$(A_0^{\lambda_0} A_2^{\lambda_2} A_3^{\lambda_3} \dots A_n^{\lambda_n}),$$

is always reductible, according to either sequence, when

$$\lambda_{n-1} > R.$$

From (VII)

$$\Omega^{R+1}(A_0^{\lambda_0} A_2^{\lambda_2} \dots A_{n-1}^{\lambda_{n-1}-R-1} A_n^{\lambda_n+R+1}) = 0,$$

and the result follows at once, for the form in question is the earliest term in the equation which appears.

THEOREM VI. *The seminvariant of order R,*

$$(A_0^{\lambda_0} A_2^{\lambda_2} \dots A_n^{\lambda_n}),$$

is always reductible, according to sequence (A), *when the first of the differences*

$$\lambda_0 - \lambda_n - R, \quad R - \lambda_{n-1}, \quad \lambda_2 - \lambda_{n-2}, \quad \dots$$

which does not vanish is positive; and, if all these differences vanish, when w is odd.

This follows at once from equation (VI),

$$(A_0^{\lambda_0} A_2^{\lambda_2} \dots A_n^{\lambda_n}) = (-)^w \frac{1}{R!} \Omega^R (A_0^{\lambda_n} A_1^{\lambda_{n-1}} \dots A_n^{\lambda_0}).$$

More generally, from

$$\frac{1}{\rho!} \Omega^\rho (A_0^{\lambda_0} A_2^{\lambda_2} \dots A_{n-1}^{\lambda_{n-1}-\rho} A_n^{\lambda_n+\rho}) = (-)^w \frac{1}{(R-\rho)!} \Omega^{R-\rho} (A_0^{\lambda_n+\rho} A_1^{\lambda_{n-1}-\rho} \dots A_n^{\lambda_0}),$$

we have

THEOREM VII. *The seminvariant of order R,*

$$(A_0^{\lambda_0} A_2^{\lambda_2} \dots A_n^{\lambda_n}),$$

is always reductible, according to sequence (A), *when the first of the differences*

$$\lambda_0 + \lambda_{n-1} - \lambda_n - R, \quad R - 2\lambda_{n-1}, \quad \lambda_2 - \lambda_{n-2}, \quad \dots$$

which does not vanish is positive; and also when w is odd and they all vanish.

14. In considering the reductibility of any form it will be usually written in the notation $\{a_0^{\lambda_1} a_2^{\lambda_2} \dots a_s^{\lambda}\}$, where here the indices are in

ascending order of magnitude and correspond to the suffixes in the notation used in the last few paragraphs. The irreductibility is ascertained first for the indices of lower magnitude. Thus, when λ_2 is odd, there is no need to go further; λ_1 must be zero. Then λ_3 is equal to $2\lambda_2$, or greater than $2\lambda_2+1$. For λ_4 we consider the form $\{a_1^{\lambda_1} a_2^{\lambda_2} a_3^{\lambda_3} a_4^{\lambda_4}\}$ as a seminvariant of degree 4 of the form of order $n = \lambda_4$. The order of this seminvariant is R_4, where

$$R_4 = 4\lambda_4 - 2(\lambda_2+\lambda_3+\lambda_4)$$

$$= 2(\lambda_4 - \lambda_2 - \lambda_3).$$

Thus $\lambda_4 \geqslant \lambda_2+\lambda_3$.

Then, writing $\lambda_2 = 2\lambda, \quad \lambda_3 = 4\lambda+\mu \quad (\mu = 0 \text{ or } > 1),$

we have $\lambda_4 = 6\lambda+\mu+\nu,$

and $R_4 = 2\nu.$

When $\lambda = 0$, it is seen from Theorem VII that the form is reductible when $\nu = 1$, for the differences are zero and w is odd. It will be seen later that, whatever λ may be, the form is reductible when $\nu = 1$; or no covariants of degree 4 and order 2 can exist, a fact easily demonstrable otherwise.

15. Another way of proceeding is as follows; operate on the seminvariant type

$$\{a_1^{\lambda_1} a_2^{\lambda} a_3^{\mu} a_4^{w-\lambda-\mu}\}$$

with a positive symmetric group G which contains all the n letters a_3, and $n-\rho$ of the letters a_4, the ρ letters omitted being in the second row of the tableau. To make matters clear we shall call B_r the positive symmetric group of the letters a_r, and consider the equation derived from

$$B_1 B_2 B_3 B_4 G \{a_1^{\lambda_1} a_2^{\lambda} a_3^{\mu} a_4^{w-\lambda-\mu}\}$$

by using G in the two different ways as in Q.S.A., II.

This yields the equation

$$\text{(VIII)} \quad \sum_u \binom{\rho+u}{\rho} \{a_1^{\lambda_1} a_2^{\lambda} a_3^{w-\rho-u-\lambda} a_4^{\rho+u}\}$$

$$= (-)^{w-\lambda-\rho} \sum_v \binom{\lambda+v}{\lambda} \{a_1^{w-\lambda-\rho-v} a_2^{\lambda-v} a_3^{\lambda_3} a_4^{\rho}\}.$$

When $\rho < \lambda$, this gives a reduction for the left-hand side according to

sequence (B); that is we have a reduction for

$$\Sigma \binom{x}{\rho} \{a_1{}^0 a_2{}^\lambda a_3^{w-\lambda-x} a_4{}^x\}$$

for the values 0, 1, 2, ..., $\lambda-1$ of ρ.

Hence we deduce a reduction for

$$\Sigma \binom{x-x_1}{\rho} \{a_1{}^0 a_2{}^\lambda a_3^{w-\lambda-x} a_4{}^x\},$$

where x_1 has any selected value.

Thus any λ consecutive terms may be expressed in terms of the rest and of earlier terms in the sequence (B). Let us choose them in the centre of the series; then, when μ_3 and μ_4 are the indices of a_3 and a_4, it is possible to express all the forms in terms of those for which the difference

$$\mu_3 \sim \mu_4 \geqslant \lambda.$$

Thus all seminvariant types of degree 4 may be expressed in terms of those of the form

$$\{a_1{}^0 a_2{}^\lambda a_3^{2\lambda+\xi} a_4^{3\lambda+\xi+\eta}\},$$

where the identity of the letters is not fixed.

Exactly the same process may be applied to seminvariant types of degree δ, where the quantics $a_{\delta-1}$, a_δ of highest index are selected in place of a_3 and a_4 in the case of degree 4, and are treated in the same way. The result may be stated thus:

THEOREM VIII. *All seminvariant types of degree δ can be expressed in terms of types of the form*

$$\{a_1{}^0 a_2{}^\lambda a_3^{2\lambda+\mu_3} a_4^{3\lambda+\mu_4} \dots a_\delta^{(\delta-1)\lambda+\mu_\delta}\},$$

where

$$\mu_\delta \geqslant \mu_{\delta-1} \geqslant \mu_{\delta-2} \dots \geqslant \mu_3 \geqslant 0.$$

Thus, when using the sequence (B), it may always be asserted that the second index is equal to or less than $w \Big/ \binom{\delta}{2}$.

III. *Relation of these forms to seminvariants expressed otherwise.*

16. Consider the representation of a gradient of a single binary n-ic in terms of the substitutional forms. The general gradient is

$$X = A_0^{\alpha_1} A_1^{\alpha_1} \dots A_n^{\alpha_n},$$

the weight w will be supposed to be not greater than $\frac{1}{2}n\delta$, where δ is the degree.

The factor $A_r^{\alpha_r}$ will be replaced by $n\alpha_r$ symbolical letters, which appear for substitutional purposes as

$$ {}_ra_{1,1}\,{}_ra_{1,2}\cdots{}_ra_{1,n};\quad {}_ra_{2,1}\cdots{}_ra_{2,n};\quad \ldots;\quad \ldots{}_ra_{\alpha_r,n}. $$

A symbolical letter γ appears usually in two ways, γ_1 and γ_2; for convenience they will here be written γ and γ'. A permutation $(\gamma\delta)$ will leave $\gamma\delta$ and $\gamma'\delta'$ unchanged, but it will interchange $\gamma\delta'$ and $\gamma'\delta$.

To express one factor A_r we have a product

$$ {}_ra'_{s,1}\,{}_ra'_{s,2}\cdots{}_ra'_{s,r}\,{}_ra_{s,r+1}\cdots{}_ra_{s,n}. $$

Then X, when written symbolically, is unchanged by any permutation of two of the undashed letters and also by any permutation of two of the dashed letters. We may therefore write it in the form

$$ \frac{1}{(n\delta-w)!\,w!}\,G_{n\delta-w}\,G_w\,X, $$

where $G_{n\delta-w}$ is the positive symmetric group of all the undashed letters, and G_w is the positive symmetric group of all the dashed letters. Using the identity

$$ X=\Sigma\,T_{\alpha_1\alpha_2\ldots\alpha_h}X, $$

where $T_{\alpha_1\alpha_2\ldots\alpha_h}$ is the substitutional expression for the $n\delta$ letters, we notice that

$$ TX=0, $$

unless $h\leqslant 2$, and $\alpha_1\geqslant n\delta-w$. The case of chief interest is

$$ T_{n\delta-w,\,w}X. $$

Here $G_{n\delta-w}G_w$ is actually a P_r formed from a tableau of this set; and the sequence of letters can always be chosen so that this P_r is standard. Since

$$ T=\Sigma\,P_sN_sM_s $$

and every $P_sN_sM_sP_r$ is zero unless $r=s$, and in this case (Q.S.A., III, 264)

$$ P_rN_rM_rP_r=P_rN_rP_r, $$

it follows that

$$ T_{n\delta-w,\,w}X=P_rN_rX. $$

That is, in the expression of a gradient X in terms of the substitutional

forms, the term arising from that T which corresponds to its weight is the seminvariant (X) defined in our symbols by the gradient itself.

The argument is stated for binary forms, but every word of it applies to forms with any number of variables; there is, consequently, no need to repeat it for higher forms, and, moreover, the expression would become too cumbersome. The only difference arises from the fact that for q-ary forms there are $q-1$ weights, and these in a "seminvariant" are equal to the number of letters in each row below the first. It is from this point of view preferable to speak of q weights, and call $n\delta - w$ the first weight in the binary case. We may then state

THEOREM IX. *In the expression of a gradient X of a single quantic in any number of variables in terms of the substitutional forms, the weights of X being in descending order of magnitude, the term arising from that T which corresponds to the weights of X is the seminvariant (X) defined in substitutional symbols by the gradient itself. The terms corresponding to all later T's are zero.*

17. It has been shown that, when X represents a gradient of weight not exceeding $\frac{1}{2}n\delta$, the seminvariants (X) of a single binary form satisfy a system of linear equations which may be expressed in the form (§ 4)

$$O \,.\, A_r / A_{r+1}(X) = 0,$$

it being understood that A_{r+1} is a factor of X. Thus, in the same notation,

$$O\Omega(X) = 0. \tag{IX}$$

Consider any seminvariant $S = \Sigma \lambda_r X_r$; when we express it in terms of these substitutional forms, we have

$$S = \Sigma \lambda_r (X_r). \tag{X}$$

Elliott* has shown that all seminvariants can be obtained by a differential operator from gradients. He gives two equivalent forms for the seminvariant derived from the gradient X, one of which is

$$\left\{1 - \frac{1}{1\,(\eta+2)}\, O\Omega + \frac{1}{1\,.\,2\,(\eta+2)(\eta+3)}\, O^2\Omega^2 - \ldots\right\} X,$$

where η is the order of the covariant, *i.e.* $n\delta - 2w$ in the notation here used.

It follows at once from (IX) and (X) that this seminvariant is (X).

---

* *Algebra of quantics* (1913), 224-225.

THEOREM X. *The seminvariant* (X) *of order* R *of a single binary form is*

$$\left\{1-\frac{1}{1\,(R+2)}\,O\Omega+\frac{1}{1\,.\,2\,(R+2)(R+3)}\,O^2\,\Omega^2-\ldots\right\}X.$$

18. The full expression for a gradient X in terms of the substitutional forms may be obtained directly by operation with the various T's. When we operate with $T_{n\delta-\varpi,\,\varpi}$, where ϖ is less than the weight w of the gradient, the result will be a sum of forms (Y) which are seminvariant forms for that particular weight. The meaning, of course, is that here we are dealing not with the seminvariant or first term of the covariant, but with a later term, in fact, the $(w-\varpi)$-th term. When (Y) is regarded as a seminvariant, then in the expression for X it must be replaced by

$$\frac{1}{(w-\varpi)\,!}\,O^{w-\varpi}(Y).$$

The expression for X may also be obtained from Theorem X as follows. It will be convenient to use the notation $(\Omega^r X)$ for the sum of the seminvariants given by the various gradients obtained by the operation, each multiplied by the corresponding numerical coefficient. By $O^s[(\Omega^r X)]$ will be meant simply the result of operating with O^s on this seminvariant.

Then, by Theorem X,

$$(\Omega X)=\left\{1-\frac{1}{1\,(R+4)}\,O\Omega+\frac{1}{1\,.\,2\,(R+4)(R+5)}\,O^2\,\Omega^2-\ldots\right\}\Omega X$$

and

$$\text{(XI)}\qquad (\Omega^s X)=\left\{1+\sum_{t=1}^{\infty}(-)^t\frac{1}{t!\,(R+2s+t+1)_t}\,O^t\,\Omega^t\right\}\Omega^s X.$$

We shall prove that

$$(X)+\frac{1}{1\,(R+2)}\,O[(\Omega X)]+\frac{1}{2!\,(R+4)_2}\,O^2[(\Omega^2 X)]+\ldots+\frac{1}{(s-1)!\,(R+2s-2)_{s-1}}\,O^{s-1}[(\Omega^{s-1}X)]$$

$$=X-(-)^s\sum_{r=s}^{\infty}(-)^r\frac{\binom{r-1}{s-1}}{r!\,(R+r+s)_r}\,O^r\,\Omega^r X.$$

When $s=1$ this is Theorem X. Let us assume it to be true for any

given s and use the value of $(\Omega^s X)$ in terms of $\Omega^s X$ given by (XI). We introduce the additional term on the left

$$\frac{1}{s!\,(R+2s)_s}\,O^s[(\Omega^s X)].$$

On the right the coefficient of

$$O^r\,\Omega^r X$$

becomes

$$(-)^{r-s}\frac{1}{s!\,(R+2s)_s}\,\frac{1}{(r-s)!\,(R+r+s+1)_{r-s}}-(-)^{r+s}\,\frac{\binom{r-1}{s-1}}{r!\,(R+r+s)_r}$$

$$=(-)^{r+s}\,\frac{\binom{r-1}{s}}{r!\,(R+r+s+1)_r}.$$

Thus our equation is true for $s+1$ if it is true for s, and hence it is always true. Hence, increasing s, we find

THEOREM XI. *The expression of a gradient X, for which $n\delta-2w=R$ is not negative, in terms of substitutional forms or of terms of covariants is*

$$X=\sum_{s=0}^{\infty}\frac{1}{s!\,(R+2s)_s}\,O^s[(\Omega^s X)].$$

19. It is useful to remember that any seminvariant given by a substitutional tableau F may be expressed in the form

$$(AB)^w\,A_1^{n\delta-2w}.$$

The letters A stand for all the letters in the upper row of F, and the letters B for those in the lower row. The actual expression of the seminvariant may then be obtained by repeated polarization with Aronhold operators, which replace the symbols A, B by symbols which directly represent the ground forms.

For actual calculation this method is too cumbersome to attempt. Its importance lies in the fact that it is theoretically possible; and hence the use of single symbols A, B and the form of the seminvariant written above is justifiable.

20. Let (X) be any seminvariant of order R of one or more binary forms; and let c_x^m be any other binary form. In the first place, let

$m = 1$, and consider the transvectant

$$K = \big((X),\, c_x\big)^1.$$

Here (X) is used for the covariant instead of the seminvariant.

As in § 2, $$(X) = GPNK,$$

where PN is given by a tableau F; this we shall write

$$a_1 a_2 \dots a_w \dots a_{w+R}$$

$$b_1 b_2 \dots b_w.$$

Then, with the numerical coefficient given in Q.S.A., IV, § 1, we have

$$(X) = \frac{f_{w+R,\,w}}{(2w+R)!}\{a_1 \dots a_{w+R}\}\{b_1 \dots b_w\}(a_1 b_1)(a_2 b_2)\dots(a_w b_w)\, a_{w+1_x} \dots a_{w+R_x},$$

where $$f_{w+R,\,R} = \frac{(2w+R)!\ (R+1)}{(w+R+1)!\ w!};$$

and

$$K = \frac{f_{w+R,\,w}}{(2w+R)!}\{a_1 \dots a_{w+R}\}\{b_1 \dots b_w\}(a_1 b_1)\dots(a_w b_w)(a_{w+1} c)\, a_{w+2_x} \dots a_{w+R_x}$$

$$= \frac{f_{w+R,\,w}}{(2w+R)!}\frac{1}{w+1}\{a_1 \dots a_{w+R}\}\{b_1 \dots b_w c\}$$

$$\times (a_1 b_1)\dots(a_w b_w)(a_{w+1} c)\, a_{w+2_x} \dots a_{w+R_x}$$

$$= \frac{R+1}{R}(XC_1).$$

For $$\frac{f_{w+R,\,w+1}}{(2w+R+1)!} = \frac{R}{(R+1)(w+1)}\,\frac{f_{w+R,\,w}}{(2w+R)!},$$

and C_1 stands for the coefficient of the quantic

$$c_x = (C_0,\ C_1 \mathbin{\text{X}} x_1,\ x_2)^1.$$

The order of (XC_1) is $R-1$, and the first transvectant of this with another linear form d_x is

$$(K,\, d_x)^1 = \frac{R}{R-1}\,\frac{R+1}{R}\,(XC_1D_1),$$

and hence the second transvectant of (X) with a quadratic $c_x{}^2$ is

$$\frac{R+1}{R-1}(XC_2).$$

Proceeding thus we see that the n-th transvectant of (X) with ${c_x}^n$ is

$$\frac{R+1}{R-n+1}(XC_n).$$

Consider next the product $(X)c_x$, or the seminvariant $(X)C_0$, where the order of the C quantic is one. This is a seminvariant of order $R+1$, and therefore expressible in terms of seminvariants $T_{w+R+1,w}$. In Q.S.A., IV, Theorem III, the result of adding one more letter to a PN was obtained; since we are here concerned only with the terms of a particular T in that result (the other terms being necessarily zero), this gives us (using a notation which exhibits the tableau)

$$\begin{Bmatrix} a_1 \dots\dots\dots a_{w+R} \\ b_1 \dots b_w \end{Bmatrix} = \left[1+\frac{1}{R+2}\Sigma(cb)\right]\begin{Bmatrix} a_1 \dots\dots\dots a_{w+R}\, c \\ b_1 \dots b_w \end{Bmatrix},$$

whence
$$(X)C_0 = (XC_0)+\frac{1}{R+2}(C_1\Omega X).$$

Then
$$(XC_n)D_0 = (XC_nD_0)+\frac{1}{R-n+2}(D_1\Omega XC_n).$$

The Ω here contains the term

$$nC_{n-1}\frac{\partial}{\partial C_n},$$

and hence

$$(XC_n)D_0 = (XC_nD_0)+\frac{n}{R-n+2}(XC_{n-1}D_1)+\frac{1}{R-n+2}(C_nD_1\Omega X).$$

And hence

$$\begin{aligned}\left((X), c_x^{n+1}\right)^n &= \left((X), {c_x}^n\right)^n c_x \\ &= \frac{(R+2)_2}{(R-n+2)_2}(XC_n)+\frac{R+1}{(R-n+2)_2}(C_{n+1}\Omega X).\end{aligned}$$

Proceeding thus we obtain

THEOREM XII. *If the covariant form (X) is of order R, then the transvectant*

$$\left((X), {C_x}^n\right)^\mu = \sum_{r=0}^{n-\mu}(R+1)\binom{n-\mu}{r}\frac{(R+n-\mu+1)_{n-\mu-r}}{(R+n-2\mu+1)_{n-\mu+1}}(C_{\mu+r}\Omega^r X).$$

The truth of this result has just been proved for $n=\mu$, and for $n=\mu+1$. We assume it to be true for $n-1$. By means of the process just

used one more symbolical c can be introduced, *i.e.* the order of the C quantic can be raised from $n-1$ to n. Then the coefficient of $(C_{\mu+r}\,\Omega^r X)$ on the right is

$$(R+1)\binom{n-\mu-1}{r}\frac{(R+n-\mu)_{n-\mu-r-1}}{(R+n-2\mu)_{n-\mu}}\left\{1+\frac{\mu+r}{R+n-2\mu+1}\right\}$$

$$+(R+1)\binom{n-\mu-1}{r-1}\frac{(R+n-\mu)_{n-\mu-r}}{(R+n-2\mu)_{n-\mu}}\frac{1}{R+n-2\mu+1},$$

which reduces to the coefficient given. Thus Theorem XII is proved by induction.

21. The last theorem may now be inverted with the result:

THEOREM XIII. *If (X) is a covariant form of order R, and C_μ represents a coefficient of a binary form of order n, then*

$$(XC_\mu)=\sum_{r=0}^{n-\mu}(-)^r\binom{n-\mu}{r}\frac{(R+n-2\mu+1)_{n-\mu-r+1}}{(R+n-\mu+r+1)_{n-\mu+1}}\left((\Omega^r X),\,c_x^{\,n}\right)^{\mu+r}.$$

To prove this we may put in the value of $\left((\Omega^r X),\,C_x^{\,n}\right)^{\mu+r}$ given by Theorem XII, and evaluate the coefficient of $(C_{\mu+s}\,\Omega^s X)$ in the sum on the right.

This is

$$\sum_{r=0}^{s}(-)^r\binom{n-\mu}{r}\frac{(R+n-2\mu+1)_{n-\mu-r+1}}{(R+n-\mu+r+1)_{n-\mu+1}}$$

$$\times(R+2r+1)\binom{n-\mu-r}{s-r}\frac{(R+n+r-\mu+1)_{n-\mu-s}}{(R+n-2\mu+1)_{n-\mu-r+1}}$$

$$=\binom{n-\mu}{s}\Sigma\,(-)^r\binom{s}{r}(R+2r+1)\frac{(R+r)!}{(R+s+r+1)!}$$

$$=\binom{n-\mu}{s}\frac{(R+1)!\;s!}{(R+2s+1)!}\Sigma\,(-)^r\binom{R+2s+1}{s-r}\left[\binom{R+r+1}{R+1}+\binom{R+r}{R+1}\right].$$

In (IV), § 7, these sums were evaluated. We write

$$\Sigma\,(-)^r\binom{R+2s+1}{s-r}\binom{R+r+1}{R+1}=\Sigma(-)^{s-t}\binom{R+2s+1}{t}\binom{R+s+1-t}{R+1},$$

and, putting $\eta = R+2s+1$, $\xi = s$, $\lambda = R+1$, we obtain for the sum

$$\binom{2s-1}{s}.$$

Similarly

$$\Sigma(-)^{s-t}\binom{R+2s+1}{t}\binom{R+s-t}{R+1} = -\binom{2s-1}{s-1}.$$

Thus the coefficient required is zero provided that $s \geqslant 1$; when $s = 0$, the coefficient is obviously unity. Thus the theorem is true.

22. Consider now a gradient X, of a single binary form f, for which $n\delta - 2w = R \geqslant 0$. Let A_μ be the coefficient of greatest weight which it contains; and let

$$X = YA_\mu.$$

Then $$(X) = \lambda_0\big((Y), f\big)^\mu + \lambda_1\big((\Omega Y), f\big)^{\mu+1} + \ldots,$$

where λ_0 is not zero.

Every term, except the first, can be expressed in forms (X'), where X' is a gradient which contains a factor A_ν, where $\nu > \mu$. Thus (X) differs from a non-zero multiple of $\big((Y), f\big)^\mu$ by earlier forms according to sequence (A). This may be generalized at once. Let

$$X = A_0 A_{\mu_2} \ldots A_{\mu_s}, \quad \mu_2 \leqslant \mu_3 \ldots \leqslant \mu_s;$$

then X differs from a non-zero multiple of the continued transvectant

$$(\ldots((f, f)^{\mu_2} f)^{\mu_3} \ldots f)^{\mu_s}$$

by earlier forms according to sequence (A). Thus the representation of covariants by continued transvectants of this nature is parallel to the representation by substitutional forms. A reduction in one case that is a representation in terms of earlier forms means also a reduction in the other case.

It should be pointed out that the object of reduction here is exactly the reverse to that hitherto used in the discussion of transvectants. It has always been considered a reduction when a transvectant is expressed in terms of lower transvectants of forms obtained from the original forms by convolution; here it is a reduction when the transvectant is expressed in terms of higher transvectants of forms obtained from the original forms by *devolution* (the reverse of convolution).

THEOREM XIV. *The covariant* (XA_μ) *of a binary form differs from a non-zero multiple of the transvectant* $\left((X), f\right)^\mu$ *by higher transvectants of* f *with forms obtained from* (X) *by devolution.*

IV. *Covariant types of degree* 4.

23. The covariant types will be written

$$\{a^0 b^\lambda c^\mu d^\nu\},$$

where the indices are arranged in general in ascending order, and each indicates the number of letters of the corresponding quantic which lie in the second row of the tableau; the form is that of § 6 with the appropriate numerical coefficients attached.

Then, by § 6,

$$\{a^0 b^\lambda c^\mu d^\nu\} - (-)^\lambda \{b^0 a^\lambda c^\mu d^\nu\}$$

is reductible according to either sequence. Thus for λ even the form belongs to $\{ab\}$ and for λ odd it belongs to $\{ab\}'$ or $\left\{\begin{matrix} a \\ b \end{matrix}\right\}$.

Theorem II, § 7, provides a reduction according to either sequence when $\mu < 2\lambda$.

When $\mu = 2\lambda$, it gives a reduction for

$$\{a^0 b^\lambda c^{2\lambda} d^\nu\} - \{b^0 c^\lambda a^{2\lambda} d^\nu\}.$$

Hence, when λ is even,

$$\{a^0 b^\lambda c^{2\lambda} d^\nu\} - \{abc\}\{a^0 b^\lambda c^{2\lambda} d^\nu\}$$

is reductible, where $\{F\}$ is the (PN) corresponding to the tableau F with the proper numerical coefficient. And, when λ is odd,

$$\left[1 - \left\{\begin{matrix} a \\ b \\ c \end{matrix}\right\}\right] \{a^0 b^\lambda c^{2\lambda} d^\nu\}$$

is reductible. When $\mu = 2\lambda + 1$,

$$\{a^0 b^\lambda c^{2\lambda+1} d^\nu\} + \{b^0 c^\lambda a^{2\lambda+1} d^\nu\} + \{c^0 a^\lambda b^{2\lambda+1} d^\nu\}$$

is reductible. Thus both

$$\{abc\}\{a^0 b^\lambda c^{2\lambda+1} d^\nu\}, \qquad \left\{\begin{matrix} a \\ b \\ c \end{matrix}\right\} \{a^0 b^\lambda c^{2\lambda+1} d^\nu\}$$

are always reductible. And the form is of the nature $\begin{pmatrix} ab \\ c \end{pmatrix}$ when λ is even, and of the nature $\begin{pmatrix} ab \\ c \end{pmatrix}'$ when λ is odd, the dash indicating that the sequence PN is changed to NP. Now the number of forms of the nature $\{F\}$, where F belongs to T, is the order f of the T matrix, *i.e.* the number of independent forms into which $\{F\}$ may be changed by permutations operating on the left. Thus there is one form $\{a^0 b^\lambda c^{2\lambda}\}$, two forms $\{a^0 b^\lambda c^{2\lambda+1}\}$, and three forms $\{a^0 b^\lambda c^\mu\}$ for $\mu > 2\lambda+1$. These latter are, for λ even, one T_3 and two $T_{2,1}$; and, for λ odd, one T_{1^3} and two $T_{2,1}$.

24. In § 15 it was shown that all seminvariant types $\{a^0 b^\lambda c^\mu d^\nu\}$ can be expressed in terms of those for which

$$\mu \geqslant 2\lambda, \quad \nu \geqslant \mu+\lambda.$$

We have to consider specially the cases

$$\nu-\mu-\lambda = 0,\ 1.$$

As in § 15, equation (VIII) yields the result that, for $\rho < \lambda$,

$$\Sigma \binom{x-x_1}{\rho} \{a^0 b^\lambda c^x d^{\mu+\nu-x}\}$$

is expressible in terms of forms whose two lowest indices are $0, \lambda'$, where $\lambda' < \lambda$. Also

$$\Sigma \binom{x}{\lambda} \{a^0 b^\lambda c^x d^{\mu+\nu-x}\} = (-)^{\lambda+\mu+\nu} \Sigma \binom{y}{\lambda} \{c^0 d^\lambda a^y b^{\mu+\nu-y}\}.$$

Hence

$$\Sigma \binom{x-x_1}{\lambda} (a^0 b^\lambda c^{\mu+\nu-x} d^x) - (-)^{\lambda+\mu+\nu} \Sigma \binom{y-y_1}{\lambda} (c^0 d^\lambda a^{\mu+\nu-y} b^y)$$

is expressible in terms of forms whose two lowest indices are $0, \lambda' (< \lambda)$; and here x_1 and y_1 may be chosen at will.

(i) Let $\mu+\nu = 5\lambda+2\xi$; we choose $x_1 = y_1 = 2\lambda+\xi$. We obtain a reduction which may be written in the form

$$\{a^0 b^\lambda c^{2\lambda+\xi} d^{3\lambda+\xi}\} - \{c^0 d^\lambda a^{2\lambda+\xi} b^{3\lambda+\xi}\} \equiv 0.$$

And, since

$$\{a^0 b^\lambda c^{2\lambda+\xi} d^{3\lambda+\xi}\} - (-)^\lambda \{b^0 a^\lambda c^{2\lambda+\xi} d^{3\lambda+\xi}\} \equiv 0,$$

it follows that

$$\{a^0 b^\lambda c^{2\lambda+\xi} d^{3\lambda+\xi}\} - (-)^\lambda \{a^0 b^\lambda d^{2\lambda+\xi} c^{3\lambda+\xi}\} \equiv 0.$$

Thus there are three irreductible forms, $(\xi > 1)$, $\{a^0 b^\lambda c^{2\lambda+\xi} d^{3\lambda+\xi}\}$. When λ is even, they are one $\{abcd\}$ and two $\begin{Bmatrix} ab \\ cd \end{Bmatrix}$. When λ is odd, they are two $T_{2,2}$ and one T_{1^4}. When $\xi = 0$, there is one form, either T_4 or T_{1^4} according as λ is even or odd.

When $\xi = 1$, we have an additional equation of the form

$$\{(abc)\} = 0.$$

In this case there are only two forms which always belong to $T_{2,2}$.

(ii) Let $\mu+\nu = 5\lambda+2\xi+1$; we again choose $x_1 = y_1 = 2\lambda+\xi$. Then, writing X for $\{a^0 b^\lambda c^{2\lambda+\xi} d^{3\lambda+\xi+1}\}$, we have

$$[1+(-)^\lambda (cd)+(ac)(bd)+(-)^\lambda (acbd)]\, X \equiv 0;$$

whence
$$[1+(ab)(cd)+(ac)(bd)+(ad)(bc)]\, X \equiv 0,$$

or
$$[T_4+T_{2,2}+T_{1^4}]\, X \equiv 0.$$

There are nine independent forms for $\xi > 1$. When λ is even, there are six forms $T_{3,1}$ and three forms $T_{2,1^2}$, when λ is odd, there are three forms $T_{3,1}$ and six forms $T_{2,1^2}$.

When $\xi = 0$, there are three forms; for λ even they are of the nature $\begin{Bmatrix} abc \\ d \end{Bmatrix}$, and for λ odd they are of the nature $\begin{Bmatrix} abc \\ d \end{Bmatrix}'$. When $\xi = 1$, there is an additional equation

$$\{(abc)\}\, X \equiv 0;$$

there are always six forms, three $T_{3,1}$ and three $T_{2,1^2}$.

25. For a single quantic, that is a form $T_4 X$, the irreductible forms are

$$\{a^0 b^{2\lambda} c^{4\lambda+\xi} d^{6\lambda+\xi+\eta}\},$$

where $\xi = 0$ or $\xi > 1$ and $\eta = 0$ or $\eta > 1$.

Indeed, the whole result may be stated thus:

THEOREM XV. *The irreductible forms of degree* 4 *are included in*

$$X = \{a^0 b^\lambda c^{2\lambda+\xi} d^{3\lambda+\xi+\eta}\},$$

where the conditions are as follows:

$$T_4X \text{ and } T_{1^4}X; \quad \xi = 0 \text{ or } \xi > 1, \quad \eta = 0 \text{ or } \eta > 1,$$

and λ *even in the first and odd in the second case.*

$$T_{2,2}X; \quad \xi > 0, \quad \eta = 0 \quad \text{or} \quad \eta > 1.$$

$$T_{3,1}X \quad \text{and} \quad T_{2,1^2}X;$$

there is one set of forms $\xi > 0$, $\eta \geqslant 0$ *with* $\lambda \geqslant 0$; *and there is an additional set* $\xi = 0$ *or* $\xi > 1$, $\eta > 0$, *with* λ *even in the first and odd in the second case.*

The generating function for T_4X is

$$\frac{1}{1-x^{12}}\,\frac{1+x^6}{1-x^4}\,\frac{1+x^3}{1-x^2} = \frac{1}{(1-x^2)(1-x^3)(1-x^4)},$$

for $T_{3,1}X$, $T_{2,2}X$, $T_{2,1^2}X$, $T_{1^4}X$ the generating functions are

$$\frac{x}{(1-x)(1-x^2)(1-x^4)}, \quad \frac{x}{(1-x^2)^2(1-x^3)}, \quad \frac{x^3}{(1-x)(1-x^2)(1-x^4)},$$

$$\frac{x^6}{(1-x^2)(1-x^3)(1-x^4)}.$$

Multiplying these functions by the numbers

$$f_4, \quad f_{3,1}, \quad f_{2,2}, \quad f_{2,1^2}, \quad f_{1^4},$$

and adding, we obtain $(1-x)^{-4}$, the generating function for all forms of degree 4, and thus verify the accuracy of the work.

V. *The cubic and quartic.*

26. This section is mainly inserted by way of illustration. For the cubic the forms $(A_0^{\lambda_0} A_2^{\lambda_2} A_3^{\lambda_3})$ have to be considered; the order of this covariant is

$$R = 3(\lambda_0 - \lambda_3) - \lambda_2.$$

(It was shown in § 5 that no gradients containing A_1 need be considered.)

By Theorem VII the form is reductible when

$$2(\lambda_2 + \lambda_3 - \lambda_0)$$

is positive, also when it is zero and the weight is odd. Thus the only forms to be considered are, for w even,

$$(A_0^{\mu_0} X_1^{\mu_1} X_2^{\mu_2}),$$

where $X_1 = A_0 A_2$, $X_2 = A_0^2 A_3^2$; and, for w odd,

$$(A_0^{\mu_0} X_1^{\mu_1} X_2^{\mu_2} X_3),$$

where $X_3 = A_0^2 A_3$.

Then (A_0) is the cubic, $(A_0 A_2)$ the Hessian, $(A_0^2 A_3)$ the cubic covariant, $(A_0^2 A_3^2)$ the invariant. The products $(A_0)^{\mu_0}(X_1)^{\mu_1}(X_2)^{\mu_2}$ are unrelated (a relation would give A_3 in terms of the other coefficients). This system is coextensive with $(A_0^{\mu_0} X_1^{\mu_1} X_2^{\mu_2})$, and hence is equivalent to it. The consideration that there are only two irreductible forms of weight 6 and degree 6, viz. $(A_0^4 A_3^2)$, $(A_0^3 A_2^3)$, leads to the syzygy

$$(A_0^2 A_3)^2 - (A_0)^2 (A_0^2 A_3^2) = \lambda (A_0 A_2)^3.$$

One is tempted to identify (XY) with $(X)(Y)$, but, though there is a connection, this is not a simple one in general. The connection, however, becomes simple and direct when the substitutional forms are transformed from the natural to the seminormal or the orthogonal forms by the transformations introduced in Q.S.A., VI; this, however, must be reserved for a later communication, but we shall return to the subject here in § 28 and in Section XII.

27. The general form for the quartic is

$$(A_0^{\lambda_0} A_2^{\lambda_2} A_3^{\lambda_3} A_4^{\lambda_4}),$$

of which the order is

$$R = 4\lambda_0 - 2\lambda_3 - 4\lambda_4.$$

By Theorem VII this is reductible when $\lambda_0 < \lambda_3 + \lambda_4$, and also when $\lambda_0 = \lambda_3 + \lambda_4$ and w is odd.

In § 10 it was seen that this form is irreductible only when $(A_0^{\lambda_0} A_2^{\lambda_2} A_3^{\lambda_3})$ is irreductible for the cubic. Thus, for w even, the irreductible forms are

$$(A_0^{\mu_0} X_1^{\mu_1} X_2^{\mu_2} X_4^{\mu_4} X_5^{\mu_5}),$$

where X_1, X_2, X_3 have the same meanings as for the cubic and

$$X_4 = A_0 A_2 A_4, \quad X_5 = A_0 A_4;$$

and, for w odd, $$(A_0^{\mu_0} X_1^{\mu_1} X_2^{\mu_2} X_4^{\mu_4} X_5^{\mu_5} X_3).$$

There is a relation between the gradients, for

$$X_1 X_5 = A_0 X_4.$$

There are only two irreductible forms of degree 4 and weight 6, viz. $(A_0^2 A_2 A_4)$ and $(A_0^2 A_3^2)$. There are also two products of forms already obtained which cannot be equal, viz. $(A_0)(A_0 A_2 A_4)$ and $(A_0 A_2)(A_0 A_4)$, so that $(A_0^2 A_3^2)$ is reducible. We see that it will be reducible for any form of order greater than 3. The syzygy obtained for the cubic will also exist for forms of order greater than 3, but its expression will be modified by expressing $(A_0^2 A_3^2)$ in terms of products of irreducible forms.

Thus the quartic forms will be

$$(A_0), \quad (A_0 A_2), \quad (A_0 A_4), \quad (A_0^2 A_3), \quad (A_0 A_2 A_4).$$

VI. *Sequences.*

28. In § 5 the seminvariant forms of a single quantic were arranged according to one or other of two sequences (A) and (B). These sequences were really sequences of gradients. Such an arrangement may be applied to the terms of a seminvariant when written out in full. According to a given sequence law, every seminvariant has a gradient which is its first term or leading gradient. Not every gradient could occupy the position of leading gradient; in fact, for any given weight and degree there must be the same number of possible leading gradients as there are linearly independent seminvariants. Thus a notation could be devised by which, for every possible leading gradient X, the seminvariant (X) is a definite seminvariant with X for leading gradient; then all other seminvariants would be linear functions of these.

The advantage of such a notation would be that a product $(X)(Y)$ would differ from (XY) by seminvariants having a later leading gradient. The main difficulty in adopting such a notation is that of determining what gradients can be leading gradients.

In all cases so far examined (and a similar statement is true for ternary and quaternary forms), the gradients X for which (X) is irreductible according to either sequence (A) or (B) are also leading gradients according to the same sequence. Although I am convinced that this is always the case, I have not yet been able to prove it in general, and hence I can state it only as an empirical law. The consequences are important, for the form (X) could be defined as a form which has X for leading gradient; with the resulting conclusions as to products. The rôle of this analysis would then be to determine which gradients can be leading gradients of a seminvariant. As has been already said in § 26, the need for this law will be removed when the seminormal forms are used. In the case of the cubic and quartic it is true not merely that there is a semin-

variant which has the leading gradient X when (X) is irreductible; but, further, when (X) is also irreductible its leading gradient is X. It follows at once that there can be no linear relation between the quartic seminvariants

$$(A_0)^{\mu_0}(X_1)^{\mu_1}(X_2)^{\mu_2}(X_4)^{\mu_4}(X_5)^{\mu_5},$$

for each has a different leading gradient.

It is certainly not true in general that the gradient X is the leading gradient of the irreductible form (X); but it seems exceedingly likely to be true for the seminormal forms (X).

29. Consider a covariant C of order R, whose leading gradient X includes the factor $A_n{}^r$, but no other coefficient of weight n. Such a covariant may be obtained thus. We put $X = X'A_n{}^r$. Then* X' is a leading gradient of a covariant of a quantic of order $n-1$. This covariant can be expressed as a symbolical product ϕ, which is of order

$$R+rn-(\delta-r).$$

Then, provided that $R \geqslant \delta-r$, the transvectant

$$\psi = (\phi, f^r)^{nr}$$

gives a perfectly definite non-zero symbolical product which, on multiplication by $\delta-r$ factors a_x, b_x, ... to raise the order of the quantics represented in ϕ from $n-1$ to n, gives a symbolical product for a covariant of the nature C whose leading gradient is X according to sequence (A). Now no restriction is placed on X' except that it is a leading gradient of the $(n-1)$-ic. Hence every gradient $X'A_n{}^r$ of degree δ and order greater than or equal to $\delta-r$ is a leading gradient provided that X' is a leading gradient.

Again, when the seminvariant $(X'A_n{}^r)$ is irreductible then (X') must be irreductible. The number of irreductible forms (X') is coextensive with that of the leading gradients X. Hence, since every gradient $X'A_n{}^r$ is a leading gradient when $R \geqslant \delta-r$, it follows that every seminvariant $(X'A_n{}^r)$ is irreductible under the same conditions.

THEOREM XVI. *When (X') is irreductible or X' is a leading gradient according to sequence* (A), *then $(X'A_n{}^r)$ is irreductible or $X'A_n{}^r$ is a leading gradient, provided that the order $R \geqslant \delta-r$.*

The same argument applies to the gradient $X = A_{n-1}^s A_n{}^r X'$, when its order R is not less than $2(\delta-s-r)$. We consider then a covariant ϕ

* Elliott, *Algebra of quantics*, 222.

of the $(n-2)$-ic whose leading gradient is X', and then take the transvectant

$$(\phi, f^{r+s})^{(r+s)\,n-s}$$

whose first term is X.

This is an additional result in the case

$$\delta - r > 2(\delta - r - s),$$

i.e. $2s > \delta - r$, otherwise it gives nothing new. This is of use only when δ is large compared with n.

30. The number of seminvariants of degree δ which do not contain the coefficient A_n' is given by the generating function

$$\frac{(1-z^n)(1-z^{n+1})\dots(1-z^{n+\delta-1})}{(1-z^2)(1-z^3)\dots(1-z^\delta)}.$$

Subtracting this from the full generating function, we find

$$z^n \frac{(1-z^{n+1})(1-z^{n+2})\dots(1-z^{n+\delta-1})}{(1-z^2)\dots(1-z^{\delta-1})}$$

as the generating function for those seminvariants which actually contain the coefficient A_n'.

Consider the effect of this subtraction on the coefficients of the expansions. Let $(\delta, n, w) \equiv \{\delta, n, R\}$ be the number of seminvariants of degree δ, extent n or less, weight w, and order R, *i.e.* the coefficient of z^w in the ordinary generating function. Then the result above gives

$$(\delta, n, w) - (\delta, n-1, w) = (\delta-1, n, w-n).$$

When $R \geqslant \delta - 1$, this expresses the fact just noticed that, for every seminvariant of weight w, of extent n, and degree $\delta - 1$, there is, under this condition, a seminvariant of weight $w+n$ and degree δ. When $R < \delta - 1$, the coefficient of z^w in

$$\frac{(1-z^n)(1-z^{n+1})\dots(1-z^{n+\delta-1})}{(1-z^2)(1-z^3)\dots(1-z^\delta)}$$

is negative and equal to minus the coefficient of

$$z^{n\delta-\delta+1-w} = z^{w+R-\delta+1}.$$

The order of a covariant of weight $w+R-\delta+1$ of the $(n-1)$-ic is $\delta-2-R$, hence, when $R < \delta - 1$,

$$\{\delta, n, R\} + \{\delta, n-1, \delta-2-R\} = \{\delta-1, n, R+n\}.$$

Thus, when (X) is an irreductible form of degree $\delta-1$ and order $R+n$, where $R<\delta-1$, (XA_n) may be reductible or irreductible, and the number of reductible forms (XA_n) is $\{\delta, n-1, \delta-2-R\}$.

31. When, for all the seminvariants (X) of a given weight, degree, and extent, which have not been reduced, it can be shown that X is a leading gradient, then it is known that every (X) is irreductible. Further, when X and Y are leading gradients, then XY is a leading gradient, and there is a strong presumption that (XY) is irreductible.

I now show how to find a seminvariant whose leading gradient is

$$X = A_0 A_{\lambda_2} A_{\lambda_3} \dots A_{\lambda_\delta}$$

when (X) is believed to be irreductible.

In § 22 a connection was established between the form (X) and the continued transvectant

$$(\dots((f, f)^{\lambda_2} f)^{\lambda_3} \dots f)^{\lambda_\delta}.$$

We look for the seminvariant in question among the symbolical product terms of this transvectant. The orthodox practice in dealing with symbolical products is to seek to express them in terms of those with the highest possible grade, the grade being the highest index to which any individual determinant factor is raised. What we need here is exactly the reverse; in fact, the lowest possible grade. The general term in this transvectant is

$$\prod_{r \neq s} (a_r a_s)^{\mu_{rs}},$$

where r, s run through the values 1, 2, ..., δ, and

$$\sum_{r=1}^{s-1} \mu_{rs} = \lambda_s.$$

Let

$$\sum_{r=s+1}^{s} \mu_{rs} = \mu_s - \lambda_s.$$

Then the coefficient of maximum weight in the seminvariant is A_{μ_r}, where μ_r is the greatest of the numbers μ_s. The coefficient (or cofactor) of A_{μ_r} is obtained by removing all factors containing a_r from the symbolical product. When $R = \delta\lambda_\delta - 2w \geqslant \delta-1$, it is always possible to choose the numbers so that $\mu_\delta > \mu_s$ $(s<\delta)$; and hence, when it is possible to find a symbolical product of degree $\delta-1$ with X/A_{λ_r} for leading gradient, it is possible to find a seminvariant with X as leading gradient, as has been proved already. When $R<\delta-1$, there will be more than one of the

numbers μ_r equal to λ_δ for

$$\sum_{r=1}^{\delta} \mu_r = 2w = \delta\lambda_\delta - R.$$

It is sometimes quite easy to arrange that of these the cofactor of A_{μ_δ}, *i.e.* the product obtained by removing all factors containing a_δ, has an earlier leading gradient than that for any other letter.

It is convenient to arrange the numbers μ_{rs} in a triangular table so that the value of μ_{rs} is on the r-th row of the s-th column, where $r < s$. The table fully represents the symbolical product. Thus, for example, the table for $\prod_{r=1}^{\delta} A_{2r-2} = X$ is

$$\begin{array}{ccccccc} 2 & 2 & 2 & \dots & 2 \\ & 2 & 2 & \dots & 2 \\ & & 2 & \dots & 2 \\ & & \dots & & \dots \\ & & & & 2 \end{array}$$

containing 2 in every position. The leading gradient is X, for the letters may be taken in any sequence with the same leading gradient always with a positive sign. And this is true whether the sequence is (A) or (B).

The table for $\prod_{r=1}^{\delta} A_{2r-2+\lambda_r}$ may be obtained by the superposition of the above on the table for ΠA_{λ_r}. Hence, when the latter is a leading gradient, so also is the former.

Other examples are

$$\begin{array}{cccc} 0 & 2 & 2 & 2 \\ & 1 & 2 & 2 \\ & & 1 & 2 \\ & & & 0 \end{array} \quad \text{and} \quad \begin{array}{cccc} 2 & 2 & 3 & 4 \\ & & 2 & 4 & 4 \\ & & & 4 & 4 \\ & & & & 0 \end{array}$$

which give leading gradients

$$A_0^2 A_3 A_5 A_6 \quad \text{and} \quad A_0 A_2 A_4 A_{11} A_{12},$$

the former according to either sequence, the latter according to sequence (A) only; both give irreductible forms (X) for sequence (A), the latter is reductible for sequence (B) by Theorem VIII.

32. The sequences (A) and (B) are not the only possible sequences. In fact, the sequence underlying the whole classical treatment of invariant theory is actually the reciprocal of these sequences. The aim in the work of earlier writers has been to express the seminvariants virtually in terms of the leading gradient according to a sequence (C), which may be thus defined. A gradient $\prod_{r=1}^{\delta} A_{\lambda_r} = X_\lambda$, in which the suffixes are in ascending order of magnitude, precedes X_μ when the first of the differences

$$\lambda_\delta - \mu_\delta, \quad \lambda_{\delta-1} - \mu_{\delta-1}, \quad \dots$$

which does not vanish is negative. From the point of view of perpetuants, this sequence has the great advantage that the necessary and sufficient condition that X may be a leading gradient is that

$$\lambda_{\delta-1} = \lambda_\delta.$$

This is in accordance with MacMahon's use of *power ending products**.

The same thing may be seen symbolically. Every perpetuant can be expressed as a symbolical product in which some given letter appears in every determinant factor. When the greatest index of these $\delta - 1$ different factors is less than twice that of the next greatest index, or when this greatest index is odd, the grade can be increased. After the grade is increased, one of the letters of the factor of greatest index may be chosen to be placed in every factor; and if necessary the process may be repeated.

Thus we need only consider a product

$$\prod_{r=2}^{\delta} (a_1 a_r)^{\lambda_r},$$

where λ_δ is even and

$$\lambda_\delta \geqslant 2\lambda_{\delta-1} \geqslant 2\lambda_{\delta-2} \dots \geqslant 2\lambda_2.$$

The leading gradient, according to sequence (C), is

$$A_{\lambda_2} \dots A_{\lambda_{\delta-1}} A_\mu^2,$$

where

$$\lambda_\delta = 2\mu;$$

this is really the method of Grace† in dealing with perpetuants.

It is easy to see that, when X is not a power ending product, then (X) is reductible according to sequence (C); for let $X = \Pi A_{\lambda_r}$, where $\lambda_\delta > \lambda_{\delta-1}$;

* Elliott, *Algebra of quantics*, 241, *et seq.*

† Grace and Young, *Algebra of invariants* (1903), 326, *et seq.*

then, by equation (I), § 4,

$$O(A_{\lambda_1} A_{\lambda_2} \dots A_{\lambda_\delta - 1}) = 0$$

is an equation which reduces (X). And since the number of linearly independent seminvariants is the same as the number of power enders, every form (X) is irreductible when X is a power ender.

Thus the law which is so far but empirical for sequences (A) and (B) is an established fact for (C).

The weakness of sequence (C) lies in the fact that there is no clear indication of the order of the quantic necessary in order that (X) may exist and be independent of other forms (Y), where X and Y are power enders. In fact, like the transvectant sequence and the symbolical product sequence, the place of difficulty also is reversed in passing from sequence (C) to sequences (A) and (B).

VII. *Invariants.*

33. The discussion of invariants is facilitated by the fact that the two rows in the tableau may be exchanged with an accompanying change of sign when the weight is odd. Thus

$$(A_0 A_{\lambda_2} \dots A_{\lambda_\delta}) = (-)^w (A_0 A_{\lambda_\delta - \lambda_{\delta-1}} A_{\lambda_\delta - \lambda_{\delta-2}} \dots A_{\lambda_\delta}),$$

the order of the quantic being here taken as the weight of the last letter.

Hence for sequence (A) the form is reductible when the first of the differences

$$\lambda_\delta - \lambda_{\delta-1} - \lambda_2, \quad \lambda_\delta - \lambda_{\delta-2} - \lambda_3, \quad \dots$$

which does not vanish is positive [negative for sequence (B)], and when all these differences are zero and w is odd. The form is also reductible when λ_r is 1 or $\lambda_\delta - 1$. The form is also reductible for sequence (A) [sequence (B)] when $(A_0 A_{\lambda_2} \dots A_{\lambda_{r-1}})$ is reductible for sequence (A) [sequence (B)], and when $(A_0 A_{\lambda_\delta - \lambda_{\delta-1}} \dots A_{\lambda_\delta - \lambda_2})$ is reductible for sequence (B) [sequence (A)]. These conditions are the complete conditions, at least so far as degree 6 is concerned, and every other form is irreductible.

The invariants of degree 5 are then

$$(A_0 A_{2\mu} A_{4\mu+\xi} A_{6\mu+\xi+\eta} A_n),$$

where ξ, η are zero or greater than 1, and

$$3n = 24\mu + 4\xi + 2\eta.$$

We also have

$$n - 6\mu - \xi - \eta \leqslant 2\mu,$$

that is

$$\xi \leqslant \eta.$$

When $\xi = \eta$, we must also have w and therefore ξ even. This gives a set of forms

$$(A_0 A_{2\mu} A_{4\mu+2\xi} A_{6\mu+4\xi} A_{8\mu+4\xi}).$$

Here μ and ξ may be any positive integers or zero. Otherwise $\eta = \xi + 3\zeta$, and the form is

$$(A_0 A_{2\mu} A_{4\mu+\xi} A_{6\mu+2\xi+3\zeta} A_{8\mu+2\xi+2\zeta}).$$

This requires that $2\mu \geqslant \zeta$; also the form is reductible when ζ is odd. All possible systems of suffixes may be obtained by adding multiples of the four schemes

$$I_4 = (0 \quad 0 \quad 2 \quad 4 \quad 4),$$

$$I_8 = (0 \quad 2 \quad 4 \quad 6 \quad 8),$$

$$I_{12} = (0 \quad 2 \quad 4 \quad 12 \quad 12),$$

$$I_{18} = (0 \quad 2 \quad 7 \quad 18 \quad 18).$$

Moreover,

$$2I_{18} = 2I_{12} + 3I_4.$$

That the corresponding products are leading gradients may be seen by the diagrams

$$I_4 \quad \begin{matrix} 0 & 1 & 1 & 2 \\ & 1 & 2 & 1 \\ & & 1 & 1 \\ & & & 0 \end{matrix}, \qquad I_{12} \quad \begin{matrix} 2 & 2 & 4 & 4 \\ & 2 & 4 & 4 \\ & & 4 & 4 \\ & & & 0 \end{matrix}, \qquad I_{18} \quad \begin{matrix} 2 & 4 & 6 & 6 \\ & 3 & 6 & 7 \\ & & 6 & 5 \\ & & & 0 \end{matrix}.$$

Moreover, these diagrams as well as the schemes are additive. Hence the former are irreductible and the gradients are leading gradients. The generating function is

$$\frac{1+x^{45}}{(1-x^{10})(1-x^{20})(1-x^{30})}.$$

The correctness of the result will again appear from the generating function.

34. The invariants of the quintic, by Hermite's law of reciprocity, should be related to those of degree 5. These are

$$(A_0^{\lambda_0} A_2^{\lambda_2} A_3^{\lambda_3} A_5^{\lambda_5}),$$

where

$$\lambda_2 - \lambda_2 = 5(\lambda_5 - \lambda_0),$$

and $\lambda_5 \geqslant \lambda_0$ for sequence (A).

Moreover, when $\lambda_5 = \lambda_0$, we have $\lambda_2 = \lambda_3$, and w must be even. Also this form is reductible when $\lambda_2+\lambda_3 > \lambda_0$ and when $\lambda_2+\lambda_3 = \lambda_0$, λ_3 odd. Thus, when (X) is an irreductible invariant, X is a product of the gradients

$$X_4 = A_0^2 A_5^2, \quad X_8 = A_0^3 A_2 A_3 A_5^3, \quad X_{12} = A_0^4 A_2^2 A_3^2 A_5^4,$$

$$X_{16} = A_0^5 A_2^5 A_5^6, \quad X_{18} = A_0^6 A_2^5 A_5^7.$$

Here $$X_{18}^2 = X_4 X_{16}^2$$

indicates the quintic syzygy; and also

$$X_8^2 = X_4 X_{12}$$

indicates a syzygy or a reduction. Consider the possible invariant forms of degree 16; they must be given by the gradients

$$X_4^4, \quad X_4^2 X_8, \quad X_8^2 = X_4 X_{12}, \quad X_{16}.$$

Also it is not difficult to obtain invariants with the above leading gradients, so that the forms (X_4), (X_8), (X_{12}) are irreductible, and the forms

$$(X_4)^4, \quad (X_4)^2(X_8), \quad (X_8)^2, \quad (X_4)(X_{12}), \quad (X_{16})$$

exist. A relation between the first four would mean that (X_4) was a factor of (X_8) or of (X_{12}), which could not be the case; hence (X_{16}) is reducible—and, indeed,

$$(X_{16}) = (X_4)(X_{12}) - (X_8)^2.$$

For the leading gradient of the difference must be later than X_8^2 and hence cannot be either X_4^4 or $X_4^2 X_8$ The generating function for the quintic invariants is then

$$\frac{1+x^{45}}{(1-x^{10})(1-x^{20})(1-x^{30})},$$

the same as for invariants of degree 5.

35. Invariants of degree 6 are given by

$$(A_0 A_{2\mu} A_{4\mu+\xi} A_{6\mu+\xi+\eta} A_{8\mu+p} A_{10\mu+q}),$$

where, since the order of the form given by the first five coefficients is positive,

$$3p \geqslant 4\xi+2\eta,$$

and $$4q = 4\xi+2\eta+2p;$$

hence $$6q \geqslant 10\xi+5\eta.$$

Also $q \leqslant p$; and, when $q = p$, $p \leqslant 2\xi+\eta$. Finally, when $q = p = 2\xi+\eta$, then

$$w = 30\mu+6\xi+3\eta$$

must be even.

It is, of course, a question of Diophantine equations; the solution is given by

$$I_2 = (0\ \ 0\ \ 0\ \ 2\ \ 2\ \ 2),$$

$$I_4 = (0\ \ 0\ \ 2\ \ 2\ \ 4\ \ 4),$$

$$I_6 = (0\ \ 0\ \ 3\ \ 3\ \ 6\ \ 6),$$

$$I_{10} = (0\ \ 2\ \ 4\ \ 6\ \ 8\ \ 10),$$

$$I_{12} = (0\ \ 2\ \ 4\ \ 6\ \ 12\ \ 12),$$

$$I_{15} = (0\ \ 2\ \ 4\ \ 9\ \ 15\ \ 15).$$

The first two are invariants of the quadratic and quartic, and are known to represent leading gradients; I_{10} is the general form dealt with in § 31; that the other three are represented by leading gradients is seen from the diagrams

$$I_6 \quad \begin{matrix} 0 & 2 & 1 & 2 & 1 \\ & 1 & 2 & 1 & 2 \\ & & 0 & 2 & 1 \\ & & & 1 & 2 \\ & & & & 0 \end{matrix}, \qquad I_{12} \quad \begin{matrix} 2 & 2 & 2 & 3 & 3 \\ & 2 & 2 & 3 & 3 \\ & & 2 & 3 & 3 \\ & & & 3 & 3 \\ & & & & 0 \end{matrix}, \qquad I_{15} \quad \begin{matrix} 2 & 2 & 3 & 5 & 3 \\ & 2 & 3 & 5 & 3 \\ & & 3 & 5 & 3 \\ & & & 0 & 6 \\ & & & & 0 \end{matrix}.$$

The following relations connect the solutions:

$$2I_6 = 3I_4,$$

$$2I_{15} = 2I_{12}+3I_2.$$

The generating function is

$$\frac{1+x^{45}}{(1-x^6)(1-x^{12})(1-x^{18})(1-x^{30})}.$$

36. The sextic invariants are forms

$$(A_0^{\lambda_0} A_2^{\lambda_2} A_3^{\lambda_3} A_4^{\lambda_4} A_6^{\lambda_6}),$$

where $$\lambda_2-\lambda_4=3(\lambda_6-\lambda_0),$$

$$\lambda_2+\lambda_3\leqslant\lambda_0,\quad \lambda_3+\lambda_4\leqslant\lambda_0,$$

and, when λ_3 is odd, $\lambda_2+\lambda_3<\lambda_0$, $\lambda_3+\lambda_4<\lambda_0$; also $\lambda_6\geqslant\lambda_0$, and when $\lambda_6=\lambda_0$, then $\lambda_4=\lambda_2$, and λ_3 is even. Then, when (X) is not reductible, X is a product of the gradients

$$X_2=A_0A_6,\quad X_4=A_0A_2A_4A_6,\quad X_6=A_0^2A_3^2A_6^2,$$
$$X_{10}=A_0^3A_2^3A_6^4,\quad X_{15}=A_0^5A_2^3A_3A_6^6;$$

with the single relation

$$X_{15}^2=X_2^2X_6X_{10}^2$$

indicating the syzygy.

The generating function is the same as for invariants of degree 6; this is as it should be, by Hermite's law of reciprocity.

VIII. *Generating functions.*

37. In a recent paper published in these *Proceedings**, certain generating functions were obtained for invariant forms. Another set is obtained from Theorem I of the present paper as follows.

Let $A_{\delta,r}$ be the number of linearly independent types of the binary q-ic which are of degree δ and order r. Let

$$\phi(y,z)=\Sigma A_{\delta,r}z^\delta y^r.$$

Let w be the weight of the type; then

$$r=q\delta-2w,$$

and we may write

$$A_{\delta,r}=B_{\delta,w},\quad \phi=\Sigma B_{\delta,w}z^\delta y^{q\delta-2w},$$

a form which puts into evidence the fact that those terms of ϕ which have z^δ in common differ by even indices of y only.

Consider the relation between coefficients of z^δ and $z^{\delta+1}$ in the light of Theorem I. We look at the tableaux of the standard forms. There must be no overlapping; the removal of the last set of letters from a standard tableau of degree $\delta+1$ must produce a standard tableau of degree δ. Thus, provided that w is not too large and that $w\geqslant q$,

$$B_{\delta+1,w}=B_{\delta,w-q}+B_{\delta,w-q+1}+\ldots+B_{\delta,w}.$$

* A. Young, "Some generating functions", *Proc. London Math. Soc.* (2), 35 (1933), 425–444.

The limitations arise when $r = q\delta - 2w$ is negative, in which case $B_{\delta, w}$ is zero, and when $w < q$. The last equation may be written

$$A_{\delta+1, r} = A_{\delta, r+q} + A_{\delta, r+q-2} + \ldots + A_{\delta, r-q}.$$

In the limiting cases we have

$$A_{\delta+1, 0} = A_{\delta, q},$$

$$A_{\delta+1, 1} = A_{\delta, q+1} + A_{\delta, q-1},$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$A_{\delta+1, r} = A_{\delta, q+r} + A_{\delta, q+r-2} + \ldots + A_{\delta, q-r} \quad (r < q).$$

Consider now

$$[1 - (y^q + y^{q-2} + \ldots + y^{-q})z]\,\phi(y, z).$$

In the coefficient of $z^{\delta+1}$ all the terms whose y index is greater than q at once disappear. For the rest we have

$$-A_{\delta, 0}(y^{q-2} + \ldots + y^{-q}) - A_{\delta, 1}(y^{q-3} + \ldots + y^{-q+1})$$
$$- \ldots - A_{\delta, r}(y^{q-r-2} + \ldots + y^{-q+r}) - \ldots - A_{\delta, q-1}y^{-1}.$$

Every alternate A is zero, but this does not affect the argument. The first term in ϕ is zy^q, Thus

$$[1 - (y^q + y^{q-2} + \ldots + y^{-q})z]\,y\phi(y, z)$$
$$= zy^{q+1} - \sum_{\delta=2}^{\infty} \sum_{r=0}^{q-1} A_{\delta, r}(y^{q-r-1} + y^{q-r-3} + \ldots + y^{-q+r+1}).$$

Replace y by y^{-1} and subtract, then

$$y\phi(y, z) - y^{-1}\phi(y^{-1}, z) = z(y^{q+1} - y^{-q-1})\left\{1 - z\,\frac{y^{q+1} - y^{-q-1}}{y - y^{-1}}\right\}^{-1}$$
$$= \sum_{\delta=1}^{\infty} z^{\delta}(y^{q+1} - y^{-q-1})^{\delta}(y - y^{-1})^{-\delta+1}.$$

THEOREM XVII. *The number of linearly independent covariant types of the binary q-ic of degree δ and order r is the coefficient of y^{r+1} in the expansion of*

$$(y^{q+1} - y^{-q-1})^{\delta}(y - y^{-1})^{-\delta+1}.$$

Exactly the same process may be applied to the case when the orders of the ground forms are not all the same. We thus obtain

THEOREM XVIII. *The number of linearly independent covariants of order r, which are of unit degree in the coefficients of each of δ different binary*

quantics whose orders are $q_1, q_2, \ldots, q_\delta$, *is the coefficient of* y^{r+1} *in the expansion of*

$$(y-y^{-1})^{-\delta+1} \prod_{s=1}^{\delta} (y^{q_s+1}-y^{-q_s-1}).$$

38. Just the same process can be applied to ternary and higher forms.

Let A_{δ, r_1, r_2} be the number of concomitant types of the ternary q-ic which are of order r_1 in x, and r_2 in u, and of degree δ. Let

$$\phi(z, y_1, y_2) = \Sigma A_{\delta, r_1, r_2} z^\delta y_1^{r_1} y_2^{r_2}.$$

Then r_1 is the difference between the lengths of the first two rows in the tableau, and r_2 is the difference between the lengths of the second and third rows. As before, we consider the relation between the coefficients of $z^{\delta+1}$ and of z^δ. Provided that neither r_1 nor r_2 is less than q, we have

$$\begin{aligned} A_{\delta+1, r_1, r_2} = \; & A_{\delta, r_1, r_2+q} + A_{\delta, r_1+1, r_2+q-2} + \ldots + A_{\delta, r_1+q, r_2-q} \\ & + A_{\delta, r_1-1, r_2+q-1} + A_{\delta, r_1, r_2+q-3} + \ldots + A_{\delta\ r_1+q-2, r_2-q+1} \\ & + \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ & \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ & + A_{\delta, r_1-q, r_2}. \end{aligned}$$

Thus we multiply ϕ, in the same way as before, by

$$1-Zz,$$

where

$$\begin{aligned} Z = \; & y_2^{-q} + y_1^{-1} y_2^{-q+2} + \ldots + y_1^{-q} y_2^{q} \\ & + y_1 y_2^{-q+1} + y_2^{-q+3} + \ldots + y_1^{-q+2} y_2^{q-1} \\ & + \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ & \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ & + y_1^{q} \\ = \; & \Delta_q / \Delta_0, \end{aligned}$$

where

$$\Delta_q = \begin{vmatrix} y_1^{q+2} & y_1^{-q-2} y_2^{q+2} & y_2^{-q-2} \\ y_1 & y_1^{-1} y_2 & y_2^{-1} \\ 1 & 1 & 1 \end{vmatrix}.$$

We must now transform the variables y_1, y_2 to ξ, η, ζ, where

$$\xi = y_1, \quad \eta = y_1^{-1} y_2, \quad \zeta = y_2^{-1},$$

i.e. y_1 is replaced by ξ and y_2 is replaced by $\xi\eta$ in ϕ. Then, when (uv) is used for x, the index of ξ represents the order in u, and the index of η the order in v. Expressed in terms of ξ, η, ζ, the quotient Δ_q/Δ_0 is the sum of homogeneous products of degree q.

The coefficient of $z^{\delta+1}$ in ϕ is

$$\sum_{r,s} A_{\delta+1,r,s}\, y_1^r y_2^s = \sum_{\rho,\sigma} A_{\delta,\rho,\sigma}\, y_1^\rho y_2^\sigma \left\{ \sum_{r,s} y_1^{r-\rho} y_2^{s-\sigma} \right\};$$

the last sum is given by the relations which express $A_{\delta+1,r,s}$ in terms of $A_{\delta,\rho,\sigma}$ and which are obtained by the removal of the last set of letters from the tableaux. Let this last set occupy λ, μ, ν places respectively in the first, second, and third rows of the tableau for a form of order r and class s; and, when this set has been removed, let ρ be the order and σ the class of the resulting form. Then

$$\xi^\lambda \eta^\mu \zeta^\nu = y_1^{r-\rho} y_2^{s-\sigma}, \quad \lambda+\mu+\nu = q.$$

This is how the term $y_1^{r-\rho} y_2^{s-\sigma}$ arises in the last sum above. In general this sum is Z, but it is necessary to take care that only such products are retained as represent not more than r removals from the first row and not more than s removals from the second row; otherwise a non-standard tableau has been introduced. It need not trouble us if a term is included which represents the removal of more letters from a row in a particular case than that row contains, for the corresponding $A_{\delta,\rho,\sigma}$ is then necessarily zero. In the coefficients of $-z^{\delta+1}$ in

$$(1-z\Delta_q/\Delta_0)\,\phi(z, y_1, y_2),$$

we replace each $A_{\delta+1,r,s}$ by its value in terms of $\Sigma A_{\delta,\rho,\sigma}$. Then $A_{\delta,\rho,\sigma}$ appears multiplied by a sum of products $\xi^\lambda \eta^\mu \zeta^\nu$, of total degree q, which are just those products which do not appear in the corresponding

$$A_{\delta+1,\,\rho+\lambda-\mu,\,\sigma+\mu-\nu}$$

terms. That is

$$\lambda > \rho+\lambda-\mu \quad \text{or} \quad \mu > \sigma+\mu-\nu.$$

Hence

$$\rho < \mu \quad \text{or} \quad \sigma < \nu.$$

Thus

$$(1-\Delta_q/\Delta_0)\,\phi(z, \xi, \xi\eta) = z\xi^q - \sum_{\delta=1}^{\infty} z^{\delta+1} \Sigma A_{\delta,\rho,\sigma} \Sigma \xi^{\rho+\sigma+\lambda} \eta^{\sigma+\mu} \zeta^\nu.$$

The second Σ extends to all values of ρ and σ which are not both greater than $q-1$, and the third sum extends to those values of λ, μ, ν for which $\lambda+\mu+\nu$ is q, and μ exceeds ρ, or ν exceeds σ.

Multiply both sides of this equation by $\xi^2\eta$ and then operate with $\{\xi\eta\zeta\}'$. Consider first the terms of the sum on the right for which $\nu > \sigma$;

they may be arranged in pairs, the sum of each pair being annihilated by the operator. A pair is

$$\lambda,\ \mu,\ \nu > \sigma\ :\ \lambda' = \lambda,\ \mu' = \nu - \sigma - 1,\ \nu' = \mu + \sigma + 1.$$

We are left with terms for which $\nu \leqslant \sigma$, $\mu > \rho$; these again can be arranged in annihilated pairs, thus:

$$\lambda,\ \mu > \rho,\ \nu\ :\ \lambda' = \mu - \rho - 1,\ \mu' = \lambda + \rho + 1,\ \nu' = \nu.$$

Thus the sum on the right disappears, and we have

$$\{\xi\eta\zeta\}'\, \xi^2 \eta (1 - z\Delta_q/\Delta_0)\, \phi(z,\ \xi,\ \xi\eta) = z\Delta_q.$$

And hence
$$\{\xi\eta\zeta\}'\, \xi^2 \eta \phi(z,\ \xi,\ \xi\eta) = \Sigma z^\delta\, \Delta_0^{-\delta+1}\, \Delta_q{}^\delta.$$

The fact that the process is just as applicable to the case when the orders of the δ ground forms are not all the same, as in the case of binary forms, is obvious. We may therefore make the following statement:

THEOREM XIX. *The number of linearly independent concomitants of order ρ and class κ, linear in the coefficients of each of δ ternary quantics of orders $q_1, q_2, \ldots, q_\delta$, is the coefficient of $y_1^{\rho+1} y_2^{\kappa+1}$ in the expansion of*

$$\Delta_0^{-\delta+1} \prod_{s=1}^{\delta} \Delta_{q_s},$$

where
$$\Delta_q = \{\xi\eta\zeta\}'\, \xi^{q+2} \eta,$$

and
$$\xi = y_1,\quad \eta = y_1^{-1} y_2,\quad \zeta = y_2^{-1}.$$

It is perhaps necessary to add a further remark. We started with the generating function

$$\phi(z,\ y_1,\ y_2) = \phi(z,\ \xi,\ \xi\eta),$$

in which the coefficient of $z^\delta y_1{}^\rho y_2{}^\kappa$ is the number of independent types of degree δ, order ρ, and class κ, there being no terms where no types correspond, *e.g.* for negative values of ρ and κ. The generating function actually obtained is

$$\{\xi\eta\zeta\}'\, \xi^2 \eta \phi(z,\ \xi,\ \xi\eta).$$

This is a sum of six functions like $\xi^2 \eta\phi$. The reason why the method is effective is that the terms arising from these six functions fall into six mutually exclusive compartments. Thus, consider a term of $\xi^2 \eta\phi$,

$$z^\delta\, \xi^\lambda\, \eta^\mu = z^\delta\, y_1^{\lambda-\mu}\, y_2{}^\mu = z^\delta\, y_1^{\rho+1}\, y_2^{\kappa+1}.$$

Then
$$\lambda > \mu > 0.$$

This term in the final sum gives rise to six terms, one belonging to each function; the corresponding values of ρ, κ may be written ρ_r, κ_r, where

$$r = 1, 2, \ldots, 6 : \rho_1 = \rho, \ \kappa_1 = \kappa.$$

These values are

$$\rho, \ \kappa : \ -\rho-\kappa-3, \ \rho : \kappa, \ -\rho-\kappa-3,$$

$$\rho+\kappa+1, \ -\kappa-2 : \ -\kappa-2, \ -\rho-2 : \ -\rho-2, \ \rho+\kappa+1.$$

and thus no confusion arises.

39. The same arguments may be applied when there are p sets of variables. We begin with the function

$$\phi(z, y_1, y_2, \ldots, y_{p-1}) = \Sigma z^\delta A_{\delta, r_1, r_2, \ldots, r_{p-1}} y_1^{r_1} y_2^{r_2} \ldots y_{p-1}^{r_{p-1}},$$

where $A_{\delta, r_1, r_2, \ldots, r_{p-1}}$ is the number of types of degree δ and of orders $r_1, r_2, \ldots, r_{p-1}$ in the different kinds of variables. The concomitants are represented by tableaux of $T_{\alpha_1, \alpha_2, \ldots, \alpha_p}$, where

$$\alpha_1-\alpha_2 = r_1, \quad \alpha_2-\alpha_3 = r_2, \quad \ldots, \quad \alpha_{p-1}-\alpha_p = r_{p-1}.$$

Then the variables $y_1, y_2, \ldots$ are changed by the transformation

$$\xi_1 = y_1, \quad \xi_2 = y_1^{-1} y_2, \quad \xi_3 = y_2^{-1} y_3, \quad \ldots, \quad \xi_p = y_{p-1}^{-1},$$

or

$$y_1 = \xi_1, \quad y_2 = \xi_1 \xi_2, \quad \ldots, \quad y_{p-1} = \xi_1 \xi_2 \ldots \xi_{p-1}, \quad 1 = \xi_1 \xi_2 \ldots \xi_p.$$

We write

$$\Delta_q = \{\xi_1 \xi_2 \ldots \xi_p\}' \, \xi_1^{p+q-1} \xi_2^{p-2} \xi_3^{p-3} \ldots \xi_{p-1}.$$

Then $A_{\delta, r_1, r_2, \ldots, r_{p-1}}$ is the coefficient of $y_1^{r_1+1} y_2^{r_2+1} \ldots y_{p-1}^{r_{p-1}+1}$ in the expansion of

$$\Delta_0^{-\delta+1} \Delta_q^{\ \delta}.$$

And similarly the generating function is obtained for the case when the orders of the ground forms are not all the same.

40. The result of § 37 may be used to obtain another form of generating function. We consider, first, types of even degree 2δ of a binary quantic. Let $A_{q,r}$ be the number of types of order r, when the order of the ground form is q; and let

$$\phi(x, y) = \Sigma A_{q,r} x^q y^r.$$

Then, by Theorem XVII,

$$y\phi(x, y)-y^{-1}\phi(x, y^{-1})$$

$$= \sum_{q=0}^{\infty} (y^{q+1}-y^{-q-1})^{2\delta}(y-y^{-1})^{-2\delta+1}x^q$$

$$= (y-y^{-1})^{-2\delta+1}\sum_{q=0}^{\infty}\left[\sum_{s=0}^{\delta-1}(-)^s\binom{2\delta}{s}\{y^{2(q+1)(\delta-s)}+y^{-2(q+1)(\delta-s)}\}+(-)^\delta\binom{2\delta}{\delta}\right]x^q$$

$$= (y-y^{-1})^{-2\delta+1}\left[\sum_{s=0}^{\delta-1}(-)^s\binom{2\delta}{s}\{y^{2(\delta-s)}(1-xy^{2(\delta-s)})^{-1}+y^{-2(\delta-s)}(1-xy^{-2(\delta-s)})^{-1}\}\right.$$

$$\left.+(-)^\delta\binom{2\delta}{\delta}(1-x)^{-1}\right].$$

Now $y\phi(x, y)$ contains only positive powers of y, and $y^{-1}\phi(x, y^{-1})$ only negative powers. Hence

(XII) $$(y-y^{-1})^{2\delta-1}y\phi(x, y)$$

$$= \sum_{r=0}^{\delta-1}(-)^r\binom{2\delta}{r}y^{2(\delta-r)}(1-xy^{2(\delta-r)})^{-1}+(-)^\delta\tfrac{1}{2}\binom{2\delta}{\delta}(1-x)^{-1}+\chi(x, y).$$

Here $\chi(x, y)$ is a function with the following properties:

$$\chi(x, y)+\chi(x, y^{-1}) = 0;$$

it only contains even powers of y; it cannot contain a term in which the index of y is less than $-2\delta+2$. Hence

$$\chi(x, y) = A_1(y^2-y^{-2})+A_2(y^4-y^{-4})+\ldots+A_{\delta-1}(y^{2\delta-2}-y^{-2\delta+2}).$$

The coefficients A are functions of x. Since the expression on the left of (XII) has the factor y^2-1 repeated $2\delta-1$ times, each of the expressions obtained by differentiating the right-hand side of (XII) 0, 1, 2, ..., $2\delta-2$ times with respect to y^2 and then putting $y = 1$ must vanish; we thus obtain $2\delta-1$ equations satisfied by $A_1, A_2, \ldots, A_{\delta-1}$.

The method is too cumbersome for calculation, but it yields at once the important fact that the denominator of $\phi(x, y)$ is

$$\prod_{r=1}^{\delta}(1-xy^{2r})(1-x)^{2\delta-2},$$

the numerator being a function $\psi(x, y)$, integral and algebraic in x, y, which contains no power of x greater than $3\delta-4$, since the coefficient of $x^{3\delta-3}$ is always zero.

41. When the degree $2\delta+1$ is odd the process is the same, but the form of the result is slightly different; for we then have

$$y\phi(x, y)-y^{-1}\phi(x, y^{-1})$$

$$=(y-y^{-1})^{-2\delta}\sum_{s=0}^{\delta}(-)^s\binom{2\delta+1}{s}\{y^{2\delta+1-s}(1-xy^{2\delta+1-s})^{-1}$$

$$-y^{-2\delta-1+s}(1-xy^{-2\delta-1+s})^{-1}\}.$$

Hence

$$(y-y^{-1})^{2\delta}y\phi(x, y)\sum_{r=0}^{\delta}(-)^r\binom{2\delta+1}{r}y^{2\delta+1-r}(1-xy^{2\delta+1-r})^{-1}+\chi(x, y),$$

where

$$\chi(x, y)=A_0+A_1(y+y^{-1})+A_2(y^2+y^{-2})+\ldots+A_{2\delta-1}(y^{2\delta-1}+y^{-2\delta+1}).$$

The denominator of ϕ is

$$(1-x^2)^{2\delta-1}\prod_{r=0}^{\delta}(1-xy^{2r+1}).$$

The numerator of ϕ is $\psi(x, y)$ in which the highest index of x is $5\delta-3$, for the coefficient of $x^{5\delta-2}$ is always zero.

42. The following are the values of $\phi(x, y)$ for the first six values of δ:

$$\delta=1,\quad \phi=\frac{1}{1-xy};$$

$$\delta=2,\quad \phi=\frac{1}{(1-x)(1-xy^2)}.$$

For higher values of δ it is sufficient to give the numerator ψ.

$\delta=3,\quad \psi=1+xy+x^2y^2;$

$\delta=4,\quad \psi=(1+xy^2)^2;$

$\delta=5,\quad \psi=1+3x^2+x^4+y(4x+2x^3-x^5)$

$$+y^2(10x^2-5x^4)+y^3(3x+4x^3-7x^5)$$

$$-y^4(-7x^2+4x^4+3x^6)-y^5(-5x^3+10x^5)$$

$$-y^6(-x^2+2x^4+4x^6)-y^7(x^3+3x^5+x^7);$$

$\delta=6,\quad \psi=1+x+x^2+y^2(8x-x^2-x^3)+y^4(4x+10x^2-11x^3)$

$$-y^6(-11x^2+10x^3+4x^4)+y^8(x^2+x^3-8x^4)-y^{10}(x^3+x^4+x^5).$$

There is a curious form of symmetry possessed by these functions, namely that the terms appear in pairs with the same coefficients, but in the last two cases with opposite signs, the products of the variables of every pair having a common value. Thus for degree 5 the common value is x^7y^7, and for degree 6 it is x^5y^{10}.

43. The generating function of Theorem XVII, § 37, may be put in another form. The coefficient of y^{r+1} in that function is the number of independent covariant types of order r. Now

$$r = q\delta - 2w.$$

If then we multiply by $y^{-q\delta-1}$ and replace y^{-2} by η, we shall have a function of η such that the coefficient of η^w is the number of independent covariant types of degree δ and of weight w. This generating function is

$$(1-\eta)^{-\delta+1}(1-\eta^{q+1})^{\delta}.$$

When q is infinite we obtain at once the well-known generating function for perpetuant types,

$$(1-\eta)^{-\delta+1}.$$

44. A similar transformation may be applied to the ternary generating function of Theorem XIX. Here we have the number of types of order r_1 and class r_2 expressed as the coefficient of $y_1^{r_1+1}y_2^{r_2+1}$. Now let the weights be w_1, w_2, w_3, *i.e.* the numbers of letters in the three rows of the tableau representing the form. Then

$$r_1 = q\delta - 2w_2 - w_3, \quad r_2 = w_2 - w_3.$$

We multiply the function obtained by $y_1^{-q\delta-1}y_2^{-1}$ and then make the transformation

$$\eta_2 = y_1^{-2}y_2, \quad \eta_3 = y_1^{-1}y_2^{-1}.$$

Hence, in the expansion of

$$\eta_2^{-1}D_0^{-\delta+1}D_q^{\delta},$$

where

$$D_q = \begin{vmatrix} 1 & \eta_2^{q+2} & \eta_3^{q+2} \\ 1 & \eta_2 & \eta_3 \\ 1 & 1 & 1 \end{vmatrix},$$

the coefficient of $\eta_2^{w_2}\eta_3^{w_3}$ is the number of types of degree δ, and of weights $\delta q - w_2 - w_3$, w_2, w_3.

This function may at once be applied to forms of infinite order and we obtain

THEOREM XX. *The number of linearly independent concomitant types, of a ternary form of infinite order, of degree* δ, *having second and third weights* w_2 *and* w_3, *is the coefficient of* $\eta_2^{w_2}\eta_3^{w_3}$ *in the expansion of*

$$(1-\eta_3\eta_2^{-1})(1-\eta_2)^{-\delta+1}(1-\eta_3)^{-\delta+1}.$$

Exactly the same process is applicable to forms with any number of variables. Thus the generating function for quaternary perpetuant types is

$$(1-\eta_3\eta_2^{-1})(1-\eta_4\eta_2^{-1})(1-\eta_4\eta_3^{-1})(1-\eta_2)^{-\delta+1}(1-\eta_3)^{-\delta+1}(1-\eta_4)^{-\delta+1}.$$

IX. *Invariant types of degree* 5.

45. A change in notation helps to bring out the essential points of the argument.

A type of degree δ has been written

$$\{a_1{}^0 a_2^{\lambda_2} \dots a_\delta^{\lambda_\delta}\}.$$

When the identity of the different quantics is immaterial, this may be written

$$\{0\ \lambda_2\ \lambda_3\ \dots\ \lambda_\delta\}.$$

Such a form will be used even with permutations; thus

$$\{a_r a_s\}\{0\ \lambda_2\ \lambda_3\ \dots\ \lambda_\delta\}$$

will indicate the sum of the type and that obtained by interchanging the quantics a_r and a_s, these being those which appear with the indices λ_r and λ_s.

The symbol $\bar{I}$ indicates the complete inversion of the sequence of quantics, thus

$$\bar{I}\{a_1{}^0 a_2^{\lambda_2} \dots a_\delta^{\lambda_\delta}\} = \{a_\delta{}^0 a_{\delta-1}^2 \dots a_1^{\lambda_\delta}\}.$$

The peculiarity of invariants lies in the fact that the two rows of the tableau may be interchanged; such interchange is accompanied with a change of sign when w, the number of letters in a row, is odd.

Thus, for invariants,

$$\{0\ \lambda_2\ \dots\ \lambda_\delta\} = (-)^w \bar{I}\{0\ \lambda_\delta - \lambda_{\delta-1}\ \lambda_\delta - \lambda_{\delta-2}\ \dots\ \lambda_\delta\},$$

the numbers $\lambda_2, \ldots, \lambda_\delta$ being in ascending order of magnitude. From our point of view the essential thing is the set of differences between the consecutive numbers $0, \lambda_2, \ldots, \lambda_\delta$. We then write

$$i_r = \lambda_{r+1} - \lambda_r,$$

and define the form in terms of these *intervals*,

$$[i_1, i_2, i_3, \ldots, i_{\delta-1}].$$

It is convenient to use Sylvester's term *excess* of a gradient for

$$\delta\lambda - 2w,$$

where δ is its degree, λ its extent (*i.e.* the maximum weight of a coefficient contained), and w its weight. Then the excess of

$$\{0 \ \lambda_2 \ \ldots \ \lambda_\delta\}$$

is

$$E_\delta = \delta\lambda_\delta - 2w.$$

We shall require also the excess for part of the form, and so shall use E_r for the excess of $\{0 \ \lambda_2 \ \ldots \ \lambda_r\}$. Then it is not difficult to show that the excess in terms of the intervals is

$$E_\delta = (\delta-2)(i_{\delta-1} - i_1) + (\delta-4)(i_{\delta-2} - i_2) + \ldots.$$

The law of reductibility of types is taken here to be the same as for covariants for a single form according to sequence (A), Then a reduction is a relation

$$[i_1, i_2, \ldots, i_{\delta-1}] = \Sigma B s\,[j_1, j_2, \ldots, j_{\delta-1}],$$

where s is a permutation of quantics, B is numerical, and the first of the differences $i_{\delta-1} - j_{\delta-1}, i_{\delta-2} - j_{\delta-2}, \ldots$ which does not vanish is positive.

46. Consider then

$$[i, i_2, i_3, i_4],$$

where

$$E_5 = 3(i_4 - i) + i_3 - i_2 = 0.$$

Then there is always a reduction when $E_r \ (r < \delta)$ is negative. Thus

$$\tfrac{1}{2} E_4 = i_3 - i = \eta \geqslant 0,$$

$$E_3 = i_2 - i = \xi \geqslant 0.$$

For invariants,

(XIII) $$[i, i_2, i_3, i_4] = (-)^w \bar{I}[i_4, i_3, i_2, i].$$

Thus the form is reductible when the first of the differences i_4-i, i_3-i_2 which does not vanish is positive.

Also, when the dexter is reductible according to sequence (B), the sinister is reductible sequence (A). Hence, by Theorem VIII, there is a reduction when any i_r is less than i_4. This does not, however, yield any fact not obtained otherwise for degree 5, but it is useful for higher degrees.

When there is a relation of the form

$$\Sigma\, Bs\,[i_1, i_2, \ldots, i_{\delta-1}] = R, \tag{XIV}$$

where R is a linear function of earlier forms, either (i) according to sequence (A) or (ii) according to sequence (B), then a similar relation is satisfied by

$$[i_1+\mu, i_2+\mu, \ldots, i_{\delta-1}+\mu], \tag{XV}$$

the sign of every transposition in s being altered when μ is odd.

This follows at once from the equations of § IV, for $\delta < 5$; that it is true for invariants of degree 5 follows from equation (XIII), which is all that is required here. I am convinced that the above statement is true for all values of δ, but cannot give a formal proof. When the intervals given refer to the leading gradient instead of to the substitutional form, its truth is clear. For, if we take K to represent the symbolical product form of (XIV), then that of (XV) may be taken as

$$K\,.\,\prod_{r>s}(a_r a_s)^{\mu},$$

whether the sequence is (A) or (B).

We need then consider only the cases

$$i_4 = 0, \quad i_3 = 4i+\xi, \quad i_2 = i+\xi,$$

that is the invariant

$$C = [i \quad i+\xi \quad 4i+\xi \quad 0].$$

When $i > 0$, $\xi > 1$, the only relations of reduction for C are (see § IV)

$$[1-(-)^i(a_1 a_2)]\,C = R, \quad [1-(a_4 a_5)]\,C = R.$$

That is, among the 5! functions obtained from C by permutation of quantics thirty are independent. Taking the permutations according to this analysis, we may represent the functions as

(i even) $\quad 1\,.\,T_5+2\,.\,T_{4,1}+2\,.\,T_{3,2}+1\,.\,T_{3,1^2}+1\,.\,T_{2^2,1}$,

(i odd) $\quad 1\,.\,T_{4,1}+1\,.\,T_{3,2}+2\,.\,T_{3,1^2}+1.\,T_{2^2,1}+1\,.\,T_{2,1^3}$.

It will be remembered that $\lambda T_{a_1 \ldots}$ stands for $\lambda f_{a_1 \ldots}$ forms belonging to this T^*.

The form is $\{0\ i\ 2i+\xi\ 6i+2\xi\ 6i+2\xi\}$, and the diagram

$$\begin{array}{cccc} i & i+\xi & 2i+\xi & 2i \\ & i & 2i+\xi & 2i+\xi \\ & & 2i & 2i+\xi \\ & & & 0 \end{array}$$

shows that there exists an invariant, with a corresponding leading gradient, which exactly fulfils the conditions.

When $i > 0$, $\xi = 0$:

i even, C belongs to $\{a_1 a_2 a_3\}$ and to $\{a_4 a_5\}$; substitutionally it contains

$$1 \,.\, T_5 + 1 \,.\, T_{4,1} + 1 \,.\, T_{4,2}.$$

i odd, C belongs to $\{a_1 a_2 a_3\}'$ and to $\{a_4 a_5\}$; substitutionally it contains

$$1 \,.\, T_{3,1^2} + 1 \,.\, T_{2,1^3}.$$

The same diagram may be used as before to establish the existence of the invariant.

When $i > 0$, $\xi = 1$, C obeys the same relations as

$$\begin{Bmatrix} a_1 a_2 \\ a_3 \end{Bmatrix} \quad \{a_4 a_5\},$$

and the forms are

$$1 \,.\, T_{4,1} + 1 \,.\, T_{3,2} + 1 \,.\, T_{3,1^2} + 1 \,.\, T_{2^2,1}.$$

When $i = 0$, $\xi > 1$.

$$[1 - (-)^w (a_1 a_5)(a_2 a_4)]\, C = 0,$$

the forms are

w even, $\quad 1 \,.\, T_5 + 1 \,.\, T_{4,1} + 1 \,.\, T_{3,2} + 1 \,.\, T_{2^2,1}$;

and w odd, $\quad 1 \,.\, T_{4,1} + 1 \,.\, T_{3,2} + 1 \,.\, T_{3,1^2}$.

Finally, when $i = 0$, $\xi = 1$, there is one form

$$(0 \quad 0 \quad 1 \quad 2 \quad 2)$$

* See Q.S.A., III, § 6.

of the nature $T_{3,1^2}$; for this we may take the diagram

$$\begin{matrix} 0 & 1 & 1 & 0 \\ & 0 & 1 & 1 \\ & & 0 & 1 \\ & & & 0. \end{matrix}$$

47. The results now may easily be verified with the generating functions. The invariant is of the form

$$[i+\mu, \quad i+\mu+\xi, \quad 4i+\mu+\xi, \quad \mu],$$

or, in the other notation,

$$\{0 \quad i+\mu \quad 2i+2\mu+\xi \quad 6i+3\mu+2\xi \quad 6i+4\mu+2\xi\}.$$

The generating function is $\Sigma A_r x^r$, where $r = 6i+4\mu+2\xi$; following Elliott (*loc. cit.*, 138), we call r the *extent* of the invariant. Its form corresponds to the common denominator of the functions

$$D = (1-x^4)(1-x^6)(1-x^8);$$

the indices corresponding to the coefficients of μ and ξ are double their coefficients, owing to the change of sign of transpositions when μ is odd, and the peculiarity of the case $\xi = 1$.

When the forms corresponding to the various T's in the last paragraph are picked out, the following generating functions are obtained; the numerator alone is given in each case, the common denominator being D.

T_5 : $1-x^6+x^{12}$. $\quad T_{4,1}$: $x^4(1+x^2)(1+x^4)$.

$T_{3,2}$: $x^4(1+x^2+2x^4+x^8)$, $\quad T_{3,1^2}$: $x^2(1+x^4+x^8)(1+x^4)$.

$T_{2^2,1}$: $x^4(1+2x^4+x^6+x^8)$. $\quad T_{2,1^3}$: $x^6(1+x^2)(1+x^4)$.

T_{1^5} : $x^4(1-x^6+x^{12})$.

If G and G' are the generating functions for two T's obtained from each other by interchanging rows and columns (*e.g.* $T_{3,2}$ and $T_{2^2,1}$), then

$$xG(x) = x^{-1} G'(x^{-1}).$$

The sum $\Sigma f_{\alpha_1, \ldots} G_{\alpha_1, \ldots} = (1-x^2)^{-3}(1+3x^2+x^4)$,

as it should be by § 42.

This shows that no reduction has been omitted, that the forms included are irreductible, and that the generating functions obtained are correct;

this also follows from the fact that leading gradients correspond to the forms obtained.

X. *Covariants of degree 5, of a single quantic.*

48. As has been proved (§ 29), the covariants of order greater than $\delta-2$ and of degree δ can be written down at once when those of lower degree are known. It is only necessary then to find those whose order is not greater than $\delta-2$. For degree 5 I have shown that all covariants are reductible other than those given, but for reasons of space I do not give the proofs here; the diagrams, by which it is seen that leading gradients correspond to the forms, are given.

Covariants of order 1:

$$C_5^1 = \{0\ \ 0\ \ 2\ \ 5\ \ 5\};\qquad C_7^1 = \{0\ \ 0\ \ 3\ \ 7\ \ 7\};$$

$$\begin{array}{ccccc} 0 & 1 & 2 & 1 & \\ & 1 & 1 & 2 & \\ & & 1 & 1 & \\ & & & 0 & \end{array} \qquad \begin{array}{ccccc} 0 & 2 & 2 & 3 & \\ & 1 & 3 & 2 & \\ & & 2 & 2 & \\ & & & 0 & \end{array}$$

$$C_{11}^1 = \{0\ \ 2\ \ 4\ \ 10\ \ 11\};\qquad C_{13}^1 = \{0\ \ 2\ \ 4\ \ 13\ \ 13\};$$

$$\begin{array}{ccccc} 2 & 2 & 4 & 3 & \\ & 2 & 3 & 4 & \\ & & 3 & 4 & \\ & & & 0 & \end{array} \qquad \begin{array}{ccccc} 2 & 2 & 5 & 4 & \\ & 2 & 4 & 5 & \\ & & 4 & 4 & \\ & & & 0 & \end{array}$$

$$C_{17}^1 = \{0\ \ 2\ \ 7\ \ 16\ \ 17\}.$$

$$\begin{array}{cccc} 2 & 4 & 5 & 6 \\ & 3 & 6 & 6 \\ & & 5 & 5 \\ & & & 0 \end{array}$$

The remaining forms are obtained by the addition of the schemes for the invariants I_4, I_8, I_{12} (§ 33) to those of one or other C^1. Incidentally, we see that

$$I_4 + C_{13}^1 = I_{12} + C_5^1.$$

Also the addition of I_{18} only produces forms obtainable otherwise.

The generating function is

$$x^5\{(1-x^2)(1-x^6)(1-x^8)\}^{-1}.$$

Covariants of order 2:

$$C_2^2 = \{0 \quad 0 \quad 0 \quad 2 \quad 2\}; \qquad C_6^2 = \{0 \quad 0 \quad 3 \quad 5 \quad 6\};$$
$$0 \quad 0 \quad 1 \quad 1 \qquad\qquad 0 \quad 2 \quad 2 \quad 2$$
$$0 \quad 1 \quad 1 \qquad\qquad 1 \quad 2 \quad 2$$
$$0 \quad 0 \qquad\qquad 1 \quad 2$$
$$0 \qquad\qquad 0$$

$$C_8^2 = \{0 \quad 0 \quad 3 \quad 8 \quad 8\}; \qquad C_{12}^2 = \{0 \quad 2 \quad 4 \quad 11 \quad 12\};$$
$$0 \quad 2 \quad 3 \quad 2 \qquad\qquad 2 \quad 2 \quad 4 \quad 4$$
$$1 \quad 3 \quad 3 \qquad\qquad 2 \quad 4 \quad 4$$
$$2 \quad 3 \qquad\qquad 3 \quad 4$$
$$0 \qquad\qquad 0$$

$$C_{12t-2}^2 = \{0 \quad 0 \quad 8t-2 \quad 10t-2 \quad 12t-2\};$$
$$0 \quad 4t-1 \quad 4t-1 \quad 4t-1$$
$$4t-1 \quad 4t-1 \quad 4t-1$$
$$2t \quad 2t$$
$$2t$$

$$C_{12t+16}^2 = \{0 \quad 0 \quad 8t+9 \quad 10t+14 \quad 12t+16\}$$
$$0 \quad 4t+5 \quad 4t+5 \quad 4t+5$$
$$4t+4 \quad 4t+5 \quad 4t+6$$
$$2t+4 \quad 2t+3$$
$$2t+2$$

As before, the other forms are obtained from I_4, I_8, I_{12} by additions.

The number sets are connected by the relations

$$I_{12}+C_{12t-2}^2 = 4I_4+I_8+C_{12(t-1)-2}^2,$$
$$I_{12}+C_{12t+16}^2 = 4I_4+I_8+C_{12(t-1)+16}^2.$$

The generating function is

$$(x^2+x^6+x^{10})\{(1-x^4)(1-x^6)(1-x^8)\}^{-1}.$$

Covariants of order 3. There is just one condition. The irreductible covariants of order 3 are those forms

$$\{0 \quad \lambda_2 \quad \lambda_3 \quad \lambda_4 \quad \lambda_5\}$$

of order 3 for which $2\lambda_4-\lambda_3-\lambda_5 \geqslant 0$, and which are irreductible for degree 4.

In every case a leading gradient corresponds.

The generating function is

$$x^3\{(1-x^2)(1-x^4)(1-x^6)\}^{-1}.$$

The forms may be given by addition of I_8 to

$$(0 \quad 0 \quad 4\sigma+\rho-3 \quad 5\sigma+2\rho-3 \quad 6\sigma+2\rho-3),$$

$$(0 \quad 2\lambda \quad 4\lambda+\rho \quad 12\lambda+2\rho+3 \quad 12\lambda+2\rho+3),$$

$$(0 \quad 2\lambda \quad 4\lambda+\rho \quad 12\lambda+2\rho \quad 12\lambda+2\rho+1).$$

It is unnecessary to give the corresponding diagrams, which may easily be written down.

XI. *The invariants of the quaternary cubic.*

49. A simple example of the general method is furnished by the invariants of a single quaternary cubic. Here the tableau has four rows of equal length. Let δ be the degree of the invariant; then the tableau contains 3δ letters and δ must be a multiple of 4. The letters occur in sets of three.

Let us suppose that there are a_1 sets which have all three letters in the first row, and that there is another set $aa'a''$ which has two letters in the first row and the other a in the second row. Then, if G is the positive symmetric group of the $3(a_1+1)$ letters of these sets, and PNX is the substitutional form obtained from the tableau*,

$$GPNX = \Sigma P'N'X,$$

where in $P'N'X$ all the letters of G are in the first row, and the other rows are the same as in PNX, except for the single letter a.

The left-hand side of this equation is simply a sum of terms, which differs from PNX only by an interchange of equivalent sets. Thus, when two letters of a set are in one row and the third in another row, the form

* Q.S.A., III, § 10.

may be expressed in terms of forms which have more sets completely in that row. This is regarded as a reduction. By repeated application of this reduction, we may express any invariant in terms of those whose tableaux only contain sets of two kinds:

(i) Sets with all three letters in the same row.

(ii) Sets with all three letters in different rows.

Let α_r be the number of sets in the r-th row; and let β_r be the number of sets of the second kind, which contain no letter in the r-th row. Now let G be the positive symmetric group of all the letters of the $\sum_{r=1}^{4} \beta_r$ sets of the second kind. We use the equation

$$GPNX = \Sigma P'N'X,$$

where all the letters of G are moved into the first row in every term on the right. When there are more letters in G than in a row the right-hand side is zero.

Consider the left-hand side. It is the sum of terms obtained by interchange of all the letters of G in PNX. In some terms, some of these sets of G are no longer of the second kind, and these either have a greater number of sets of the first kind or are equal to forms with a greater number of sets of the first kind. An increase in the number of sets of the first kind is regarded as a reduction. In the remaining terms the sets of G are all of the second kind, and, the number of places in each row being unaltered, it is easy to see that the numbers β_1, β_2, β_3, β_4 are necessarily unchanged; that is the terms differ from PNX by an interchange of equivalent sets.

On the right there are $\Sigma\beta_r$ sets with all three letters in the first row, and the α_2, the α_3, and the α_4 sets are unchanged; hence there is a reduction when

$$\Sigma\beta_r > \alpha_1.$$

Again, in the equation used, the sign* of the terms on the right is plus or minus according as the number of letters moved up into the first row is even or odd. Thus there is still a reduction when

$$\Sigma\beta_r = \alpha_1,$$

and this sign is minus, that is $3\beta_1+2\beta_2+2\beta_3+2\beta_4$ is odd; that is β_1 must be odd.

* *Loc. cit.*

Any other row might be taken instead of the first, with a similar result. Thus, for an irreductible form,

$$\alpha_s \geqslant \Sigma\beta,$$

and if $\alpha_s = \Sigma\beta$, β_s is even.

The number of letters in the first row is

$$3\alpha_1+\beta_2+\beta_3+\beta_4 = \tfrac{3}{4}\delta.$$

Hence

$$3\alpha_1-\beta_1 = 3\alpha_2-\beta_2 = 3\alpha_3-\beta_3 = 3\alpha_4-\beta_4.$$

When $\frac{1}{4}\delta$ is odd, the interchange of a pair of rows changes the sign; if then $\alpha_1 = \alpha_2$ in this case, the first two rows are identical and the form is zero. Thus, when $\frac{1}{4}\delta$ is odd, no two of the numbers α can be equal. These facts give the whole theory.

We write an invariant in the notation

$$\begin{Bmatrix}\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4\\ \beta_1 & \beta_2 & \beta_3 & \beta_4\end{Bmatrix},$$

and we confine ourselves to forms whose schemes cannot be expressed as the sum of the schemes of forms already obtained. Then we have

$$I_8 = \begin{Bmatrix}2 & 2 & 2 & 2\\ 0 & 0 & 0 & 0\end{Bmatrix},$$

$$I_{16} = \begin{Bmatrix}4 & 3 & 3 & 3\\ 3 & 0 & 0 & 0\end{Bmatrix},$$

$$I_{24} = \begin{Bmatrix}5 & 5 & 5 & 5\\ 1 & 1 & 1 & 1\end{Bmatrix},$$

$$I_{32} = \begin{Bmatrix}7 & 7 & 6 & 6\\ 3 & 3 & 0 & 0\end{Bmatrix},$$

$$I_{40} = \begin{Bmatrix}8 & 8 & 8 & 8\\ 2 & 2 & 2 & 2\end{Bmatrix},$$

$$I_{48} = \begin{Bmatrix}10 & 10 & 10 & 9\\ 3 & 3 & 3 & 0\end{Bmatrix},$$

$$I_{100} = \begin{Bmatrix}22 & 21 & 20 & 19\\ 9 & 6 & 3 & 0\end{Bmatrix}.$$

In adding schemes, a convention about the identity of rows is necessary; this is effected by agreeing that $\alpha_1 \geqslant \alpha_2 \geqslant \alpha_3 \geqslant \alpha_4$.

It is a simple arithmetical problem to show that all schemes not reductible according to the rules above can be obtained as sums from the above seven. Moreover, we have

$$2I_{100} = 2I_{48} + 2I_{32} + 2I_{16} + I_2,$$

which indicates a syzygy for I_{100}^2.

50. Wakeford* has shown that the quantic of order n in any number of variables x_r, in which the coefficients of all terms

$$x_r^{n-1} x_s$$

are zero, is canonical.

We consider then a quaternary cubic of this form, which exactly corresponds to the substitutional schemes used here.

Consider the complex

$$C_{4,p}^6 = \begin{pmatrix} 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(which resembles the invariant of a binary cubic).

Let $$D_{p,q} = \Sigma \frac{\partial^2}{\partial p_r \partial q_r}.$$

Then $$D_{p,q}^6 C_{4,p}^6 C_{4,q}^6$$

is an invariant of the form I_8, and its leading gradient is that of the scheme for I_8, the sequence of gradients following the convention about sequence of rows in a scheme. Let

$$E_{xu} = \Sigma \frac{\partial^2}{\partial x_r \partial u_r}.$$

Also let f be the cubic and let

$$\Delta_{4u} = (abcu)(bcdu)(cdau)(dabu).$$

Then the leading gradient of

$$E_{xu}^{12} f^4 (\Delta_{4u})^3$$

is given by I_{16}.

Consider the complex

$$C_{12,p}^{14} = \begin{pmatrix} 5 & 5 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}.$$

* *Proc. London Math. Soc.* (2), 18 (1920), 403–410 (408).

Then $$D^{14}_{p,q}\,C^{14}_{12,p}\,C^{14}_{12,q}$$

is an invariant whose leading gradient is given by I_{24}. For I_{32} we use in the same way the complexes

$$C^{18}_{20,p} = \begin{Bmatrix} 7 & 7 & 0 & 0 \\ 3 & 3 & 0 & 0 \end{Bmatrix},$$

$$C^{18}_{12,p} = \begin{Bmatrix} 6 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{Bmatrix},$$

and take the invariant

$$D^{18}_{p,q}\,C^{18}_{20,p}\,C^{18}_{12,q}.$$

For I_{40} we use $$\begin{Bmatrix} 8 & 8 & 0 & 0 \\ 2 & 2 & 0 & 0 \end{Bmatrix}.$$

Up to this point we are assured that the invariants actually exist, and that no reduction has been missed. Later the difficulties increase. But for I_{48} an indirect proof can be given. By addition of the schemes,

$$2I_{24} = I_8 + I_{40}.$$

In other words, using the invariants in the sense of those given by the leading gradients,

$$I^2_{24} - I_8 \,.\, I_{40}$$

is an invariant with a later leading gradient.

It cannot be zero, for then I_8 must be reducible or a factor of I_{24}, neither of which is possible. It must then be I_{48}, which consequently exists and is reducible. The direct verification of the existence of I_{100} I have not carried out. But we conclude that there are six irreducible invariants of the quaternary cubic. That of the highest degree is *gauche* in the sense used for the invariant of degree 18 of the binary quantic, since, in that case, its square is expressible in terms of the other invariants. There is thus one syzygy.

XII. *Concluding remarks.*

51. Certain conclusions of a general nature may be mentioned, as the leading results of the foregoing work. And in connection with them, I would call attention to certain results which, as yet, I have been unable to prove; but they have been verified in so many cases that I feel justified in recording them as empirical theorems. In the first place, substitutional analysis and the tableau notation provide a means of

writing any concomitant of quantics in any number of variables as a function of a certain number of arguments; each argument is a positive integer and the number of arguments is, in the first place, that of the coefficients of the ground forms. The concomitants thus defined are not all independent, and, indeed, the set of functions may be limited to those in which certain of the arguments are zero. There is a (1, 1) correspondence between arguments and coefficients, and it would appear that there is a connection between coefficients which may be omitted in a canonical form and arguments which may be omitted as zero, at least in the case of invariants.

There are still relations among the functions when they are limited to those which have these particular arguments zero. And the first object has been to define exactly the arguments as sets of integers, so that they give a linearly independent set of concomitants. This has been done completely for covariants of a binary form up to degree 5, and in certain other cases.

When the arguments are such a set, the form is called irreductible, otherwise it is called reductible.

Now, in all cases considered, the sets are found to be additive. That is, the addition of arguments of two irreductible forms gives an irreductible form. This, indeed, must necessarily follow from the fact that the relations which fix the sets are Diophantine inequalities. The arguments may be said to belong to a compound additive "field" in the sense of the definition of Dickson*.

When it is remembered that such a form obtained by addition of arguments from two others has the same characteristics, such as degree, order, and so on, as the product of these other forms, one is tempted to look upon it as actually the product, and to regard the set of functions as providing a kind of logarithmic theory of forms, in the sense that products are represented by a sum of arguments. But investigation shows that this is not the case in general for the substitutional forms as introduced. However, in the sixth paper on this subject, the substitutional units have been transformed into normal and into semi-normal units. And I have established a theorem (not yet published) which shows that, at least in the binary case, the suggested result is actually true when normal or semi-normal units are used as the basis of the covariant functions instead of the natural units. I know of no reason to anticipate insurmountable difficulty in extending the theorem to the general case. This must be left for a future communication.

* *Linear groups* (1901), 5.

The introduction of the notion of the leading gradient of a concomitant provides another way of proceeding to a like result.

It is to be remembered that both leading gradient and field of arguments presuppose the definition of an agreed sequence, and this sequence is naturally taken to be the same in both cases.

EMPIRICAL THEOREM I. *In general, the complete set of leading gradients is defined in the same way as the complete set of irreductible forms.*

I believe that it is possible to construct a sequence law for which this is not the case, but that such a law is exceptional. The theorem has been proved true for binary forms with sequence (C). It has stood all tests that have been applied in all cases so far. It is certainly true for sequence (A) when the degree is not greater than 5; also I have proved it for degree 6 except for covariants of order 4 where the work is not yet complete. It is true also for sequence (A), for degree δ when the order is greater than $\delta-2$, provided that it is true for degree $\delta-1$. This suggests the definition of the forms by their leading gradients, but as different forms have a common leading gradient something more is needed, in addition to the discovery of what gradients are to be included.

I believe that the semi-normal form which follows naturally from sequence (A) actually has the leading gradient defined by its arguments.

Another advantage of the semi-normal forms arises from the fact that the product of a permutation which affects no letter later than a_r with a semi-normal unit Q is expressed in terms of units which are the same as Q so far as letters later than a_r are concerned. It follows that the expression of the binary semi-normal form as continued transvectants, as in Theorem XIII, contains just one term; *i.e.* it is the continued transvectant whose transvection indices are the arguments of the form.

52. A point of less general interest is touched on in § 46. It may be stated thus:

EMPIRICAL THEOREM II. *A permutational equation of reduction for a covariant type of a binary form, defined as in* § 45 *by the intervals* $i_1, i_2, \ldots$, *—when the sequence is* (A) *or* (B)*—still holds good when all the intervals are increased by the same arbitrary number; when, however, this number is odd the sign of every transposition in the equation must be changed.*

This theorem is true, as was shown, for the leading gradients, and hence follows from the first theorem, provided that that theorem is true.

53. An invariant of a binary quantic is a sum of products of the differences of the roots; a covariant is a sum of products of the differences of the roots and of differences of the variable and the roots. This leads to a substitutional expression for invariants and covariants of a nature exactly similar to that already developed. If the quantic is of order n and the covariant of degree δ, there are $n\delta$ letters in the substitutional form, as before, but now they are grouped into n sets of δ letters each, each set representing one of the n roots. Thus we have two different expressions for a covariant, one by means of the symbolical calculus and the coefficients, in which the $n\delta$ letters appear in δ equivalent sets, each set containing n letters which represent one letter of that calculus. In the other expression, the $n\delta$ letters appear in n sets of δ letters, each set representing one root of the form. In both cases, the form belongs to the same substitutional set $T_{n\delta-w,\,w}$. This, of course, leads at once to Hermite's law of reciprocity. According to the first empirical theorem, the irreductible form—sequence (A)—

$$\{0\ \lambda_2\ \dots\ \lambda_\delta\}$$

has for its leading gradient

$$\Pi\, A_{\lambda_r}.$$

This may be expressed in terms of a sum of products of the roots, which sum may be taken to begin with a term which we write

$$\begin{array}{cccccc} \alpha_1 & \alpha_2 & \dots & \dots & \dots & \alpha_{\lambda_\delta} \\ \alpha_1 & \alpha_2 & \dots & \dots & \alpha_{\lambda_{\delta-1}} & \\ \dots & \dots & \dots & \dots & & \\ \alpha_1 & \alpha_2 & \dots & \alpha_\lambda, & & \end{array}$$

where $\alpha_1, \alpha_2, \dots$ are the roots in a definite sequence. The rows of this tableau indicate the leading gradient. The columns give us the term in the roots

$$\alpha_1^{\delta-1}\alpha_2^{\delta-1}\dots\alpha_{\lambda_2}^{\delta-1}\alpha_{\lambda_2+1}^{\delta-2}\dots.$$

This is the first term according to a sequence (D) of a covariant form in the differences of the roots. The sequence (D) is the reciprocal of the sequence (A), and is defined by the law that $\{\lambda_1\lambda_2\dots\lambda_\delta\}$ precedes $\{\mu_1\mu_2\dots\mu_\delta\}$ when the first of the differences

$$\lambda_1-\mu_1,\quad \lambda_2-\mu_2,\quad \dots$$

which does not vanish is negative.

The sequence (C) is the reciprocal of the sequence (B).

When a covariant is expressed in terms of the coefficients as K, and in terms of the roots as L, we must look for the relations between K and L to be developed in such a way that they follow reciprocal sequences; either (A) and (D) or (B) and (C). The actual forms irreductible for sequence (D) are the same as those for sequence (C), the argument being the same, and, further, the forms represent leading gradients in both cases. The above is merely an indication of what is to be expected. Development on these lines should follow directly from the use of semi-normal units. Obviously, the expression for the discriminant in terms of the roots is, sequence (A) or (B),

$$\{0\ 2\ 4\ \ldots\ 2n-2\}.$$

In terms of the coefficients this is the catalecticant of the $(2n-2)$-ic.

Reciprocally the expression, sequence (C) or (D), is

$$\{1,\ 1,\ \ 2,\ 2,\ \ \ldots,\ \ n-1,\ n-1\}.$$

From the well-known determinant expression for the discriminant it is easy to see that this gives its leading gradient according to sequences (C) or (D). Two reciprocal expressions, one in the coefficients and one in the roots, are not in general equivalent, at least for natural units. They will differ by earlier forms of one or other sequence.

ON THE SOLVING NUCLEI OF INTEGRAL EQUATIONS WHOSE NUCLEI ARE HOMOGENEOUS AND OF DEGREE −1

By A. C. DIXON.

[Received and read 19 May, 1932.]

Introduction.

With the notation of hypergeometric series, let $\sigma(z)$ denote

$$z^{\alpha+\beta+\frac{1}{2}} F(\alpha,\ \beta;\ \gamma;\ 1-z^2);$$

then $\sigma(z)$ has no singularity in the neighbourhood of $z=1$, but near the origin $\sigma(z)$ can be expanded in a series containing the powers

$$\alpha+\beta+\tfrac{1}{2},\quad \alpha+\beta+\tfrac{5}{2},\quad \ldots$$

$$2\gamma-\alpha-\beta+\tfrac{1}{2},\quad 2\gamma-\alpha-\beta+\tfrac{5}{2},\quad \ldots,$$

so that

$$e^{-i\pi\gamma}\sigma(ze^{i\pi})-2i\sigma(z)\cos\pi(\alpha+\beta-\gamma)-e^{i\pi\gamma}\sigma(ze^{-i\pi})=0.$$

On the other hand, near infinity the expansion of $\sigma(z)$ contains the powers

$$\tfrac{1}{2}\pm(\alpha-\beta),\quad -\tfrac{3}{2}\pm(\alpha-\beta),\quad -\tfrac{7}{2}\pm(\alpha-\beta),\quad \ldots,$$

and

$$\sigma(ze^{i\pi})-2i\sigma(z)\cos\pi(\alpha-\beta)-\sigma(ze^{-i\pi})=0.$$

Thus $\sigma(z)$ satisfies two different functional equations, the one inside, the other outside, the unit circle; it has singular points at 0, -1, ∞, but is otherwise regular in the plane cut along the negative real axis.

The present paper discusses (§§ 1–6) functions with such properties as have been stated for $\sigma(z)$, particularly (§§ 4–6) those which have simple

poles at assigned points, and (§§ 7–10) the use of such functions in the theoretical solution of integral equations (§ 8). In a former paper on the subject* the singularity of $\sigma(z)$ at $z=-1$ was logarithmic, the value of γ being integral (S.N., 238); here a more general value is taken for γ and the singularity is that of a non-integral power; the form of the results is somewhat changed.

The integral equations concerned include (§ 9) those in which the nucleus is an algebraic fraction, homogeneous of degree -1, as in S.N.M., 435; when a fraction has the difference of the variables as a simple factor in the denominator it falls within the scope of this paper, though in the former paper such a factor in the denominator was excluded; for such a factor repeated, further modification will be necessary.

In this paper, as in the former papers, the results concerning integral equations (8.4) (8.5) follow from an important theorem (7.6) which has wider applications.

Resolution of the operators into factors.

1. Let $S(z)$ denote

$$e^{i\pi z}-e^{-i\pi z}+\Sigma\, Ce^{cz}, \tag{1.1}$$

where the summation extends over a finite number of terms in each of which $|I(c)|\leqslant\pi$; but in the terms for which $|I(c)|=\pi$, $R(c)$ and $I(c)$ are to have the same sign. Any coefficient C or $C(z)$ may be a polynomial in z. Let $S_0(z)=e^{i\pi z}-e^{-i\pi z}$.

On account of the assumptions made, all the points c lie within a strip bounded by the lines through $\pm i\pi$ at a positive angle δ† with the real axis, and, if z describes the line through $\frac{1}{2}$ making an angle $\frac{1}{2}\pi-\delta$ with the real axis, then near infinity the dominant terms are the same in $S(z)$ and $S_0(z)$. Hence the net increase in the logarithm of $S(z)/S_0(z)$ as z traverses this line, with deviations if necessary to avoid the zeros of $S(z)$, is an integral multiple of $2i\pi$, say $2ni\pi$. Let P be a path of integration, following this line in general, but deviating additionally‡, if n is positive, round the nearest n zeros of $S(z)/S_0(z)$ on the left, or if n is negative, round the nearest $-n$ zeros of $S(z)/S_0(z)$ on the right; P is still to have the points 1, 2, 3, ... on its right and 0, -1, -2, ... on its left.

---

* *Proc. London Math. Soc.* (2), 27 (1926), 233–272. That paper will be referred to as S.N., and the paper *Proc. London Math. Soc.* (2), 30 (1929), 433–468, as S.N.M.

† The restriction that c shall not be the logarithm of a negative real quantity (S.N., § 11, S.N.M., 435) is avoided by the introduction of the angle δ.

‡ The restriction (S.N. 8.12) that n shall be 0 is avoided by these deviations.

There is then a value of the logarithm which is exponentially small near each limit, and, if that value is taken, the integral

$$\int_P \log \frac{S(z)}{S_0(z)} \frac{dz}{\lambda - z} \tag{1.2}$$

represents a function of λ which is regular so long as λ does not cross P.

When λ is on the right of P, denote this function by

$$2i\pi \log \eta(\lambda) - 2i\pi \log \eta_0(\lambda), \tag{1.21}$$

where $\eta_0(\lambda) = \{\Gamma(\lambda)\}^{-1}$, so that $\eta_0(\lambda)$ has the zeros of $S_0(\lambda)$ on the left of the path, but none on the right.

The function $\eta(\lambda)$ can be continued to the left of the path, but when λ crosses to the left the integral comes to represent

$$2i\pi \{\log \eta(\lambda) - \log \eta_0(\lambda) + \log S_0(\lambda) - \log S(\lambda)\},$$

which is, therefore, regular when λ is on the left. In either position of λ the integral is $O(\lambda^{-1})$ at infinity*. Hence $\eta(\lambda)$ is an integral function of λ, having no zeros to the right of P and only those of $S(\lambda)$, each to its proper order, on the left; thus $S(\lambda)$ has been resolved into the two factors $\eta(\lambda)$ and $S(\lambda)/\eta(\lambda)$, each of which is an integral function. Near infinity on the right, $\eta(\lambda)/\eta_0(\lambda)$ differs from 1 by $O(\lambda^{-1})$; on the left the same is true for $\{S(\lambda)/\eta(\lambda)\} \div \{S_0(\lambda)/\eta_0(\lambda)\}$. Since the angle δ is not fixed, but only subject to inequalities, these statements hold when the amplitude of λ† is less than $\frac{1}{2}\pi$ and greater than $-\frac{1}{2}\pi - \delta$; the path may, in fact, run from infinity at amplitude $-\frac{1}{2}\pi - \delta''$ to infinity at amplitude $\frac{1}{2}\pi - \delta'$ if δ', δ'' are any angles between 0 and δ; the effect of a change is not on the function represented but on the region of validity of the representation.

The function σ_{12}.

2. Let there be four expressions of the form (1.1) distinguished by suffixes and let the corresponding coefficients C, paths P, and factors $\eta(z)$ be similarly distinguished, with the understanding that some of the coefficients C may be zero; also let $S_3(z) \equiv S_1(1-z)$, $S_4(z) \equiv S_2(1-z)$.

---

* S.N., 254, S.N.M., 445-6. The resolution is discussed there without the use of the gamma-function.

† Or $-\lambda$, when λ is on the left of P.

Then we may take it that, as z describes P_1, $1-z$ describes P_3 backwards, and, if λ is on the right of P_3, $1-\lambda$ is on the left of P_1,; thus

$$2i\pi \log \frac{\eta_3(\lambda)}{\eta_0(\lambda)} = \int_{P_3} \log \frac{S_3(z)}{S_0(z)} \frac{dz}{\lambda - z}$$

$$= \int_{P_3} \log \frac{S_1(1-z)}{S_0(1-z)} \frac{dz}{\lambda - z}$$

$$= -\int_{P_1} \log \frac{S_1(z)}{S_0(z)} \frac{dz}{1-z-\lambda}, \quad \text{by the change of } z \text{ into } 1-z,$$

$$= -2i\pi \log \left\{ \frac{\eta_1(1-\lambda)}{\eta_0(1-\lambda)} \frac{S_0(1-\lambda)}{S_1(1-\lambda)} \right\}. \tag{2.1}$$

Now
$$2i\pi \eta_0(\lambda)\eta_0(1-\lambda) = S_0(\lambda) = S_0(1-\lambda), \tag{2.21}$$

and hence
$$2i\pi \eta_1(1-\lambda)\eta_3(\lambda) = S_1(1-\lambda) = S_3(\lambda), \tag{2.22}$$

$$2i\pi \eta_1(\lambda)\eta_3(1-\lambda) = S_1(\lambda) = S_3(1-\lambda). \tag{2.23}$$

In
$$\int_{P_{12}} \frac{e^{\lambda t} d\lambda}{\eta_1(\lambda)\eta_4(1-\lambda+\gamma)}, \tag{2.3}$$

let P_{12} be a path from infinity in the third quadrant* to infinity in the first†, separating the zeros of $\eta_1(\lambda)$ and $\eta_2(\lambda-\gamma)$ from those of $\eta_3(1-\lambda)$ $\eta_4(1-\lambda+\gamma)$; then the logarithm of the subject differs by $O(\lambda^{-1})$ from

$$\lambda t - \log\{\eta_0(\lambda)\eta_0(1-\lambda+\gamma)\}$$

$$= \lambda t + \log\Gamma(\lambda) + \log\Gamma(1-\lambda+\gamma), \tag{2.41}$$

or from

$$\lambda t + (\lambda - \tfrac{1}{2})\log\lambda - \lambda + (\tfrac{1}{2} - \lambda + \gamma)\log(1-\lambda+\gamma) - (1-\lambda+\gamma) + \log 2\pi$$

$$= \lambda(t \pm i\pi) + \gamma \log(\mp i\lambda) \mp \tfrac{1}{2}(\gamma+1)i\pi + \log 2\pi, \tag{2.42}$$

the upper sign referring to great values of λ in the upper half-plane, the lower sign to those in the lower half-plane.

Hence, if $-\frac{1}{2}\pi - \delta''$, $\frac{1}{2}\pi - \delta'$ are the initial and final amplitudes on P_{12}, the integral (2.3) converges absolutely, and represents a regular function of t, in the wedge or strip in which

$$-\pi + \delta'' < \operatorname{amp}(t - i\pi) < \delta'',$$

$$\delta' < \operatorname{amp}(t + i\pi) < \pi + \delta'. \tag{2.43}$$

---

* At an amplitude between $-\frac{1}{2}\pi - \delta$ and $-\frac{1}{2}\pi$.

† At an amplitude between $\frac{1}{2}\pi - \delta$ and $\frac{1}{2}\pi$.

Let the function there represented by (2.3) be denoted by $2i\pi\sigma_{12}(t)$. Then $\sigma_{12}(t)$ is regular so long as $|I(t)|<\pi$, since δ', δ'' may be made arbitrarily small.

When $t\to i\pi$, the singular part of $\sigma_{12}(t)$ is of the form

$$\int_R e^{\lambda(t-i\pi)}\lambda^\gamma\left\{A'+\frac{B'}{\lambda}+\frac{C'}{\lambda^2}+\ldots\right\}d\lambda, \tag{2.51}$$

where R is the part of P_{12} in the lower half-plane; it is, therefore,

$$(t-i\pi)^{-\gamma-1}\{A+B(t-i\pi)+C(t-i\pi)^2+\ldots\}, \tag{2.52}$$

where A, B, C, ... are constants. A similar result holds when $t\to -i\pi$, with $t+i\pi$ raised to the powers $-\gamma-1$, $-\gamma$, $-\gamma+1$, ...

The angles δ', δ'' are restricted only by inequalities and either of them may be the greater. If $\delta'>\delta''$, the region of validity (2.43) is wedge-shaped, opening to the left, and contains all the arguments of σ_{12} in $S_4(1+\gamma-d/dt)\sigma_{12}(t)$ so long as amp t lies between $\pi+\delta'$ and $\pi+\delta''$. Hence

$$S_4\left(1+\gamma-\frac{d}{dt}\right)\sigma_{12}(t)=\frac{1}{2i\pi}\int_{P_{12}}\frac{e^{\lambda t}S_4(1+\gamma-\lambda)}{\eta_1(\lambda)\eta_4(1+\gamma-\lambda)}d\lambda$$

$$=\int_{P_{12}}\frac{e^{\lambda t}\eta_2(\lambda-\gamma)}{\eta_1(\lambda)}d\lambda. \tag{2.6}$$

Here the integrand has no poles on the right of the path and the path may be deformed into the line at infinity on the right between the amplitudes $-\frac{1}{2}\pi-\delta''$, $\frac{1}{2}\pi-\delta'$, so that the integral vanishes, that is,

$$S_4\left(1+\gamma-\frac{d}{dt}\right)\sigma_{12}(t)=S_2\left(\frac{d}{dt}-\gamma\right)\sigma_{12}(t)=0 \tag{2.61}$$

in the left half-plane. [Compare S.N., (9.3).] Similarly, in the right half-plane,

$$S_1\left(\frac{d}{dt}\right)\sigma_{12}(t)=0. \tag{2.62}$$

Thus $\sigma_{12}(t)$ can be continued over the whole plane, but it is not uniform in the neighbourhood of its singular points $\pm i\pi$, $\pm 2i\pi-c$, ...; c, c', ... are not singular points.

3. Let $$\ldots, \lambda_{-1}, \lambda_0, \lambda_1, \ldots$$

be the zeros of $S_1(\lambda)$ in the order in which they would be passed over by

a line inclined at an angle $\frac{1}{2}\pi-\delta$ to the real axis, moving from right to left. Let*

$$\lambda_1, \lambda_2, \ldots \quad \text{be the zeros of } \eta_1(\lambda),$$

$$\ldots, \mu_{-1}, \mu_0, \mu_1, \ldots \quad \text{those of } S_2(\lambda) \text{ similarly arranged,}$$

and

$$\mu_1, \mu_2, \ldots \quad \text{those of } \eta_2(\lambda).$$

By shifting the path P_{12} to the right over the points

$$\mu_0+\gamma, \quad \mu_{-1}+\gamma, \quad \ldots, \quad \mu_{-q}+\gamma$$

into a new position P', we have

$$2i\pi\sigma_{12}(t) = -\sum_{s=-q}^{0} \frac{2i\pi\, e^{t(\mu_s+\gamma)}}{\eta_1(\mu_s+\gamma)\,\eta_4'(1-\mu_s)} + \int_{P'} \frac{e^{\lambda t}\,d\lambda}{\eta_1(\lambda)\,\eta_4(1-\lambda+\gamma)}. \quad (3.1)$$

When P' is a straight line on the right of all the zeros of $\eta_1(\lambda)$, let it meet the real axis at a distance g from the origin; then, if $\delta \leqslant \operatorname{amp}(i\pi \pm t) \leqslant \pi+\delta$, the integral in (3.1) is $O(e^{g(t+0)})$ when t or g becomes large.

In (3.1) we have then an expansion for $\sigma_{12}(t)$, such that the error is of the order of the first term omitted, when t is on the left in the strip between $\pm i\pi$ at slope δ; this will be called the *primary strip*. In particular, $\sigma_{12}(t)$ itself is of the order of the first term $e^{t(\mu_0+\gamma)}$ when t tends to $-\infty$ at slope δ.

There is a similar expansion for use when t is on the right in the primary strip. The terms are multiples of

$$e^{\lambda_1 t}, e^{\lambda_2 t}, \ldots,$$

and $\sigma_{12}(t)$ itself is of the order of the first of these when t is large.

When $c = 0$ and $S_1(\lambda)$, $S_2(\lambda)$ are both of the form

$$e^{i\pi\lambda} - e^{-i\pi\lambda} + C,$$

the expansions are hypergeometric series in ascending or descending powers of e^{2t}, as in S.N., 237–8, and, since there are three constants C_1, C_2, γ involved, the results relate to the most general hypergeometric function.

If c, c', ... are $i\pi - 2ri\pi/n$ ($r = 1, 2, \ldots, n-1$) and C, C', ... are constants, then $e^{i\pi\lambda} S_1(\lambda)$ and $e^{i\pi\lambda} S_2(\lambda)$ are n-ics in $e^{2i\pi\lambda/n}$, so that, in substance, $S_1(\lambda)$, $S_2(\lambda)$ are products, each of n factors such as $\sin(\lambda-a)\pi/n$, and $\eta_1(\lambda)$, $\eta_2(\lambda)$ are products of such factors as $1/\Gamma\{(\lambda-a)/n\}$.

* Some changes of detail must be made in the work if, for instance, no straight line can be drawn separating the zeros of $\eta_1(\lambda)$ from those of $\eta_3(1-\lambda)$

Thus $\sigma_{12}(t)$ is a hypergeometric function of e^{nt} of the n-th order, and, being regular at the origin, it is that one which can be expanded in a regular series of the powers of $1-e^{nt}$ or $1-e^{-nt}$.

When the argument of such a function as $\sigma_{12}(t)$ tends to infinity in the following work, it is to be understood to do so in the primary strip or within a finite distance from it; when other strips occur they are parallel to the primary. The *median* will mean the median line of the primary strip.

The functions I, J.

4. Let $\sigma_{34}(t)$ be formed from S_3, S_4, and $-\gamma$ as $\sigma_{12}(t)$ is formed from S_1, S_2, and γ; that is, let

$$\sigma_{34}(t) = \frac{1}{2i\pi}\int_{P_{12}} \frac{e^{(1-\lambda)t}\,d\lambda}{\eta_3(1-\lambda)\,\eta_2(\lambda-\gamma)}. \tag{4.1}$$

Then, when $|\theta|$ is large and $\operatorname{amp}\theta = \delta$,

$$\sigma_{12}(t+\theta), \quad \sigma_{34}(t+\theta), \quad \sigma_{12}(t-\theta), \quad \sigma_{34}(t-\theta)$$

are of the orders of

$$e^{\lambda_1\theta}, \quad e^{(1-\lambda_0)\theta}, \quad e^{-(\mu_0+\gamma)\theta}, \quad e^{(\mu_1+\gamma-1)\theta},$$

and $\sigma_{12}(t+\theta)\,\sigma_{34}(u+\theta)\,e^{-\theta}, \quad \sigma_{12}(t-\theta)\,\sigma_{34}(u-\theta)\,e^{\theta}$

are of the orders of $e^{(\lambda_1-\lambda_0)\theta}, \quad e^{(\mu_1-\mu_0)\theta}$.

Thus we may put

$$I(t,u) = \frac{e^{-u}}{2i\pi}\int_0^\infty e^{-\theta}\,\sigma_{12}(t+\theta)\,\sigma_{34}(u+\theta)\,d\theta, \tag{4.21}$$

$$J(t,u) = \frac{e^{-u}}{2i\pi}\int_0^\infty e^{\theta}\,\sigma_{12}(t-\theta)\,\sigma_{34}(u-\theta)\,d\theta, \tag{4.22}$$

both of these integrals being absolutely convergent for any positions of t, u in the primary strip. Then

$$(2i\pi)^3 e^u\, I(t,u) = \int_0^\infty e^{-\theta}\left[\left(\sum_{r=1}^{p}\int_{(\lambda_r)+} + \int_{P'}\right)\frac{e^{\lambda(t+\theta)}\,d\lambda}{\eta_1(\lambda)\,\eta_4(1-\lambda+\gamma)}\right]$$
$$\times\left[\left(\sum_{s=-q}^{0}\int_{(\lambda_s)-} + \int_{P'}\right)\frac{e^{(1-\mu)(u+\theta)}\,d\mu}{\eta_3(1-\mu)\,\eta_2(\mu-\gamma)}\right]d\theta, \tag{4.31}$$

$$(2i\pi)^3 e^u\, J(t,u) = \int_0^\infty e^{\theta}\left[\left(\sum_{s=-q'}^{0}\int_{(\mu_s+\gamma)-} + \int_{P'}\right)\frac{e^{\lambda(t-\theta)}\,d\lambda}{\eta_1(\lambda)\,\eta_4(1-\lambda+\gamma)}\right]$$
$$\times\left[\left(\sum_{r=1}^{p'}\int_{(\mu_r+\gamma)+} + \int_{P'}\right)\frac{e^{(1-\mu)(u-\theta)}\,d\mu}{\eta_3(1-\mu)\,\eta_2(\mu-\gamma)}\right]d\theta, \tag{4.32}$$

where

$P`$ is a straight path at the slope $\frac{1}{2}\pi-\delta$, on the left of P_{12} and of $\mu_0+\gamma$, λ_0,

P' ,, ,, ,, ,, right ,, ,, $\mu_1+\gamma$, λ_1,

λ_{-q}, ..., λ_0, λ_1, ..., λ_p are all of the series ..., λ_{-1}, λ_0, λ_1, ..., and

$$\mu_{-q'}+\gamma, \ldots, \mu_0+\gamma, \mu_1+\gamma, \ldots, \mu_{p'}+\gamma$$

are all of the series $\ldots, \mu_{-1}+\gamma, \mu_0+\gamma, \cdots$

that fall between $P`$ and P'.

These expressions contain triple integrals, all of which converge absolutely, so that the integration with respect to θ may be taken first, and then the path P_{12} may be restored. Thus $(2i\pi)^3 e^u I(t, u)$ and $-(2i\pi)^3 e^u J(t, u)$ both become

$$\int_{P_{12}}\int_{P_{12}} \frac{e^{\lambda t+u-\mu u}\, d\mu}{(\mu-\lambda)\eta_1(\lambda)\eta_4(1-\lambda+\gamma)\eta_3(1-\mu)\eta_2(\mu-\gamma)}\, d\lambda, \qquad (4.4)$$

but in the first case the μ-path must deviate to pass λ on the right, and in the second case on the left. The two paths differ by a circuit about λ, and thus

$$I(t, u)+J(t, u) = \frac{1}{(2i\pi)^2}\int_{P_{12}} \frac{e^{\lambda(t-u)}\, d\lambda}{\eta_1(\lambda)\eta_4(1-\lambda+\gamma)\eta_3(1-\lambda)\eta_2(\lambda-\gamma)}$$

$$= \int_{P_{12}} \frac{e^{\lambda(t-u)}\, d\lambda}{S_1(\lambda)\, S_2(\lambda-\gamma)}. \qquad (4.5)$$

So long as t, u are both within the primary strip, $I(t, u)$ and $J(t, u)$ are both free from singularity. If they approach the edges, the θ-path must deviate to avoid the occurrence of $\pm i\pi$ as arguments of σ_{12}, σ_{34}, and this is possible unless $t-u = \pm 2i\pi$. Then, if t, u are in the right half-plane, the path in $I(t, u)$ need not be changed and $I(t, u)$ is regular, but, by (4.5), $J(t, u)$ has simple poles, the singular parts being

$$\frac{e^{i\pi\gamma}}{t-u-2i\pi} - \frac{e^{-i\pi\gamma}}{t-u+2i\pi}.$$

If t, u are in the left half-plane, these poles belong to $I(t, u)$ and $J(t, u)$ is regular in the neighbourhoods.

If t approaches $\pm i\pi$, each of the functions has terms in

$$(t\mp i\pi)^{-\gamma}, \quad (t\mp i\pi)^{-\gamma+1}, \quad \ldots$$

as well as regular terms, and the sum of the functions is regular by (4.5).

If u approaches $\pm i\pi$, each of the functions has terms in

$$(u \mp i\pi)^{\gamma}, \quad (u \mp i\pi)^{\gamma+1}, \quad \ldots$$

as well as regular terms, the sum again being regular.

When t is in the right half-plane, $S_1\left(\frac{d}{dt}\right) I(t, u) = 0,$

,, ,, ,, left ,, $S_2\left(\frac{d}{dt} - \gamma\right) J(t, u) = 0,$

,, u ,, right ,, $S_1\left(-\frac{d}{du}\right) I(t, u) = 0,$

,, ,, ,, left ,, $S_2\left(-\frac{d}{du} - \gamma\right) J(t, u) = 0.$

The elementary function F.

5. In $S_r(\lambda)$ $(r = 1, 2, 3, 4)$ the coefficients of λ in the exponents are $\pm i\pi$, $\pm c$, ... Let lines at slope δ be drawn through the points $\pm c$, ... dividing the primary strip into partial strips. Suppose c', c'', c''' to be any three of the series $i\pi$, c, ..., $-c$, $-i\pi$, such that the parallels through them are consecutive, so that there is a partial strip between c', c'', and the partial strip below it is between c''', c''''.

Put $S_r = T_r + U_r$, where $T_1(\lambda)$, $T_2(\lambda)$ contain the terms in $e^{i\pi\lambda}$, ..., $e^{c'\lambda}$,

$U_1(\lambda)$, $U_2(\lambda)$,, ,, $e^{c''\lambda}$, ..., $e^{-i\pi\lambda}$,

$T_3(\lambda)$, $T_4(\lambda)$,, ,, $e^{i\pi\lambda}$, ..., $e^{-c''\lambda}$,

$U_3(\lambda)$, $U_4(\lambda)$,, ,, $e^{-c'\lambda}$, ..., $e^{-i\pi\lambda}$.

Thus $T_3(1-\lambda) = U_1(\lambda)$, and so on.

Let

$$\begin{aligned} S_5(\lambda) &= T_4(\lambda+\gamma)\, U_3(\lambda) - T_3(\lambda)\, U_4(\lambda+\gamma) \\ &= S_4(\lambda+\gamma)\, U_3(\lambda) - S_3(\lambda)\, U_4(\lambda+\gamma) = T_4(\lambda+\gamma)\, S_3(\lambda) - T_3(\lambda)\, S_4(\lambda+\gamma) \\ &= S_2(1-\lambda-\gamma)\, T_1(1-\lambda) - S_1(1-\lambda)\, T_2(1-\lambda-\gamma) \\ &= U_2(1-\lambda-\gamma)\, S_1(1-\lambda) - U_1(1-\lambda)\, S_2(1-\lambda-\gamma). \end{aligned}$$

Then

$$S_5\left(\frac{\partial}{\partial u}\right)\{e^u I(t,u)+c'' J(t,u)\} = \frac{1}{2i\pi}\int_{-\infty}^{\infty} e^{-\theta}\sigma_{12}(t+\theta)\, S_5\left(\frac{\partial}{\partial u}\right)\sigma_{34}(u+\theta)\,d\theta$$

$$= \int_{P_{12}} e^{\lambda t+(1-\lambda)u}\,\frac{S_5(1-\lambda)\,d\lambda}{S_1(\lambda)\,S_2(\lambda-\gamma)} \quad \text{by (4.5)}$$

$$= \int_{P_{12}} e^{\lambda t+(1-\lambda)u}\left[\frac{U_2(\lambda-\gamma)}{S_2(\lambda-\gamma)}-\frac{U_1(\lambda)}{S_1(\lambda)}\right]d\lambda$$

$$= \int_{P_{12}} e^{\lambda t+(1-\lambda)u}\left[\frac{T_1(\lambda)}{S_1(\lambda)}-\frac{T_2(\lambda-\gamma)}{S_2(\lambda-\gamma)}\right]d\lambda. \quad (5.1)$$

The exponents in $S_5(\lambda)$ are $(i\pi-c')\lambda$, ..., $(-i\pi-c'')\lambda$, so that the arguments of σ_{34} in the second member of (5.1) are

$$u+i\pi-c'+\theta, \quad \ldots, \quad u-i\pi-c''+\theta.$$

The result (5.1) holds if all these are in the primary strip, that is, if u is in the partial strip between c', c'', and if t is anywhere in the primary strip.

If t is in the partial strip between u and $i\pi$, we have, from (5.1),

$$-\int_{P_{12}} e^{\lambda(t-u)}\,\frac{U_1(\lambda)}{S_1(\lambda)}\,d\lambda - S_5\left(\frac{\partial}{\partial u}+1\right)I(t,u)$$

$$= -\int_{P_{12}} e^{\lambda(t-u)}\,\frac{U_2(\lambda-\gamma)}{S_2(\lambda-\gamma)}\,d\lambda + S_5\left(\frac{\partial}{\partial u}+1\right)J(t,u), \quad (5.2)$$

while, if t is in the partial strip between $-i\pi$ and u,

$$\int_{P_{12}} e^{\lambda(t-u)}\,\frac{T_1(\lambda)}{S_1(\lambda)}\,d\lambda - S_5\left(\frac{\partial}{\partial u}+1\right)I(t,u)$$

$$= \int_{P_{12}} e^{\lambda(t-u)}\,\frac{T_2(\lambda-\gamma)}{S_2(\lambda-\gamma)}\,d\lambda + S_5\left(\frac{\partial}{\partial u}+1\right)J(t,u). \quad (5.3)$$

The four expressions in (5.2), (5.3) are representations of a function of t which satisfies the functional equations (2.61), (2.62) in the two halves of the plane respectively and which has no singularity in the primary strip but a pole at u with residue 1. [Compare S.N., (12.41), (12.42).]

Let this function be called $F(t,u)$ or $F_{12}(t,u)$ or $F_{12}(t,u,c',c'')$. The first term on the left in (5.3) or (5.2) may be written as

$$T_1\left(-\frac{\partial}{\partial u}\right)\sigma_{11}(t-u), \quad \text{where} \quad \sigma_{11}(t)=\int_{P_{12}}\frac{e^{\lambda t}\,d\lambda,}{S_1(\lambda)}, \quad (5.41)$$

and the first term on the right in (5.3) or (5.2) as

$$e^{\gamma(t-u)}\,T_2\Big(-\frac{\partial}{\partial u}\Big)\,\sigma_{22}(t-u), \quad \text{where} \quad \sigma_{22}(t)=\int_{P_{12}}\frac{e^{(\lambda-\gamma)t}\,d\lambda}{S_2(\lambda-\gamma)}. \quad (5.42)$$

The first member of (5.2) may be written

$$-U_1\Big(-\frac{\partial}{\partial u}\Big)\,S_2\Big(-\gamma-\frac{\partial}{\partial u}\Big)\{I(t,u)+J(t,u)\}-S_5\Big(\frac{\partial}{\partial u}+1\Big)\,I(t,u),$$

so that

$$F(t,u)=-U_2\Big(-\gamma-\frac{\partial}{\partial u}\Big)\,S_1\Big(-\frac{\partial}{\partial u}\Big)\,I(t,u)$$
$$-U_1\Big(-\frac{\partial}{\partial u}\Big)\,S_2\Big(-\gamma-\frac{\partial}{\partial u}\Big)\,J(t,u). \quad (5.4)$$

In the right half-plane (u), the first term on the right in (5.4) vanishes, and in the left half-plane the second term vanishes. Similarly, from (5.3),

$$F(t,u)=T_1\Big(-\frac{\partial}{\partial u}\Big)\,S_2\Big(-\gamma-\frac{\partial}{\partial u}\Big)\,J(t,u)+T_2\Big(-\gamma-\frac{\partial}{\partial u}\Big)\,S_1\Big(-\frac{\partial}{\partial u}\Big)\,I(t,u), \quad (5.5)$$

and in (5.5) the second term on the right vanishes in the right half-plane (u), the first in the left.

The functions G, H.

6. When $c'=i\pi$, $c''=c$, we have $T_1(\lambda)=T_2(\lambda)=e^{i\pi\lambda}$, and thus

$$F_{12}(t,u,i\pi,c)=S_2\Big(-\gamma-\frac{\partial}{\partial u}\Big)\,J(t,u-i\pi)+e^{-i\pi\gamma}\,S_1\Big(-\frac{\partial}{\partial u}\Big)\,I(t,u-i\pi). \quad (6.1)$$

In the right half of the u-plane

$$F_{12}(t,u+i\pi,i\pi,c)=S_2\Big(-\gamma-\frac{\partial}{\partial u}\Big)\,J(t,u)=G(t,u), \quad (6.11)$$

say, and in the left half

$$F_{12}(t,u+i\pi,i\pi,c)=e^{-i\pi\gamma}\,S_1\Big(-\frac{\partial}{\partial u}\Big)\,I(t,u)=e^{-i\pi\gamma}\,H(t,u), \quad \text{say.} \quad (6.12)$$

Similarly, when $c'=-c$, $c''=-i\pi$, we have $U_1(\lambda)=U_2(\lambda)=-e^{-i\pi\lambda}$, and

$$F_{12}(t,u,-c,-i\pi)=e^{i\pi\gamma}\,S_1\Big(-\frac{\partial}{\partial u}\Big)\,I(t,u+i\pi)+S_2\Big(-\gamma-\frac{\partial}{\partial u}\Big)\,J(t,u+i\pi), \quad (6.2)$$

so that $$F_{12}(t, u-i\pi, -c, -i\pi) = G(t, u) \quad \text{on the right,} \quad (6.21)$$

$$= e^{i\pi\gamma} H(t, u) \quad \text{on the left.} \quad (6.22)$$

Now F_{12} is a regular function of u in the partial strip, and a path in the strip from $\pm i\pi+u$ in the right half-plane to $\pm i\pi-u$ in the left changes u into $ue^{\mp i\pi}$, so that, when u is positive,

$$e^{-i\pi\gamma} H(t, -u) = G(t, ue^{-i\pi}) \quad (6.13)$$

and $$e^{i\pi\gamma} H(t, -u) = G(t, ue^{i\pi}). \quad (6.23)$$

Thus $G(t, u)u^{-\gamma}$ and $H(t, u)u^{-\gamma}$ are uniform functions of u near the origin; they are, in fact, the same function except for a constant multiplier, but the primary domain for G is the right half-plane and for H the left; this function of u has no singularity in the strip between $\pm(i\pi-c)$, except a pole at $t\pm i\pi$ if $t\pm i\pi$ is in that strip. As a function of t, $G(t, u)$ has the pole at $u\pm i\pi$ and no other within the primary strip; in the domain of $\pm i\pi$ it has a regular part and also terms in $(t\mp i\pi)^{-\gamma}$, $(t\mp i\pi)^{-\gamma+1}$, ...

The value of F in any other partial strip may be expressed in terms of G or H. We have, in the right half of the u-plane,

$$F(t, u, c', c'') = T_1\left(-\frac{\partial}{\partial u}\right) G(t, u) = -U_1\left(-\frac{\partial}{\partial u}\right) G(t, u), \quad (6.31)$$

but in the left half

$$F(t, u, c', c'') = T_2\left(-\gamma-\frac{\partial}{\partial u}\right) H(t, u) = -U_2\left(-\gamma-\frac{\partial}{\partial u}\right) H(t, u). \quad (6.32)$$

When u crosses the bounding line of the strip, the meaning of F changes. From (5.4),

$$F(t, u, c', c'') = \Phi_1(u)+\Phi_2(u)+\mathrm{X}_3(u)+\mathrm{X}_4(u), \quad (6.41)$$

$$F(t, u, c'', c''') = \Phi_1(u)+\Phi_2(u)+\Psi_3(u)+\Psi_4(u), \quad (6.42)$$

where

$$\Phi_1(u) = -U_2\left(-\gamma-\frac{\partial}{\partial u}\right) U_1\left(-\frac{\partial}{\partial u}\right)\{I(t, u)+J(t, u)\},$$

$$\Phi_2(u) = -U_2'\left(-\gamma-\frac{\partial}{\partial u}\right) T_1\left(-\frac{\partial}{\partial u}\right) I(t, u) - U_1'\left(-\frac{\partial}{\partial u}\right) T_2\left(-\gamma-\frac{\partial}{\partial u}\right) J(t, u),$$

U_1' and U_2' being the expressions which take the places of U_1 and U_2 for

the partial strip between c'' and c''',

$$X_3(u) = -e^{-c''\gamma} C_2'' \left(-\gamma - \frac{\partial}{\partial u}\right) T_1\left(-\frac{\partial}{\partial u}\right) I(t, u-c''),$$

$$X_4(u) = -C_1'' \left(-\frac{\partial}{\partial u}\right) T_2\left(-\gamma - \frac{\partial}{\partial u}\right) J(t, u-c''),$$

$$\Psi_3(u) = e^{-c''\gamma} C_2'' \left(-\gamma - \frac{\partial}{\partial u}\right) U_1\left(-\frac{\partial}{\partial u}\right) I(t, u-c''),$$

$$\Psi_4(u) = C_1'' \left(-\frac{\partial}{\partial u}\right) U_2\left(-\gamma - \frac{\partial}{\partial u}\right) J(t, u-c'').$$

Now, from the form (4.5), $I(t, u)+J(t, u)$ is reduced to zero by

$$S_2\left(-\gamma - \frac{\partial}{\partial u}\right) S_1\left(-\frac{\partial}{\partial u}\right)$$

and it has poles at $t \pm 2i\pi$ with no singularity in the strip between.

$\Phi_1(u)$ in the form written involves $I+J$ with second arguments $u+2i\pi$, ..., $u-2c''$, and it may also be expressed as

$$\left[S_2\left(-\gamma - \frac{\partial}{\partial u}\right) S_1\left(-\frac{\partial}{\partial u}\right) - U_2\left(-\gamma - \frac{\partial}{\partial u}\right) U_1\left(-\frac{\partial}{\partial u}\right)\right]\{I(t, u)+J(t, u)\},$$

in which form it involves second arguments

$$u+i\pi-c', \quad \ldots, \quad u-2i\pi.$$

Thus it has a pole at t, but has no other singularity in the partial strips from t down to $t-2i\pi+2c''$ by the first form, from t up to $t+i\pi+c'$ by the second. Since t is in the primary strip it follows that, apart from the pole at t, $\Phi_1(u)$ has no singularity in the partial strip between $2c''-i\pi$ and c', and therefore none on the parallel through c''.

In $\Phi_2(u)$ the second arguments of I and J are

$$u-i\pi-c''', \quad \ldots, \quad u+i\pi-c',$$

and these are in the primary strip when u is near the parallel through c'', so that $\Phi_2(u)$ is regular along that parallel.

$X_3(u)$ is regular on the parallel through c'' except at c'' itself, where it branches. On the right it is equal to $\Psi_3(u)$, but the function which is $X_3(u)$ on the left above the parallel through c'' is continued as $\Psi_3(u)$ on the left after a clockwise circuit about c''.

$X_3(u+c'')$ involves I with second arguments

$$u-i\pi, \quad \ldots, \quad u-c',$$

and, therefore, when u is near the origin it may be written

$$u^\gamma P(u)+Q(u),$$

where P, Q are uniform.

Then, if u is in the left half-plane,

$$\Psi_3(u+c'') = u^\gamma e^{-2i\pi\gamma} P(u)+Q(u),$$

and, by subtraction,

$$u^\gamma(1-e^{-2i\pi\gamma}) P(u) = -e^{-c''\gamma} C_2'' \left(-\gamma-\frac{\partial}{\partial u}\right) S_1\left(-\frac{\partial}{\partial u}\right) I(t, u)$$

$$= -e^{-c''\gamma} C_2'' \left(-\gamma-\frac{\partial}{\partial u}\right) H(t, u). \qquad (6.51)$$

Thus the part of $X_3(u+c'')$ or $\Psi_3(u+c'')$ that is not uniform near the origin may be written, on the right,

$$-e^{-c''\gamma} C_2'' \left(-\gamma-\frac{\partial}{\partial u}\right) \frac{G(t, u)}{e^{i\pi\gamma}-e^{-i\pi\gamma}}. \qquad (6.52)$$

Similarly, if we write

$$X_4(u+c'') = u^\gamma S(u)+T(u),$$

where S, T are uniform near the origin, we have, in the right half-plane,

$$\Psi_4(u+c'') = u^\gamma e^{2i\pi\gamma} S(u)+T(u)$$

and, by subtraction,

$$u^\gamma(1-e^{2i\pi\gamma}) S(u) = -C_1'' \left(-\frac{\partial}{\partial u}\right) S_2\left(-\gamma-\frac{\partial}{\partial u}\right) J(t, u)$$

$$= -C_1'' \left(-\frac{\partial}{\partial u}\right) G(t, u), \qquad (6.61)$$

and the part of $X_4(u+c'')$ or $\Psi_4(u+c'')$ that is not uniform near the origin is, on the left,

$$C_1'' \left(-\frac{\partial}{\partial u}\right) \frac{H(t, u)}{e^{i\pi\gamma}-e^{-i\pi\gamma}}. \qquad (6.62)$$

A theorem proved by contour integration.

7. Let t have any value within the primary strip, except for the present at $c, c', \ldots$. Let $f(u)$ denote a function of u regular in the primary strip and on its edges except for poles at points $k, \ldots$ with residues $A, \ldots$ and for possible singularities at $\pm i\pi$. The poles $k, \ldots$ are not to fall at $c, c', \ldots$

and the slope δ of the strip is to be so adjusted that none of the points t, k, ... lies on a line parallel to the edges of the strip through $i\pi$, c, c', ..., $-i\pi$.

Let α, β be points on the median line of the primary strip at a sufficiently great distance from the origin; then, as in S.N.M., § 16,

$$\frac{1}{2i\pi}\left[\int_{c''-\beta}^{c''+\alpha}+\int_{c''+\alpha}^{c'+\alpha}-\int_{c'-\beta}^{c'+\alpha}-\int_{c''-\beta}^{c'-\beta}\right]F(t, u, c', c'')f(u)\,du$$

is equal to the sum of the residues of $F(t, u, c', c'')f(u)$ in the parallelogram whose vertices are

$$c''-\beta, \quad c''+\alpha, \quad c'+\alpha, \quad c'-\beta,$$

which, when $\alpha, \beta \to +\infty$, becomes the partial strip between c' and c''.

One of these residues is $-f(t)$ if t falls within the partial strip and another is $AF(t, k)$ if k falls within it. Hence, by summation over all the partial strips,

$$2i\pi f(t)-2i\pi\,\Sigma\,AF(t, k)+\int_{-i\pi+\alpha}^{i\pi+\alpha}F(t, u)\,f(u)\,du-\int_{-i\pi-\beta}^{i\pi-\beta}F(t, u)\,f(u)\,du$$

$$=\int_{i\pi-\beta}^{i\pi+\alpha}F(t, u, i\pi, c)\,f(u)\,du-\ldots-\int_{c'-\beta}^{c''+\alpha}F(t, u, c', c'')\,f(u)\,du$$

$$+\int_{c''-\beta}^{c''+\alpha}F(t, u, c'', c''')\,f(u)\,du-\ldots-\int_{-i\pi-\beta}^{-i\pi+\alpha}F(t, u, -c, -i\pi)\,f(u)\,du, \quad (7.1)$$

where on the left the summation in the second term is over all the poles k, and in the third term the path goes by way of $-c+\alpha$, $-c'+\alpha$, ..., $c+\alpha$, F having different meanings in the different segments, and so also in the fourth term.

On the right the parallel through c'' occurs twice and contributes

$$-\int_{-\beta}^{\alpha}F(t, u+c'', c', c'')f(u+c'')\,du+\int_{-\beta}^{\alpha}F(t, u+c'', c'', c''')\,f(u+c'')\,du, \quad (7.2)$$

the path being the median line, but with a deviation above the origin in the first term, below in the second.

To evaluate this, take $F(t, u)$ in the forms (6.41), (6.42). The parts* $\Phi_1(u)$ and $\Phi_2(u)$, being regular on the parallel through c'', contribute

* If t was on the parallel through c'' it would not lie within one of the partial strips, but the term $2i\pi f(t)$ in (7.1) would be contributed by the term $\Phi_1(u+c'')f(u+c'')$ of the integrands in (7.2), since the path runs below t in the second term, but above t in the first. Hence there is no need to exclude t from such positions, as was done provisionally.

zero. The uniform parts of X_3 and Ψ_3 cancel, and the other parts cancel on the right, so that X_3 and Ψ_3 contribute a loop-integral

$$-\int_{-\beta}^{(0)+} \frac{e^{-c''\gamma}}{e^{i\pi\gamma}-e^{-i\pi\gamma}} C_2'' \left(-\gamma-\frac{\partial}{\partial u}\right) G(t, u) f(u+c'')\, du, \qquad (7.31)$$

and similarly X_4 and Ψ_4 contribute

$$\int_{\alpha}^{(0)+} \frac{1}{e^{i\pi\gamma}-e^{-i\pi\gamma}} C_1'' \left(-\frac{\partial}{\partial u}\right) H(t, u) f(u+c'')\, du, \qquad (7.32)$$

In each of these loop integrals the turn of the path falls within the primary domain of the function, G or H as the case may be, and at the turn the function is to have its primary value.

Now there are two expressions $\omega_r''\{f(u), g(u)\}$ $(r = 1, 2)$, bilinear in f, g and their derivatives, such that

$$\frac{d}{du}\omega_2''\{f(u), g(u)\} = f(u) C_2'' \left(-\gamma-\frac{d}{du}\right) g(u) - g(u) C_2'' \left(-\gamma+\frac{d}{du}\right) f(u),$$

$$\frac{d}{du}\omega_1''\{f(u), g(u)\} = f(u) C_1'' \left(-\frac{d}{du}\right) g(u) - g(u) C_1'' \left(\frac{d}{du}\right) f(u),$$

and thus (7.31) and (7.32) become respectively

$$-\frac{e^{-c''\gamma}}{e^{i\pi\gamma}-e^{-i\pi\gamma}} \int_{-\beta}^{(0)+} G(t, u) C_2'' \left(-\gamma+\frac{d}{du}\right) f(u+c'')\, du$$

$$-e^{-c''\gamma}[\omega_2''\{f(u+c''), H(t, u)\}]_{u=-\beta} \qquad (7.41)$$

and

$$\frac{1}{e^{i\pi\gamma}-e^{-i\pi\gamma}} \int_{\alpha}^{(0)+} H(t, u) C_1'' \left(\frac{\partial}{\partial u}\right) f(u+c'')\, du + [\omega_1''\{f(u+c''), G(t, u)\}]_{u=\alpha}. \qquad (7.42)$$

The other terms on the right in (7.1) are

$$\int_{-\beta}^{\alpha} F(t, u+i\pi, i\pi, c) f(u+i\pi)\, du - \int_{-\beta}^{\alpha} F(t, u-i\pi, -c, -i\pi) f(u-i\pi)\, du, \qquad (7.5)$$

and in the first the path is below the origin, in the second above.

In the first we have

$$F(t, u+i\pi) = G(t, u) = e^{-i\pi\gamma} H(t, u) = \{H(t, u) - e^{-i\pi\gamma} G(t, u)\}/\{e^{i\pi\gamma}-e^{-i\pi\gamma}\},$$

the last being an expression which vanishes on a path above the origin, so

that nothing is changed if we subtract from the first term of (7.5) the vanishing expression

$$\int_{-\beta}^{\alpha} \frac{H(t,u)-e^{-i\pi\gamma}G(t,u)}{e^{i\pi\gamma}-e^{-i\pi\gamma}} f(u+i\pi)\,du, \tag{7.51}$$

in which the path is taken above the origin; in this vanishing expression the values assigned to f for arguments outside the primary strip are immaterial, since the other factor of the integrand vanishes. The first term of (7.5) is thus reduced to

$$\frac{1}{e^{i\pi\gamma}-e^{-i\pi\gamma}}\int_{\alpha}^{(0)+} H(t,u)f(u+i\pi)\,du-\frac{e^{-i\pi\gamma}}{e^{i\pi\gamma}-e^{-i\pi\gamma}}\int_{-\beta}^{(0)+} G(t,u)f(u+i\pi)\,du, \tag{7.52}$$

the primary value being taken for G or H at the turn of the path in either case.

In the second part of (7.5) we take

$$F(t,u-i\pi)=G(t,u)=e^{i\pi\gamma}H(t,u)=\frac{e^{i\pi\gamma}G(t,u)-H(t,u)}{e^{i\pi\gamma}-e^{-i\pi\gamma}},$$

which vanishes on a path below the origin, so that by subtracting a vanishing expression we reduce the second part of (7.5) to

$$-\frac{1}{e^{i\pi\gamma}-e^{-i\pi\gamma}}\int_{\alpha}^{(0)+} H(t,u)f(u-i\pi)\,du+\frac{e^{i\pi\gamma}}{e^{i\pi\gamma}-e^{-i\pi\gamma}}\int_{-\beta}^{(0)+} G(t,u)f(u-i\pi)\,du. \tag{7.53}$$

Thus (7.1) becomes*

$$\begin{aligned}2i\pi f(t)-2i\pi\Sigma AF(t,k)+\int_{-i\pi+\alpha}^{i\pi+\alpha} F(t,u)\,f(u)\,du-\int_{-i\pi-\beta}^{i\pi-\beta} F(t,u)\,f(u)\,du\\ -[\Sigma\omega_1''\{f(u+c''),\,G(t,u)\}]_{u=\alpha}+[\Sigma e^{-c''\gamma}\omega_2''\{f(u+c''),\,H(t,u)\}]_{u=-\beta}\\ =\frac{1}{e^{i\pi\gamma}-e^{-i\pi\gamma}}\left\{\int_{\alpha}^{(0)+} H(t,u)\,S_1\left(\frac{d}{du}\right)f(u)\,du-\int_{-\beta}^{(0)+} G(t,u)\,S_2\left(\frac{d}{du}-\gamma\right)f(u)\,du\right\}.\end{aligned} \tag{7.6}$$

* If $\gamma=0$, as in S.N.M., § 16, the expressions must be modified. $G(t,u)\,u^{-\gamma}$ is uniform, and thus in the loop integrals we may write $G(t,u)(1-u^{-\gamma})$ and $H(t,u)(1-u^{-\gamma})$ for $G(t,u)$ and $H(t,u)$. Since

$$\lim_{\gamma\to 0}\frac{1-u^{-\gamma}}{e^{i\pi\gamma}-e^{-i\pi\gamma}}=\frac{1}{2i\pi}\log u,$$

the last expression is to be substituted for $(e^{i\pi\gamma}-e^{-i\pi\gamma})^{-1}$ in the loop integrals when $\gamma=0$. This may be verified if the values of $\log u$, at points such that $u\pm i\pi$ is outside the primary strip, are the same in both loops, so that those parts of the loops will still cancel independently of the values arbitrarily assigned to $f(u)$ at points outside the primary strip.

The right side of (7.6) may also be written as

$$\frac{1}{e^{i\pi\gamma}-e^{-i\pi\gamma}}\left[\int_{-\beta+0i}^{\alpha+0i} G(t,u)\left\{S_2\left(\frac{d}{du}-\gamma\right)-e^{-i\pi\gamma}S_1\left(\frac{d}{du}\right)\right\}f(u)\,du\right.$$
$$\left.-\int_{-\beta-0i}^{\alpha-0i} G(t,u)\left\{S_2\left(\frac{d}{du}-\gamma\right)-e^{i\pi\gamma}S_1\left(\frac{d}{du}\right)\right\}f(u)\,du\right], \quad (7.61)$$

in which form it does not involve any arguments for f that are outside the primary strip.

A third form, corresponding to S.N.M., (16.8), can be deduced if the circles about the origin may be diminished without limit, as, for instance, if $R\gamma > -1$ and $f(u)$ is regular at $\pm i\pi$. The right side of (7.6) may then be written

$$\int_{-\beta}^{0} H(t,u)\,S_2\left(\frac{d}{du}-\gamma\right)f(u)\,du+\int_0^{\alpha} G(t,u)\,S_1\left(\frac{d}{du}\right)f(u)\,du. \quad (7.62)$$

The consequences of the equation (7.6) are like those of S.N.M. (16.8) and need not be further discussed here, except the integral equations whose solution is made possible.

Making α and β infinite, we have

$$2i\pi f(t)-2i\pi\Sigma AF(t,k)$$
$$=\int_{-\infty}^{0} H(t,u)\,S_2\left(\frac{d}{du}-\gamma\right)f(u)\,du+\int_0^{\infty} G(t,u)\,S_1\left(\frac{d}{du}\right)f(u)\,du; \quad (7.7)$$

provided that, in addition to the conditions already stated, $f(z)e^{-\lambda_0 z}$, $f(-z)e^{z(\gamma+\mu_1)}$ and their derivatives, as far as they occur, tend to zero uniformly in the strip when $z\to+\infty$. The arguments of f are generally inside the strip, except $u\pm i\pi$, which are generally on the boundary; these are to be taken as approached from the inside of the strip.

The integral equations.

8. Let $f(t)$ be $e^{\gamma' t}G'(t,v)$, where G' is formed as G was with a constant γ' and operators S_1' and S_2'. Take $1+\gamma'$, $1+\gamma$, and $1\pm(\gamma'-\gamma)$ to have positive real parts. Then near the origin $G'(u,v)$ may have terms regular in u as well as multiples of $u^{-\gamma'}$, $u^{-\gamma'+1}$, ... and $G(t,u)$ is $O(u^{\gamma})$, so that the loops near the origin are unnecessary.

When u tends towards ∞ in the right half-plane, $e^{\nu u}G'(u,v)\,G(t,u)$ and its derivatives are $O(e^{(\nu+\lambda_1'-\lambda_0)u})$, and in the left half-plane they are $O(e^{(\nu+\mu_0'+\gamma'-\mu_1-\gamma)u})$.

The integrals are therefore convergent if ν is such that

$$(\nu+\lambda_1`-\lambda_0)e^{i\delta} \quad \text{and} \quad (-\nu+\mu_1+\gamma-\mu_0`-\gamma`)e^{i\delta}$$

have negative real parts, and such a choice of ν can be made in ordinary cases*, that is, when P_{12}, $P_{12}`$ are straight lines, since $P_{12}$ is assumed to separate $\mu_1+\gamma$ on its left from $\lambda_0$ on its right and $P_{12}`$ to separate $\lambda_1`$ on its left from $\mu_0`+\gamma`$ on its right.

Hence, when ν has been so chosen, if v is above the median, so that the pole of $G`$ is $v-i\pi$, (7.7) gives

$$2i\pi e^{\nu t}G`(t, v)-2i\pi e^{\nu(v-i\pi)}G(t, v)$$

$$=\int_{-\infty}^{0} H(t, u)\, S_2\left(\frac{\partial}{\partial u}-\gamma\right)\{e^{\nu u}G`(u, v)\}\,du+\int_{0}^{\infty} G(t, u)\, S_1\left(\frac{\partial}{\partial u}\right)\{e^{\nu u}G`(u, v)\}\,du. \tag{8.1}$$

Since $H(t, u)$ is the same function as $G(t, u)$ save for a constant factor, (8.1) shows $G(t, u)$ as satisfying an integral equation, and, in fact, a class of integral equations; the parameters involved in $G`$ may be varied with much freedom.

For definiteness we have taken v above the median, so that $v+i\pi$, although a pole of $G`$, is ignored as being outside the primary strip; if $G$, $G`$ are to be uniform functions of their second arguments, a cut must be made from the origin, say in the third quadrant just below the median. Accordingly $H(t, u)=e^{-i\pi\gamma}G(t, u)$.

Then, if κ is a constant, $e^{\kappa(t-v)}G(t, v)$ is a value of $\phi(v)$ satisfying the integral equation

$$\phi(v)+f(v)=\int_{-\infty}^{\infty} K`(u, v)\,\phi(u)\,du, \tag{8.2}$$

where
$$f(v)=-e^{\kappa(t-v)+\nu(t-v+i\pi)}G`(t, v) \tag{8.21}$$

and
$$2i\pi K`(u, v)=-e^{\kappa(u-v)}S_1\left(\frac{\partial}{\partial u}\right)\{e^{\nu(u-v+i\pi)}G`(u, v)\}, \tag{8.22}$$

when u runs from 0 to $+\infty$, but

$$2i\pi K`(u, v)=-e^{\kappa(u-v)-i\pi\gamma}S_2\left(\frac{\partial}{\partial u}-\gamma\right)\{e^{\nu(u-v+i\pi)}G`(u, v)\} \tag{8.23}$$

when u runs from $-\infty$ to 0.

* For exceptional cases, see § 10 below.

To bar circuits for t, u, v about the origin and each other we shall take the first arguments of G, G' to be below the median; points on the median itself are possible positions for either argument, but when u is on the median the values $u-i\pi$ for the first argument and $u+i\pi$ for the second are to be treated as not belonging to the primary strip. Hence we put

$$K'(u,v)=\frac{1}{2i\pi}\left\{e^{-i\pi\nu}S_1'\left(\frac{\partial}{\partial u}-\kappa-\nu\right)-S_1\left(\frac{\partial}{\partial u}-\kappa\right)\right\}\{e^{\kappa(u-v)+\nu(u-v+i\pi)}G'(u,v)\} \tag{8.31}$$

on the part of the range in the right half-plane, but

$$K'(u,v)=\frac{1}{2i\pi}\left\{e^{-i\pi(\nu+\gamma')}S_2'\left(\frac{\partial}{\partial u}-\kappa-\nu-\gamma'\right)-e^{-i\pi\gamma}S_2\left(\frac{\partial}{\partial u}-\kappa-\gamma\right)\right\}\{e^{\kappa(u-v)+\nu(u-v+i\pi)}G'(u,v)\} \tag{8.32}$$

in the rest, or, if $\kappa+\nu=\kappa'$, we may write

$$K'(u,v)=\frac{1}{2i\pi}\left\{e^{-i\pi\kappa'}S_1'\left(\frac{\partial}{\partial u}-\kappa'\right)-e^{-i\pi\kappa}S_1\left(\frac{\partial}{\partial u}-\kappa\right)\right\}\{e^{\kappa'(u-v+i\pi)}G'(u,v)\} \tag{8.33}$$

in the right half of the u-plane, but

$$K'(u,v)=\frac{1}{2i\pi}\left\{e^{-i\pi(\kappa'+\gamma')}S_2'\left(\frac{\partial}{\partial u}-\kappa'-\gamma'\right)-e^{-i\pi(\kappa+\gamma)}S_2\left(\frac{\partial}{\partial u}-\kappa-\gamma\right)\right\}\{e^{\kappa'(u-v+i\pi)}G'(u,v)\} \tag{8.34}$$

in the left.

Then, if K is derived from K' by interchanging G, S, γ, κ with G', S', γ', κ', it follows from (8.1) that

$$K(t,v)+K'(t,v)=\int_{-\infty}^{\infty}K(t,u)\,K'(u,v)\,du, \tag{8.4}$$

and similarly

$$K(t,v)+K'(t,v)=\int_{-\infty}^{\infty}K'(t,u)\,K(u,v)\,du, \tag{8.5}$$

so that of the pair K, K' each is a *solving nucleus* for the other, if convergence is secured by the conditions that

$$(\kappa'+\lambda_1'-\kappa-\lambda_0)e^{i\delta},\quad(\kappa+\lambda_1-\kappa'-\lambda_0')e^{i\delta},$$

$$(\kappa+\mu_1+\gamma-\kappa'-\mu_0'-\gamma')e^{i\delta},\quad(\kappa'+\mu_1'+\gamma'-\kappa-\mu_0-\gamma)e^{i\delta} \tag{8.6}$$

have negative real parts, and the conditions already stated that the real parts of $1+\gamma$, $1+\gamma`$, $1\pm(\gamma-\gamma`)$ are positive.

The nuclei K, $K`$ in (8.4), (8.5) are formed from $G$, $G`$ as in (8.33), (8.34). $G(t, v)$ is a function of t, satisfying the functional equations (2.61), (2.62) in the two halves of the plane and having in the primary strip one simple pole at $v \pm i\pi$ with residue unity. The formation of this function by means of σ_{12} is explained in §§ 4–6.

The path of u is supposed to separate t on its right from v on its left.

Case of a rational algebraic nucleus.

9. An important particular case is that in which

$$S_1`(\lambda) = S_2`(\lambda) = e^{i\pi\lambda} - e^{-i\pi\lambda}, \quad \gamma` = 0, \quad \kappa` = 0.$$

Thus $G`(t, v) = e^t/(e^t+e^v)$ and

$$K`(u, v) = \frac{1}{2i\pi}\left\{\frac{(1-e^{-2i\pi\kappa})\,e^u}{e^u-e^v} - \Sigma\, C_1\left(\frac{\partial}{\partial u}-\kappa\right)\frac{e^{u-\kappa c-i\pi\kappa}}{e^u+e^{v-c}}\right\}$$

in the right half-plane, the expression in the left half-plane being the same with C_2 for C_1 and $\kappa+\gamma$ for κ.

If x, y, z are written for e^t, e^u, e^v, the new nucleus is $K`(u, v)e^{-u}$ or, on the right,

$$\frac{1}{2i\pi}\left\{\frac{1-e^{-2i\pi\kappa}}{y-z} + \Sigma\, C_1\left(y\,\frac{\partial}{\partial y}-\kappa+1\right)\frac{e^{-\kappa c-i\pi\kappa}}{y+ze^{-c}}\right\},$$

Then $K(\log x, \log y)/y$ is the solving function for an integral equation in which the nucleus is equal to two different rational algebraic fractions in the two parts $(0, 1)$, $(1, \infty)$ of the range.

The fractions are homogeneous functions of degree -1 and may be any functions whatever, provided that in the denominators the factors y, z do not occur at all and the factor $y-z$ occurs only to the first degree.

If no factor $y-kz$ occurs in which k is real and positive, then in all the terms of ΣCe^{cz}, (1.1), $|I(c)| < \pi$ and δ may be zero, so that the median of the primary strip may be taken to be the real axis. But if such a factor $y-kz$ does occur, so that, for instance, $I(c) = \pi$, then the functions $\sigma_{12}(t)$, $G(t, u)$ have singularities

$$2i\pi-c, \quad 3i\pi-2c, \quad 4i\pi-3c, \quad \ldots;$$

to exclude these we take the strip at a positive slope δ, and the corresponding path for y or e^u as a spiral cutting its radii at a constant angle δ.

At such singularities the expansions of § 3 would cease to converge; in fact, if C_2 was not zero, there would be a sequence of zeros of $S_2(\lambda)$ tending to infinity at amplitude $-\frac{1}{2}\pi$ and, in the expansion (3.1), $e^{t(\mu_s+\gamma)}$ would not tend to zero as s increased if t were not below the real axis. Also the first term would not be dominant in the expansion.

If in each part of the range the denominator of the nucleus is y^n-z^n, then σ_{12} is the hypergeometric function of order n, as in § 3.

Cases of exception.

10. If $e^{vt}G(t, v)$ is too large at infinity for convergence, take

$$f(t) = e^{vt}\{G(t, v)+A_0\sigma_{12}(t)+A_1\sigma'_{12}(t)+\ldots\},$$

choosing the coefficients $A_0, A_1, \ldots$ so that the offending terms are destroyed. This secures convergence at infinity but introduces new singular terms at $\pm i\pi$. Thus cases arise where the solution of the integral equation fails unless additional conditions are fulfilled. Compare S.N., (14.7).

In other cases the inequalities (8.6) leave some liberty with regard to the separation among the zeros of S_1, S_2 to be effected by P_{12}, and the solution of the integral equations will contain a complementary function.

AN EXTENDED FORM OF DIRICHLET'S DIVISOR PROBLEM

By J. R. WILTON.

[Received 15 August, 1931.—Read 12 November, 1931.]

1. *Introduction and notation.*

The problem with which I am concerned in the present paper is to some extent classical, but, in its general form, it seems first to have been considered, quite recently, by Oppenheim*.

Let s be a complex number, $s=\sigma+it$, $|t|<A$, where, as throughout the paper, A denotes a positive absolute constant, not the same at different occurrences; and let

$$\sigma_s(x)=\sum_{d|x} d^s \quad \text{when } x \text{ is a positive integer,} \tag{1.11}$$

$$\sigma_s(x)=0 \qquad \text{otherwise.} \tag{1.12}$$

We consider the summatory function†

$$D_s^{(0)}(x)=\sum_{n\leqslant x}{}' \sigma_s(n)=\sum_{n\leqslant x}\sigma_s(n)-\tfrac{1}{2}\sigma_s(x), \tag{1.13}$$

and the integrated function of order λ (>-1),

$$D_\lambda(x)=D_s^{(\lambda)}(x)=\frac{1}{\Gamma(\lambda+1)}\sum_{n\leqslant x}{}' \sigma_s(n)(x-n)^\lambda \tag{1.14}$$

$$=\frac{1}{\Gamma(\lambda+1)}\sum\sum_{hk\leqslant x}{}' h^s(x-hk)^\lambda.$$

The function

$$\Phi_\lambda(x)=\Phi_s^{(\lambda)}(x) \tag{1.15}$$

$$=\frac{x^{s+\lambda+1}}{\Gamma(s+\lambda+2)}\Gamma(s+1)\zeta(s+1)-\tfrac{1}{2}\frac{x^\lambda}{\Gamma(\lambda+1)}\zeta(-s)+R_\lambda(x),$$

where

$$R_\lambda(x)=2\sum_{m=0}^{[\frac{1}{2}\lambda+\frac{1}{2}]}(-)^{m-1}\frac{x^{\lambda-2m+1}}{\Gamma(\lambda-2m+2)}\frac{\zeta(2m)}{(2\pi)^{2m}}\zeta(-2m+1-s), \tag{1.151}$$

---

* 11. (The reference is to the list of books and papers at the end.)

† Throughout the paper h, j, k, l, m, n, p are integers, and $n\geqslant 1$.

is defined* for all values of s when λ is an integer, and for all values of s in the half plane $\sigma > -\eta-1$, where

$$\eta = 1+2[\tfrac{1}{2}\lambda+\tfrac{1}{2}], \tag{1.152}$$

when λ is not an integer. If λ is not an integer, and $\sigma \leqslant -\eta-1$, let

$$\Phi_\lambda(x) = R_\lambda(x)-\tfrac{1}{2}\{x^\lambda/\Gamma(\lambda+1)\}\zeta(-s). \tag{1.153}$$

Further, let

$$\Delta_\lambda(x) = \Delta_s^{(\lambda)}(x) = D_\lambda(x)-\Phi_\lambda(x), \tag{1.16}$$

where $\Phi_\lambda(x)$ is defined by (1.15) or (1.153), as the case may be. The ζ-function is, of course, Riemann's function defined by $\zeta(z) = \Sigma n^{-z}$ when $Rz > 1$, and we have

$$\zeta(z)\,\zeta(z-s) = \Sigma\,\sigma_s(n)\,n^{-z} \quad (Rz > 1,\ \ Rz > 1+\sigma). \tag{1.17}$$

I am concerned chiefly with an exact representation of $\Delta_\lambda(x)$ by an infinite series, the terms of which are multiples of Bessel functions if $\lambda\ (\geqslant 0)$ is an integer, and are otherwise slight extensions of Bessel functions with very similar properties. The approximate representation of $\Delta_\lambda(x)$, as $x \to \infty$, by elementary functions of x is only incidentally required, and I prove from first principles the little that I use in this field.

Let

$$\rho = \lambda+s+1, \tag{1.21}$$

and, when λ is an integer, let

$$G_\rho(z) = -J_\rho(z)\sin\tfrac{1}{2}s\pi-\{Y_\rho(z)+(-)^\lambda(2/\pi)\,K_\rho(z)\}\cos\tfrac{1}{2}s\pi, \tag{1.22}$$

where Y_ρ, K_ρ are the Bessel functions of the second and third kinds, as defined in Watson's standard treatise†. Then, if $l = 0$,

$$G_\rho(z)\sin\tfrac{1}{2}s\pi = -\sum_{m\geqslant l}\frac{(\frac{1}{2}z)^{\rho+4m}}{(2m)!\,\Gamma(\rho+2m+1)} + \sum_{m\geqslant -\frac{1}{2}\lambda-\frac{1}{2}}\frac{(\frac{1}{2}z)^{-s+\lambda+4m+1}}{\Gamma(2m+\lambda+2)\,\Gamma(-s+2m+1)}. \tag{1.23}$$

For non-integral values of λ we define $G_\rho(z)$ by (1.23), where

$$l = \mathrm{Max}([-\tfrac{1}{2}\sigma]-[\tfrac{1}{2}\lambda+\tfrac{1}{2}],\ [\tfrac{1}{2}k+\tfrac{1}{2}]), \tag{1.24}$$

* As a limit, if s is zero or a negative integer. It will save constant repetition if it is understood, once for all, that every function of s which occurs in the paper is defined as a limit when s is integral.

† 16, 64 and 78.

and $k=0$. The choice of l secures the existence of the limit as s tends to a negative even integer (from the left). Since $\lambda > -1$ we have $l=0$ if $\sigma > -2$.

The principal theorem of the paper may now be stated, as follows:

THEOREM 4. *If $x>0$, ρ is given by* (1.21), *$G_\rho(z)$ by* (1.22) *or* (1.23) *according as λ is or is not an integer, and if $\lambda > -\frac{1}{2}$* when $|\sigma|<1$, and otherwise $\lambda > -1$, then*

$$(1.3) \qquad (2\pi)^\lambda \Delta_\lambda(x) \sim \sum_{n=1}^{\infty} \left(\frac{x}{n}\right)^{\frac{1}{2}\rho} \sigma_s(n)\, G_\rho\{4\pi\sqrt{(nx)}\},$$

where the series on the right of (1.3)

(i) *is absolutely convergent if $\lambda > |\sigma| + \frac{1}{2}$;*

and, provided that, when λ is not an integer, $\sigma > -\eta$, and that, when $\lambda < 0$, x is not an integer, the series (1.3)

(ii) *is conditionally convergent, if $|\sigma| - \frac{1}{2} < \lambda \leqslant |\sigma| + \frac{1}{2}$,*

(iii) *is summable $(R, n, q > |\sigma| - \frac{1}{2} - \lambda)$, if $\lambda \leqslant |\sigma| - \frac{1}{2}$,*

(iv) *is not summable $(R, n, |\sigma| - \frac{1}{2} - \lambda)$;*

when λ is not an integer and $\sigma \leqslant -\eta$, the series (1.3)

(v) *is not summable (R, n, q) for any q.*

If $\lambda > 0$, $A < x < A$, convergence and summability are uniform; and if $-1 < \lambda \leqslant 0$, $A < x < A$, convergence and summability are uniform in the set I of intervals

$$(1.31) \qquad m + \vartheta \leqslant x \leqslant m + 1 - \vartheta \quad (\tfrac{1}{2} > \vartheta > 0,\ m = 0, 1, 2, \ldots).$$

The condition $\sigma > -\eta$ confines attention to the case $l=0$.

Theorem 1 is the part of the above theorem which relates to absolute convergence when λ is an integer; Theorem 2 gives an explicit expression for the partial sums of (1.3); and Theorem 3 determines the order of magnitude of these partial sums $(n \leqslant N)$ as x and N tend independently to infinity. Theorems 2 and 3 are new except for the particular case $s=0$, which I have given elsewhere†.

Theorem 4, when λ is an integer and s is real, with the exception of clause (iv), is due to Oppenheim‡, who uses complex analysis. Oppenheim's

* See foot-note * on p. 424.

† 20.

‡ 11, Theorems 1, 5, and 5′.

theorems include also the case of negative integral values* of λ, in which my analysis fails. References to earlier work, all of it concerned with the special cases $s = -1, 0, 1$, are given by Oppenheim in the paragraphs following the statement of his Theorem 1 on page 296, and Theorem 5 on page 299. A few later references may be added to these†.

Throughout the paper I employ real analysis, but the method cannot be said to be entirely real, for, as a labour-saving device, I make an occasional appeal to the principle of analytic continuation; and, further, the only published proof of Lemma 2, the integral on which the proof of Theorem 2 is based, is by complex analysis‡. The proof of Theorem 1 follows closely the first section of Landau's paper just referred to§; and the proofs of Theorems 2, 3, and 4 resemble those of corresponding theorems, in the related lattice-points problem, which I have given recently‖. But I assume no knowledge of these papers, nor of any earlier work on the subject.

In Theorems A and A′, a backwater from the main current of the argument, I give integral equations for $\Delta_s^{(\lambda)}(x)$ in the case in which λ is an integer¶.

2. *The case of absolute convergence.*

THEOREM 1. *If*

$$\lambda = \kappa = [|\sigma| + \tfrac{3}{2}] \quad \textit{so that} \quad \rho = \kappa + s + 1, \tag{2.11}$$

---

* **11**, Theorem 3, and the theorem stated in §7.8 on p. 332. There would presumably be no great difficulty in extending my analysis to include the case $\lambda \leqslant -1$; one would consider (3.31), for instance, as a Cesàro or Riesz mean $(R, n, q > -\lambda - 1)$. But the fact that, when $\frac{1}{2} \leqslant |\sigma| < 1$, it is necessary to suppose that $\lambda > -\frac{1}{2}$ shows that other difficulties would be encountered besides that of the mere interpretation of a summable integral.

† Walfisz, **13** and **15**. These papers contain, as very special cases of a much more general theorem, so much of Theorem 4 as is applicable when $s = -1$, 0, 1 and λ $(\geqslant -1)$ is an integer.

Landau, **10**. This paper contains, in §§1 and 2, simple proofs by real analysis when $s = 0$ and $\lambda = 1$ or $\lambda = 0$.

Wilton, **20**. A new proof by real analysis when $s = 0$, $\lambda = 0$.

Dixon and Ferrar, **5**. In this paper (which reached me after my present paper was complete, but before it was written out for publication) fractional integrals, in the case $s = 0$, are for the first time considered, and some properties (when $s = 0$) of the functions defined in (1.23) are derived. I have made no change in my presentation, which develops only the minimum, relative to $G_\rho(z)$, that is required for the proof of Theorems 2-4. Dixon and Ferrar prove Theorem 4, as far as it is applicable when $s = 0$.

‡ Dixon and Ferrar, **4**.

§ **10**, Satz 10.

‖ **18**, for the analogues of Theorems 2 and 4; **19**, for the analogue of Theorem 3.

¶ The case $s = 0$, $\lambda = 0$ is due to Hardy, **7**.

then

$$(2.12)\qquad (2\pi)^{\kappa}\Delta_{\kappa}(x)=\sum_{n=1}^{\infty}\left(\frac{x}{n}\right)^{\frac{1}{2}\rho}\sigma_{s}(n)\,G_{\rho}\{4\pi\sqrt{(nx)}\},$$

and the series is absolutely convergent.

I assume, in the first place, that σ is negative, and that

$$(2.13)\qquad \tfrac{1}{2}<\kappa+\sigma<1.$$

Then, from (1.14), we have, by the Euler-Maclaurin sum-formula*,

$$(2.21)\quad \kappa!\,D_{s}^{(\kappa)}(x)=\sum_{h\leqslant x}h^{s}\sum_{k\leqslant x/h}(x-hk)^{\kappa}$$

$$=\sum_{h\leqslant x}h^{s}\left\{-\tfrac{1}{2}x^{\kappa}+\int_{0}^{x/h}(x-hv)^{\kappa}\,dv+2\sum_{k=1}^{\infty}\int_{0}^{x/h}(x-hv)^{\kappa}\cos 2\pi kv\,dv\right\}$$

$$=\sum_{\delta\leqslant h\leqslant x/\delta}h^{s}\left\{-\tfrac{1}{2}x^{\kappa}+\frac{x^{\kappa+1}}{(\kappa+1)h}+\sum_{k=1}^{\infty}\frac{(\frac{1}{2}h)^{\kappa}\kappa!}{(\pi k)^{\kappa+1}}\left[\frac{w^{\kappa-1}}{\kappa-1!}-\frac{w^{\kappa-3}}{\kappa-3!}+\cdots\right.\right.$$

$$\left.\left.-w\cos\tfrac{1}{2}\kappa\pi+\sin\tfrac{1}{2}\kappa\pi+\sin(w-\tfrac{1}{2}\kappa\pi)\right]\right\},$$

where $w=2\pi kx/h$ and $0<\delta<1$. In (2.21) we have used the fact that $\sum_{k\leqslant x/h}$ is null if $h>x$. By (2.21), (2.13), and a further use of the Euler-Maclaurin sum-formula in the case of the sum

$$\sum_{h\leqslant x/\delta}h^{\kappa+s}\{\sin(w-\tfrac{1}{2}\kappa\pi)+\sin\tfrac{1}{2}\kappa\pi\}$$

(in which we note that the summand vanishes when $h=0$), we have†, as $\delta\to 0$,

$$D_{\kappa}(x)=-\tfrac{1}{2}\frac{x^{\kappa}}{\kappa!}\left\{\zeta(-s)+\frac{x^{1+s}}{1+s}\,\delta^{-1-s}\right\}+\frac{x^{\kappa+1}}{\kappa+1!}\,\zeta(1-s)+o(1)$$

$$(2.22)\quad +2\sum_{k=1}^{\infty}\left\{\frac{x^{\kappa-1}}{\kappa-1!}\frac{\zeta(-1-s)}{(2\pi k)^{2}}-\cdots-\frac{x\cos\frac{1}{2}\kappa\pi}{(2\pi k)^{\kappa}}\,\zeta(1-\kappa-s)\right\}$$

$$(2.23)\quad +2\sum_{k=1}^{\infty}\frac{1}{(2\pi k)^{\kappa+1}}\left[\int_{0}^{x/\delta}u^{\kappa+s}\left\{\sin\left(\frac{2\pi kx}{u}-\tfrac{1}{2}\kappa\pi\right)\right.\right.$$

$$\left.\left.+\sin\tfrac{1}{2}\kappa\pi-\frac{2\pi kx}{u}\cos\tfrac{1}{2}\kappa\pi\right\}du\right.$$

$$(2.24)\quad \left.+2\sum_{h=1}^{\infty}\int_{0}^{x/\delta}u^{\kappa+s}\left\{\sin\left(\frac{2\pi kx}{u}-\tfrac{1}{2}\kappa\pi\right)+\sin\tfrac{1}{2}\kappa\pi\right\}\cos 2\pi hu\,du\right],$$

* Due, in this form, to Dirichlet, 2, 474–475, Equations (1) and (2).

† But not uniformly with respect to x.

where the last term of (2.23) arises from the fact that

$$\sum_{h<x/\delta} h^{\kappa+s-1} - \int_0^{x/\delta} u^{\kappa+s-1}\,du = \zeta(1-\kappa-s)+o(1),$$

and δ^{-1-s} is without significance unless $\kappa = 1$.

In (2.23) put $x/u = w$, and integrate by parts; we obtain

$$(2.25)\quad \frac{2x^{\kappa+s+1}}{\kappa+s+1}\sum_{k=1}^{\infty}\left\{\frac{\sin(2\pi k\delta-\frac{1}{2}\kappa\pi)+\sin\frac{1}{2}\kappa\pi-2\pi k\delta\cos\frac{1}{2}\kappa\pi}{(2\pi k)^{\kappa+1}\delta^{\kappa+s+1}}\right.$$

$$(2.26)\qquad \left.+\int_\delta^\infty \frac{\cos(2\pi kw-\frac{1}{2}\kappa\pi)-\cos\frac{1}{2}\kappa\pi}{(2\pi k)^\kappa}\,\frac{dw}{w^{\kappa+s+1}}\right\},$$

and this, when $\kappa = 1$, is equal to

$$\frac{2x^{s+2}}{s+2}\sum_{k=1}^{\infty}\left\{\frac{1-\cos 2\pi k\delta}{(2\pi k)^2}\,\delta^{-s-2}+\frac{\sin 2\pi k\delta}{2\pi k}\,\frac{\delta^{-s-1}}{s+1}+\frac{1}{s+1}\int_\delta^\infty \frac{\cos 2\pi kw}{w^{s+1}}\,dw\right\}$$

$$=\frac{\frac{1}{2}x^{s+2}}{s+1}\,\delta^{-1-s}+\frac{x^{s+2}}{(s+1)(s+2)}\lim_{n\to\infty}\left(\sum_1^n m^{-1-s}-\int_\delta^n \frac{dw}{w^{s+1}}\right)+o(1),$$

by Poisson's summation formula; hence, when $-\frac{1}{2}<\sigma<0$, and $\kappa = 1$,

$$(2.27)\qquad \Delta_s^{(\kappa)}(x)+\frac{2\zeta(\kappa+1)}{(2\pi)^{\kappa+1}}\,\zeta(-\kappa-s)\sin\tfrac{1}{2}\kappa\pi = (2.24)+o(1).$$

When $\kappa > 1$ (2.25) is easily seen, by actual summation*, to be $o(1)$, and (2.26) is equal to

$$\frac{x^{\kappa+s+1}}{\kappa+s+1}\,2(2\pi)^s\zeta(-s)\,\Gamma(-\kappa-s)\cos(\kappa+\tfrac{1}{2}s)\pi+O(\delta^{1-\kappa-\sigma})$$

$$=\frac{x^{\kappa+s+1}}{\Gamma(\kappa+s+2)}\,\Gamma(s+1)\,\zeta(s+1)+o(1),$$

so that (2.27) is true in general, if σ satisfies (2.13).

It remains to consider (2.24). We show first that the upper limit of integration may be replaced by infinity, with an error $O(\delta^{1-\kappa-\sigma})$. To this end consider the series of integrals from x/δ to ∞, put $x/u = w$, and integrate by parts; we obtain a multiple of

$$\sum_{k=1}^{\infty}\frac{1}{k^{\kappa+1}}\left\{\frac{\sin(2\pi k\delta-\frac{1}{2}\kappa\pi)+\sin\frac{1}{2}\kappa\pi}{\delta^{\kappa+s}}\,\psi\left(\frac{x}{\delta}\right)-\int_0^\delta[2\pi kw\cos(2\pi kw-\tfrac{1}{2}\kappa\pi)\right.$$

$$\left.-(\kappa+s)\{\sin(2\pi kw-\tfrac{1}{2}\kappa\pi)+\sin\tfrac{1}{2}\kappa\pi\}]\,\psi(x/w)\,w^{-\kappa-s-1}\,dw\right\}$$

$$= O(\delta^{1-\kappa-\sigma})\quad\text{if}\quad \kappa>1,$$

where

$$\psi(u) = -\sum_{k=1}^{\infty}\frac{\sin 2\pi ku}{\pi k} = u-[u]-\tfrac{1}{2}.$$

* See, for instance, Bromwich, **1**, 256. There are some errors of sign in the formulae there given.

When $\kappa = 1$ we have to examine separately the term

$$\sum_{k=1}^{\infty} \frac{1}{k^2} \int_0^{\delta} \frac{1-\cos 2\pi kw}{w^{2+s}} \psi\left(\frac{x}{w}\right) dw = \sum_{k-1}^{\infty} \frac{1}{k^{1-s}} \int_0^{k\delta} \frac{1-\cos 2\pi w}{w^{2+s}} \psi\left(\frac{kx}{w}\right) dw$$

$$= O\left\{ \sum_{k \leqslant \delta^{-1}} \frac{1}{k^{1-\sigma}} \int_0^{k\delta} w^{-\sigma} dw + \sum_{k > \delta^{-1}} \frac{1}{k^{1-\sigma}} \int_0^{\infty} \frac{1-\cos 2\pi w}{w^{2+\sigma}} dw \right\} = O(\delta^{-\sigma}).$$

From the last result and (2.27) it follows, on making $\delta \to 0$, that, subject to (2.13),

$$(2\pi)^{\kappa} \Delta_s^{(\kappa)}(x) = \frac{4}{(2\pi)^{\kappa+s+2}} \sum_{k=1}^{\infty} \frac{1}{k^{\kappa+1}} \sum_{h=1}^{\infty} \frac{1}{h^{\kappa+s+1}}$$

$$\times \int_0^{\infty} \left\{ \sin\left(\frac{4\pi^2 hkx}{u} - \tfrac{1}{2}\kappa\pi\right) + \sin \tfrac{1}{2}\kappa\pi \right\} u^{\kappa+s} \cos u \, du$$

$$- (1/\pi)\, \zeta(\kappa+1)\, \zeta(-\kappa-s) \sin \tfrac{1}{2}\kappa\pi$$

$$(2.28) \qquad = \sum_{k=1}^{\infty} k^s \sum_{h=1}^{\infty} \frac{x^{\frac{1}{2}\rho}}{(hk)^{\frac{1}{2}\rho}} G_{\rho}\{4\pi \sqrt{(hkx)}\}$$

$$(2.29) \qquad = \sum_{n=1}^{\infty} \left(\frac{x}{n}\right)^{\frac{1}{2}\rho} \sigma_s(n)\, G_{\rho}\{4\pi\sqrt{(nx)}\},$$

where the rearrangement of (2.28) as (2.29) is justified since, by the asymptotic expansions of the Bessel functions, (2.28) is absolutely convergent*. This proves (2.12) when s is in the strip (2.13); and, since, for a given integer κ, both sides of (2.12) are analytic functions of s in the strip

$$(2.291) \qquad \tfrac{1}{2} - \kappa < \sigma < \kappa - \tfrac{1}{2},$$

Theorem 1 is proved. The convergence of (2.12) is uniform with respect to s in the finite part of the strip

$$\tfrac{1}{2} - \kappa + \vartheta \leqslant \sigma \leqslant \kappa - \tfrac{1}{2} - \vartheta \quad (\vartheta > 0).$$

3. *Preliminary lemmas.*

3.1. It would be natural to take (1.23) as the definition of $G_{\rho}(z)$ for variable λ, but owing to the form of the sequence relations this would be very inconvenient; I therefore take (1.23) or (1.22) as defining $G_{\rho}(z)$

* *Cf.* (3.15) below. For the derivation of (2.28) from the previous line see Watson, **16**, 184. (The formulae given by Watson require supplementing to a certain extent.)

for fixed λ, the λ of Theorem 4, and define $G_{\rho+k}(z)$, when k is an integer (positive or negative), by

$$(3.11)\quad (-)^k G_{\rho+k}(z)\sin\tfrac{1}{2}s\pi = -\sum_{m\geqslant l}\frac{(\frac{1}{2}z)^{\rho-k+4m}}{2m-k!\,\Gamma(\rho+2m+1)} + \sum_{m\geqslant -\frac{1}{2}\lambda-\frac{1}{2}}\frac{(\frac{1}{2}z)^{-s-k+\lambda+4m+1}}{\Gamma(2m+\lambda+2)\,\Gamma(-s-k+2m+1)},$$

where l is defined by (1.24), unless λ is an integer, in which case $l=[\frac{1}{2}k+\frac{1}{2}]$.

It is then immediately verifiable that, if $\alpha=\rho+k$,

$$(3.12)\quad \frac{d}{dt}\left[\left(\frac{x}{t}\right)^{\frac{1}{2}\alpha}G_\alpha\{4\pi\sqrt{(xt)}\}\right] = -2\pi\left(\frac{x}{t}\right)^{\frac{1}{2}\alpha+\frac{1}{2}}G_{\alpha+1}\{4\pi\sqrt{(xt)}\}.$$

It is this property which justifies the rather awkward definition (3.11). It will be used repeatedly. We shall also require (but much less frequently) another sequence relation, in which it is convenient to adopt a different notation. I define $F_{k+s+1}(z)$, for every integer k, by the equation

$$(3.13)\qquad F_{k+s+1}(z)=G_\rho(z)\quad\text{when}\quad\lambda=k,$$

i.e. $F_{k+s+1}(z)$ is the result of putting $\lambda=k$ on the right of (1.22). We then have, immediately, if $\gamma=k+s+1$,

$$(3.14)\quad \frac{d}{dt}\left[\left(\frac{t}{n}\right)^{\frac{1}{2}\gamma}F_\gamma\{4\pi\sqrt{(nt)}\}\right] = 2\pi\left(\frac{t}{n}\right)^{\frac{1}{2}\gamma-\frac{1}{2}}F_{\gamma-1}\{4\pi\sqrt{(nt)}\}.$$

Besides the relations (3.12) and (3.14) we shall need an asymptotic expression for $G_\rho(z)$ when z is large. This may be derived from the properties of Lommel's functions* associated with Bessel functions. We observe in the first place that, when λ is an integer, $G_{\rho+k}(z)$ is given by the right-hand side of (1.22) with $\rho+k$ instead of ρ (λ being unchanged), *i.e.*, if $z>A$,

$$(3.15)\qquad G_{\rho+k}(z)=\sqrt{(2/\pi z)}\cos(z-\beta)+O(z^{-\frac{3}{2}}),$$

where $\beta=\frac{1}{2}\pi(\lambda+k+\frac{1}{2})$. It is easy to show, by the process of fractional integration, that (3.15) is true for all values of λ if $\lambda+\sigma+k>\frac{1}{2}$, but it is not easy by this method to obtain a result as definite as (3.15) for other values of k.

Let

$$\alpha=\rho+k=\lambda+s+k+1,\quad \delta=\lambda-[\lambda],\quad \mu=2\delta-\alpha-1;$$

* Watson, **16**, 345–352.

then it is evident that the first sum on the right of (3.11) is equal to

$$-\tfrac{1}{2}I_\alpha(z)-\tfrac{1}{2}(-)^k J_\alpha(z)+(-)^k P_\alpha(z)\sin\tfrac{1}{2}s\pi, \tag{3.16}$$

where

$$(-)^k P_\alpha(z)\sin\tfrac{1}{2}s\pi=\sum_{l>m\geqslant\frac{1}{2}k}\frac{(\frac{1}{2}z)^{\rho-k+4m}}{(2m-k)!\,\Gamma(\rho+2m+1)}; \tag{3.161}$$

and the second sum is closely associated with Lommel's function $S_{\mu,\alpha}(z)$. More precisely, the second sum on the right of (3.11) is equal to*

$$\begin{aligned}-\frac{e^{\frac{1}{2}\pi i(1-\mu)}S_{\mu,\alpha}(ze^{\frac{1}{2}\pi i})+(-)^{[\lambda]}S_{\mu,\alpha}(z)}{2^\mu\,\Gamma(\delta)\,\Gamma(\delta-\alpha)}&\\ +\tfrac{1}{2}I_\alpha(z)-(1/\pi)e^{-i\pi\delta}K_\alpha(z)\sin(\delta-\alpha)\pi&\\ +(-)^{[\lambda]+1}\tfrac{1}{2}\{J_\alpha(z)\cos(\delta-\alpha)\pi+Y_\alpha(z)\sin(\delta-\alpha)\pi\}.&\end{aligned} \tag{3.17}$$

Hence we have

LEMMA 1. *If* $\alpha=\rho+k$, $\delta=\lambda-[\lambda]$, $\mu=2\delta-\alpha-1$, *then*

$$G_\alpha(z)=-J_\alpha(z)\sin\tfrac{1}{2}s\pi-\{Y_\alpha(z)+e^{-i\pi\lambda}(2/\pi)K_\alpha(z)\}\cos\tfrac{1}{2}s\pi \tag{3.18}$$

$$+P_\alpha(z)+(-)^{k+1}\frac{e^{\frac{1}{2}\cdot i(1-\mu)}S_{\mu,\alpha}(ze^{\frac{1}{2}\pi i})+(-)^{[\lambda]}S_{\mu,\alpha}(z)}{2^s\,\Gamma(\delta)\,\Gamma(-s-[\lambda]-k-1)\sin\frac{1}{2}s\pi}; \tag{3.181}$$

and, if $z>A$, $\beta=\frac{1}{2}\pi(\lambda+k+\frac{1}{2})$, *then*

$$G_\alpha(z)=\sqrt{(2/\pi z)}\cos(z-\beta)+O(z^{-\frac{3}{2}}) \tag{3.19}$$

$$+P_\alpha(z)+(-)^{k+1}\frac{z^{\mu-1}\operatorname{cosec}\frac{1}{2}s\pi}{2^\mu\,\Gamma(\delta)\,\Gamma(-s-[\lambda]-k-1)}\times\left[1+(-1)^{[\lambda]}+\frac{(\mu-1)^2-\alpha^2}{z^2}\{1-(-1)^{[\lambda]}\}+O(z^{-4})\right] \tag{3.191}$$

$$=O(z^{-\frac{1}{2}})+O(z^{2\lambda-2\eta-\alpha}|\log z|), \tag{3.192}$$

where η *is defined in* (1.152) *and* $a=R\alpha=\sigma+\lambda+k+1$.

Here (3.191) arises from the asymptotic expansion† of $S_{\mu,\alpha}(z)$; it vanishes when $\delta=0$, so that we then have (3.15), and it is significant in (3.192) only if

$$\sigma\leqslant\lambda-k-\tfrac{1}{2}-2\eta. \tag{3.193}$$

In particular $G_\alpha(z)$ always satisfies (3.15) if $k\geqslant\kappa-1$ $(\lambda>-1)$.

---

* Watson, **16**, 345 and 347. In spite of the appearance of the asymptotic expansion, (3.17) is not real unless (in the case considered) s is an integer.

† Watson, **16**, 352.

In (3.181) $P_\alpha(z)=0$ unless $l\geqslant 1+[\frac{1}{2}k+\frac{1}{2})$; when this condition is satisfied the index of the highest power of z in $P_\alpha(z)$ corresponds to $m=l-1$, and its real part is

$$2\lambda-2\eta-\alpha+2\sigma+4\,[-\tfrac{1}{2}\sigma]\leqslant 2\lambda-2\eta-\alpha,$$

and in case s is integral the series of which the first two terms are given in (3.191) terminates; when

$$-s=2n\geqslant k+\eta+1,$$

(3.181) would vanish if it were not for the factor cosec $\frac{1}{2}s\pi$, hence the limit as $s\to -2n$ exists; it is only in this case, and in the process of proceeding to the limit, that the factor $\log z$ arises in (3.192). When $-s$ is an odd integer,

$$-s=\eta+2h\quad(h>\tfrac{1}{2}k+\tfrac{1}{2}),$$

the two parts of (3.181) or (3.191) together make up the initial terms of $J_\alpha(z)\sin\frac{1}{2}s\pi$.

3.2. The proof of all three of Theorems 2–4 depends upon a discontinuous integral evaluated recently by Dixon and Ferrar in terms of a particular form of the Weber-Schafheitlin integral* which has more than once been employed in an analogous problem†. In order to avoid difficulties caused by the divergence of the functions at the origin, it is necessary to modify somewhat $G_\alpha(z)$, $F_{k+s}(z)$; for this purpose I define

$$g_\alpha(z)=G_\alpha(z)-\mathcal{G}_\alpha(z),\quad f_\gamma(z)=F_\gamma(z)-\mathcal{F}_\gamma(z), \tag{3.21}$$

where‡ $\gamma=\kappa+s+1$,

$$\begin{cases}\mathcal{F}_\gamma(z)=\dfrac{1}{\sin\frac{1}{2}s\pi}\displaystyle\sum_{m=1}^{[\frac{1}{4}\kappa+\frac{1}{2}]}\frac{(\frac{1}{2}z)^{-s+\kappa-4m+1}}{\kappa+1-2m\,!\;\Gamma(-s-2m+1)} & \text{if}\quad \sigma\geqslant 0,\\ \mathcal{F}_\gamma(z)=\dfrac{1-(-1)^\kappa}{2\sin\frac{1}{2}s\pi}\dfrac{(\frac{1}{2}z)^{-\gamma}}{\Gamma(1-\gamma)} & \text{if}\quad \sigma<0,\end{cases} \tag{3.22}$$

and (since $\alpha=\rho+k$)

$$\mathcal{G}_\alpha(z)=\frac{(-)^k}{\sin\frac{1}{2}s\pi}\sum_{m=-p}^{[\frac{1}{4}\lambda+\frac{1}{2}]}\frac{(\frac{1}{2}z)^{-s-k-4m+\lambda+1}}{\Gamma(-2m+\lambda+2)\,\Gamma(-s-k-2m+1)}, \tag{3.23}$$

where, in the last equation,

$$p=\mathrm{Max}(0,\,[\tfrac{1}{2}\sigma]), \tag{3.231}$$

---

* Watson, **16**, 398-410.

† Hardy, **6**, Wilton, **18** and **20**.

‡ κ is defined in (2.11).

and, in case s is an even integer and*

$$-p \leqslant m \leqslant -\tfrac{1}{2}s-\tfrac{1}{2}k,$$

every term of the first series on the right of (3.11) in which the index of the power of z is equal to the index in a term of (3.23) is to be subtracted from the corresponding term of the series in (3.23) before proceeding to the limit. With this understanding $\mathcal{G}_{\mathcal{J}_\alpha}(z)$ is defined for all values of s, and it is clear that g_α and $\mathcal{G}_{\mathcal{J}_\alpha}$ satisfy the sequence relation (3.12) and that f_γ satisfies (3.14). It is, moreover, easy to verify that the integrals

$$(3.24) \qquad \int^\infty \mathcal{G}_{\mathcal{J}_\alpha}(az)\, f_\gamma(bz)\, \frac{dz}{z^\lambda}, \quad \int^\infty g_\alpha(az)\, \mathcal{J}_\gamma(bz)\, \frac{dz}{z^\lambda}, \quad \int_0 g_\alpha(az)\, f_\gamma(bz)\, \frac{dz}{z^\lambda}$$

are absolutely convergent if $k=\kappa+1$ $(\lambda>-1)$; and it is not difficult to see that (with $k=\kappa+1$), if $\lambda\,(\geqslant 0)$ is an integer, the conditions necessary for the truth of the formula†

$$(3.241) \qquad \int_0^\infty g_\alpha(az)\, f_\gamma(bz)\, \frac{dz}{z^\lambda} = \int_0^\infty J_\alpha(az)\, J_\gamma(bz)\, \frac{dz}{z^\lambda}$$

are all satisfied. Hence we have‡

LEMMA 2. *If $\lambda\,(\geqslant 0)$ is an integer, $x>0$, $n>0$, and*

$$(3.25) \qquad \alpha=\rho+\kappa+1=\gamma+\lambda+1,$$

then

$$(3.26) \qquad \int_0^\infty g_\alpha\{4\pi\sqrt{(xt)}\}\, f_\gamma\{4\pi\sqrt{(nt)}\}\, t^{\frac{1}{2}\lambda+\frac{1}{2}}\, \frac{dt}{t} = \frac{(2\pi)^{\lambda-1}}{\Gamma(\lambda+1)}\, \frac{n^{\frac{1}{2}\gamma}}{x^{\frac{1}{2}\alpha}}\, (x-n)^\lambda,$$

if $n<x$; the value of the integral is zero if $n\geqslant x$, unless $\lambda=0$ and $n=x$, in which case its value is $(4\pi\sqrt{x})^{-1}$.

It will appear that Lemma 2 remains true when λ is not an integer, if $\lambda>-1$, with the additional condition that $n\neq x$ when $0>\lambda>-1$. This might be proved directly§ from the definition of $g_\alpha(z)$, but, in order to keep the analysis as far as possible "real", I prefer to deduce it from (3.26) by the process of fractional integration. For this purpose I employ part of the following lemma, which is useful in the proof of Theorem A and is required in the proof of Theorem 3.

---

* This case cannot arise when $k\geqslant\kappa$.

† Dixon and Ferrar, **4**, 135-6.

‡ In the statement of Lemma 2, n need not be integral, but in applications n is an integer greater than or equal to 1.

§ Dixon and Ferrar, **5**, §6.

LEMMA 3. *If* $\alpha=\rho+k$, $\gamma=s+k$, $\lambda>-1$, $k\geqslant\kappa$, $N>A$, $Nx>A$, $n\geqslant 1$, *and if*

$$(3.31)\qquad J_{\alpha,\gamma}=2\pi\int_N^\infty\left(\frac{x}{t}\right)^{\frac{1}{2}\alpha}G_\alpha\{4\pi\sqrt{(xt)}\}\left(\frac{t}{n}\right)^{\frac{1}{2}\gamma}F_\gamma\{4\pi\sqrt{(nt)}\}\,dt,$$

then (i), *if* $n>(1+A)x$,

$$(3.32)\qquad J_{\alpha,\gamma}=O\{(x/N)^{\frac{1}{2}\lambda+\frac{1}{2}}(x/n)^{\frac{1}{2}(\sigma+k-\frac{1}{2})}n^{-1}\};$$

(ii) *if* $n>Ax$,

$$(3.33)\qquad J_{\alpha,\gamma}=\frac{(4\pi|\sqrt{x}-\sqrt{n}|)^\lambda x^{\frac{1}{2}\alpha+\frac{1}{4}}}{\pi n^{\frac{1}{2}\gamma+\frac{3}{4}}}\int_{4\pi\sqrt{N}|\sqrt{x}-\sqrt{n}|}^\infty\sin\{t\,\mathrm{sgn}\,(x-n)-\tfrac{1}{2}\lambda\pi\}\frac{dt}{t^{\lambda+1}}$$
$$+O\{(3.32)\}^*\qquad(n\neq x),$$

$$(3.331)\qquad J_{\alpha,\gamma}=-\frac{\sin\frac{1}{2}\lambda\pi}{\lambda\pi}\left(\frac{x}{N}\right)^{\frac{1}{2}\lambda}+O\left(\frac{x^{\frac{1}{2}\lambda-\frac{1}{2}}}{N^{\frac{1}{2}\lambda+\frac{1}{2}}}\right)\qquad(n=x,\ \lambda>0),$$

$$(3.332)\qquad J_{\alpha,\gamma}=O\{(Nx)^{-\frac{1}{4}}\}\qquad(n=x,\ \lambda=0);$$

(iii) *if* $1\leqslant n\leqslant(1-A)x$,

$$(3.34)\qquad J_{\alpha,\gamma}=O\{(x/N)^{\frac{1}{2}\lambda+\frac{1}{2}}(x/n)^{\frac{1}{2}(\sigma+k+\frac{1}{2})}x^{-1}\}.$$

The proof follows very closely that of a lemma employed by Hardy† in the corresponding problem of the lattice points of a circle; we integrate (3.31) by parts (in one direction or the other) and employ (3.19), remembering that $k-1\geqslant\kappa-1$. A brief indication will suffice. We have, by (3.12), (3.14), and (3.19),

$$J_{\alpha,\gamma}=-\left(\frac{x}{N}\right)^{\frac{1}{2}\alpha}G_\alpha\{4\pi\sqrt{(Nx)}\}\left(\frac{N}{n}\right)^{\frac{1}{2}\gamma+\frac{1}{2}}F_{\gamma+1}\{4\pi\sqrt{(Nn)}\}$$
$$+2\pi\int_N^\infty\left(\frac{x}{t}\right)^{\frac{1}{2}\alpha+\frac{1}{4}}G_{\alpha+1}\{4\pi\sqrt{(xt)}\}\left(\frac{t}{n}\right)^{\frac{1}{2}\gamma+\frac{1}{4}}F_{\gamma+1}\{4\pi\sqrt{(nt)}\}\,dt$$

$$(3.35)\qquad=\frac{x^{\frac{1}{2}\alpha+\frac{1}{4}}}{\pi n^{\frac{1}{2}\gamma+\frac{3}{4}}}\int_{\sqrt{N}}^\infty[\sin\{4\pi(\sqrt{x}-\sqrt{n})t-\tfrac{1}{2}\lambda\pi\}$$
$$+(-)^{k+1}\cos\{4\pi(\sqrt{x}+\sqrt{n})t-\tfrac{1}{2}\lambda\pi\}]\frac{dt}{t^{\lambda+1}}+O\{(3.32)\},$$

---

* Here $O\{(3.32)\}$ means "a function of at most the order indicated in (3.32)". I use similar abbreviations freely; thus (p. 407)

$$(4.51)=(4.45)+(4.48)+(4.49)$$

means "the integral (4.51) is equal to the sum of the three series (4.45), (4.48), (4.49)".

† 6, Lemma 2 (with the correction on p. 369 of vol. 20).

and this is equivalent to (3.33–3.332). The integral in (3.33) falls under (3.32) when $n>(1+A)x$. To obtain (3.34) we integrate $J_{\alpha,\gamma}$ by parts so that $G_{\alpha-1}$ and $F_{\gamma-1}$ remain under the sign of integration.

LEMMA 4. *If* $\lambda>-1$, $x>0$, $n>0$, *and* α, γ *are given by* (3.25), *then* (3.26) *is true, provided that* $x\neq n$ *when* $-1<\lambda<0$.

We evidently have

$$x^{\frac{1}{2}\alpha}g_\alpha(2\sqrt{x})=\frac{1}{\Gamma(\delta)}\int_0^x(x-\xi)^{\delta-1}\xi^{\frac{1}{2}\alpha'}g_{\alpha'}(2\sqrt{\xi})\,d\xi,$$

where $\alpha'=[\lambda]+s+\kappa+2$, $\delta=\lambda-[\lambda]$; and therefore, if $\lambda>0$,

$$\int_0^\infty g_\alpha\{2\sqrt{(xt)}\}f_\gamma\{2\sqrt{(nt)}\}\frac{dt}{t^{\frac{1}{2}\lambda+\frac{1}{2}}}$$

$$(3.41)\qquad =\frac{x^{-\frac{1}{2}\alpha}}{\Gamma(\delta)}\int_0^\infty\frac{f_\gamma\,dt}{t^{\frac{1}{2}[\lambda]+\frac{1}{2}}}\int_0^x(x-\xi)^{\delta-1}\xi^{\frac{1}{2}\alpha'}g_{\alpha'}\{2\sqrt{(\xi t)}\}\,d\xi$$

$$=\frac{x^{-\frac{1}{2}\alpha}n^{\frac{1}{2}\gamma}}{\Gamma(\delta)\Gamma([\lambda]+1)}\int_n^x(x-\xi)^{\delta-1}(\xi-n)^{[\lambda]}\,d\xi=\frac{n^{\frac{1}{2}\gamma}}{x^{\frac{1}{2}\alpha}}\frac{(x-n)^\lambda}{\Gamma(\lambda+1)}.$$

If, in the lower bound of summation in the second sum on the right of (3.11) and in the upper bound of the sum (3.23), we replace $\frac{1}{2}\lambda+\frac{1}{2}$ by 0, the function $g_\alpha(z)$ is an analytic function of λ in the strip

$$(3.42)\qquad -1<R(\lambda)<1;$$

and if, in Lemma 1, $[\lambda]$ is replaced by 0 wherever it occurs, the lemma is true (with $\eta=1$) in (3.42), and consequently the integral (3.26) is an analytic function of λ in the strip (3.42). Hence Lemma 4 will have been proved when the change of order of integration in (3.41) has been justified.

In consequence of the absolute convergence of the integrals (3.24), it is sufficient to justify the change of order in

$$\int_N^\infty F_\gamma\{2\sqrt{(nt)}\}\frac{dt}{t^{\frac{1}{2}[\lambda]+\frac{1}{2}}}\int_0^x(x-\xi)^{\delta-1}\xi^{\frac{1}{2}\alpha'}G_{\alpha'}\{2\sqrt{(\xi t)}\}\,d\xi\quad(N>A).$$

On account of Lemma 3 the change presents no difficulty if $\lambda\geqslant 1$, or if $[\lambda]=0$ and $n>x$*. If $n\leqslant x$, and $[\lambda]=0$, the change of order will have been justified when it has been shown that†

$$I(\vartheta)=\int_{n-\vartheta}^{n+\vartheta}(x-\xi)^{\delta-1}\xi^{\frac{1}{2}\alpha'}\,d\xi\int_N^\infty F_\gamma\,G_{\alpha'}\frac{dt}{\sqrt{t}}\quad(\tfrac{1}{2}>\vartheta>0)$$

---

* See, for example, Hobson, **9**, 348 (1) and (2).

† The upper limit of integration in $I(\vartheta)$ is n, if $n=x$.

tends to zero as $\vartheta \to 0$. But this is evident since, by (3.33),

$$|I(\vartheta)| < \int_{n-\vartheta}^{n+\vartheta} (x-\xi)^{\delta-1} d\xi (C_1 + C_2 N^{-\frac{1}{2}}) < C_3 \frac{\vartheta^\delta}{\delta},$$

where C_1, C_2, C_3 are constants depending on n and s.

4. *Proof of Theorem 2.*

We are now in a position to state and prove

THEOREM 2. *If $x > 0$, $N > 0$, $\lambda > -1$, and $\Delta_\lambda(x)$ is defined by* (1.16), *then*

$$\Sigma(\lambda, N) = (2\pi)^\lambda \Delta_\lambda(x) - \sum_{n=1}^{N}{}' \left(\frac{x}{n}\right)^{\frac{1}{2}\rho} \sigma_s(n)\, G_\rho\{4\pi\sqrt{(nx)}\} \tag{4.10}$$

$$= 2\pi \sum_{n=1}^{\infty} \sigma_s(n) \int_N^\infty \left(\frac{x}{t}\right)^{\frac{1}{2}(\rho+\kappa)} G_{\rho+\kappa}\{4\pi\sqrt{(xt)}\} \left(\frac{t}{n}\right)^{\frac{1}{2}(s+\kappa)} \times F_{s+\kappa}\{4\pi\sqrt{(nt)}\}\, dt \tag{4.11}$$

$$- \sum_{k=-1}^{\kappa-1} (2\pi)^k \left(\frac{x}{N}\right)^{\frac{1}{2}(\rho+k)} G_{\rho+k}\{4\pi\sqrt{(Nx)}\} \Delta_k(N) \tag{4.12}$$

$$+ \zeta(1+s) \left[\sum_{k=2}^{\theta} \frac{s(s-1)\dots(s-k+2)}{(2\pi)^k} \left(\frac{x}{N}\right)^{\frac{1}{2}(\rho-k)} \times G_{\rho-k}\{4\pi\sqrt{(Nx)}\} N^{s-k+\frac{1}{2}} \right. \tag{4.13}*$$

$$\left. + \frac{s(s-1)\dots(s-\theta+1)}{(2\pi)^\theta} \int_N^\infty \left(\frac{x}{t}\right)^{\frac{1}{2}(\rho-\theta)} \times G_{\rho-\theta}\{4\pi\sqrt{(xt)}\} t^{s-\theta}\, dt \right], \tag{4.14}$$

where

$$\Delta_{-1}(N) = -N^s \zeta(1+s) - \zeta(1-s), \tag{4.15}$$

and

$$\theta = \mathrm{Max}\,([\sigma - \lambda + \tfrac{1}{2}],\ 1).$$

If α and γ are given by (3.25), we have, by (1.14) and (3.26),

$$(2\pi)^\lambda D_\lambda(x) = 2\pi \sum_{n=1}^{\infty} \sigma_s(n) \int_0^\infty \left(\frac{x}{t}\right)^{\frac{1}{2}\alpha} g_\alpha\{4\pi\sqrt{(xt)}\} \left(\frac{t}{n}\right)^{\frac{1}{2}\gamma} f_\gamma\{4\pi\sqrt{(nt)}\}\, dt,$$

provided that $n \neq x$ when $0 > \lambda > -1$. Divide the interval $(0, \infty)$ into $(0, 1)$ and $(1, \infty)$; in the second sub-interval employ (3.21), and in the

* This sum is, of course, null if $\theta = 1$, *i.e.* if $\lambda > \sigma - \frac{3}{2}$.

case of that one of the four resulting integrals which involves the product $G_\alpha F_\gamma$ replace the interval* $(1, \infty)$ by $(1, N)$ and (N, ∞); in all cases except that of the integral over (N, ∞), change the order of summation and integration† and use Theorem 1 and (1.16), remembering that, when $t<1$, $D_k(t)=0$. We thus obtain

$$(4.21)\quad (2\pi)^\lambda D_\lambda(x) = 2\pi \sum_1^\infty \sigma_s(n) \int_N^\infty \left(\frac{x}{t}\right)^{\frac{1}{2}\alpha} G_\alpha\left(\frac{t}{n}\right)^{\frac{1}{2}\gamma} F_\gamma\, dt$$

$$(4.22)+(4.23)\qquad +(2\pi)^{\kappa+1}\int_1^N \left(\frac{x}{t}\right)^{\frac{1}{2}\alpha} G_\alpha\{D_\kappa(t)-\Phi_\kappa(t)\}\,dt$$

$$(4.24)\qquad +(2\pi)^{\kappa+1}\int_0^1 \left(\frac{x}{t}\right)^{\frac{1}{2}\alpha} g_\alpha\{S_\kappa(t)-\Phi_\kappa(t)\}\,dt$$

$$(4.25)+(4.26)\qquad +(2\pi)^{\kappa+1}\int_1^\infty \left(\frac{x}{t}\right)^{\frac{1}{2}\alpha} \{g_\alpha S_\kappa(t)-\mathcal{G}_\alpha \Delta_\kappa(t)\}\,dt,$$

where the argument of G_α, g_α, and $\mathcal{G}_\alpha$ is $4\pi\sqrt{(xt)}$, that of F_γ is $4\pi\sqrt{(nt)}$, and

$$S_\kappa(t) = -\frac{1}{(2\pi)^\kappa}\sum_{n=1}^\infty \sigma_s(n)\left(\frac{t}{n}\right)^{\frac{1}{2}\gamma} \mathcal{F}_\gamma\{4\pi\sqrt{(nt)}\}.$$

By (3.22) and (1.17), if $\sigma<0$, $S_\kappa(t)=0$ when κ is even, and

$$S_\kappa(t) = -\frac{(2\pi)^{-\gamma-\kappa}\zeta(\gamma)\zeta(\gamma-s)}{\Gamma(1-\gamma)\sin\frac{1}{2}s\pi}$$

$$= (-)^{\frac{1}{2}\kappa-\frac{1}{2}} 2(2\pi)^{-\kappa-1}\zeta(\kappa+1)\zeta(-\kappa-s)$$

when κ is odd, *i.e.*, by (1.151), $S_\kappa(t)$ is the term independent of t in $\Phi_\kappa(t)$. If $\sigma \geqslant 0$,

$$(4.27)\quad S_\kappa(t) = 2\sum_{m=1}^{[\frac{1}{2}\kappa+\frac{1}{2}]} (-)^{m-1}\frac{t^{\kappa+1-2m}}{\kappa+1-2m!}\frac{\zeta(2m)}{(2\pi)^{2m}}\zeta(-2m+1-s)$$

$$= R_\kappa(t)-\zeta(1-s)\,t^{\kappa+1}/\kappa+1!,$$

i.e.

$$(4.28)\quad \Phi_\kappa(t)-S_\kappa(t) = \frac{t^{\kappa+s+1}}{\Gamma(\kappa+s+2)}\Gamma(1+s)\zeta(1+s)+\frac{t^{\kappa+1}}{\kappa+1!}\zeta(1-s)$$

$$-\tfrac{1}{2}\frac{t^\kappa}{\kappa!}\zeta(-s).$$

---

* $N>0$, but N need not be a point of the interval $(1, \infty)$.

† All the changes of order are justified by the condition of absolute convergence.

For the present we assume that $\sigma \geqslant 0$ and that $\lambda > -\frac{1}{2}$; then the polynomial $S_\kappa(t)$ is of degree $\kappa-1$, and all the derivatives of (4.28) of order not exceeding $\kappa+\theta$ vanish when $t=0$, except

$$\Phi_0(0) = -\tfrac{1}{2}\zeta(-s) \quad \text{and} \quad \Phi_{-1}(0) = \zeta(1-s),$$

unless $\sigma = 0$, in which case $\Phi_{-1}(t) = O(\log 1/t)$ as $t \to 0$.

From (3.11) and (3.23) it is clear that, if s is not an even integer,

$$\lim_{t\to 0}\left(\frac{x}{t}\right)^{\frac{1}{2}(\rho+k)} g_{\rho+k}\{4\pi\sqrt{(xt)}\} = \frac{-(2\pi x)^{\rho+k}}{\Gamma(\rho+k+1)\sin\frac{1}{2}s\pi}$$

when $k(\leqslant 0)$ is even, and that the limit is zero when k is odd. We now integrate the integrals (4.22–4.25) by parts, κ times in the case of (4.22), $\kappa+\theta+1$ times in the case of the other three integrals. The result is

$$(4.31)\quad 2\pi\int_1^N \left(\frac{x}{t}\right)^{\frac{1}{2}\rho+\frac{1}{2}} G_{\rho+1}\{4\pi\sqrt{(xt)}\} \sum_{n\leqslant t}\sigma_s(n)\,dt$$

$$(4.32)\quad -\sum_{k=1}^{\kappa}(2\pi)^k\left(\frac{x}{N}\right)^{\frac{1}{2}\rho+\frac{1}{2}k} G_{\rho+k}\{4\pi\sqrt{(Nx)}\}\Delta_k(N)$$

$$(4.33)\quad +\sum_{k=0}^{\theta}(2\pi)^{-k}\left(\frac{x}{N}\right)^{\frac{1}{2}\rho-\frac{1}{2}k} G_{\rho-k}\{4\pi\sqrt{(Nx)}\}\Phi_{-k}(N)$$

$$(4.34)\quad -\sum_{k=-\theta}^{\kappa}(2\pi)^k x^{\frac{1}{2}\rho+\frac{1}{2}k}\mathcal{J}_{\rho+k}(4\pi\sqrt{x})\Phi_k(1)$$

$$(4.35)\quad -\frac{(2\pi x)^\rho\zeta(-s)}{2\Gamma(\rho+1)\sin\frac{1}{2}s\pi}$$

$$(4.36)\quad -\frac{1}{(2\pi)^\theta}\int_0^1\left(\frac{x}{t}\right)^{\frac{1}{2}\rho-\frac{1}{2}\theta} g_{\rho-\theta}\{4\pi\sqrt{(xt)}\}\Phi_{-\theta-1}(t)\,dt$$

$$(4.37)+(4.38)\quad -\frac{1}{(2\pi)^\theta}\left(\int_1^\infty - \int_N^\infty\right)\left(\frac{x}{t}\right)^{\frac{1}{2}\rho-\frac{1}{2}\theta} G_{\rho-\theta}\{4\pi\sqrt{(xt)}\}\Phi_{-\theta-1}(t)\,dt,$$

where $$\Phi_{-\theta-1}(t) = \frac{t^{s-\theta}}{\Gamma(s-\theta+1)}\Gamma(s+1)\zeta(s+1).$$

Consider now the integral (4.26). This, by (2.12) and (3.23), is absolutely convergent, but we are not at liberty to integrate by parts more than once; we therefore evaluate, instead of (4.26), the integral

$$(4.41)\quad -(2\pi)^{\kappa+1}\int_1^\infty\left(\frac{x}{t}\right)^{\frac{1}{2}\beta}\mathcal{J}_\beta\{4\pi\sqrt{(xt)}\}\Delta_\kappa(t)\,dt,$$

where $\beta = \rho + h$, and h is a positive integer sufficiently large to ensure the absolute convergence of all the integrals in which it occurs*. With this understanding (4. 41) is equal to

$$(4.42)\quad \sum_{k=1}^{\kappa} (2\pi)^k x^{\frac{1}{2}(\beta-\kappa+k-1)} \mathcal{J}_{\beta-\kappa+k-1}(4\pi\sqrt{x})\,\Phi_k(1)$$

$$(4.43)+(4.44)\qquad -2\pi \int_1^\infty \left(\frac{x}{t}\right)^{\frac{1}{2}(\beta-\kappa)} \mathcal{J}_{\beta-\kappa}\{4\pi\sqrt{(xt)}\} \left\{ \sum_{n\leqslant t} \sigma_s(n) - \Phi_0(t) \right\} dt.$$

In the case of the integral (4.44) integrate by parts $\theta+1$ times further; then (4.41) is equal to

$$(4.45)\quad \sum_{k=-\theta}^{\kappa} (2\pi)^k x^{\frac{1}{2}(\rho+k+\beta-\alpha)} \mathcal{J}_{\rho+k+\beta-\alpha}(4\pi\sqrt{x})\,\Phi_k(1)$$

$$(4.46)\qquad +\frac{1}{(2\pi)^\theta} \int_1^\infty \left(\frac{x}{t}\right)^{\frac{1}{2}(\rho-\theta+\beta-\alpha)} \mathcal{J}_{\rho-\theta+\beta-\alpha}\{4\pi\sqrt{(xt)}\}\,\Phi_{-\theta-1}(t)\,dt$$

$$(4.47)\qquad -2\pi \sum_{n=1}^{\infty} \sigma_s(n) \int_n^\infty \left(\frac{x}{t}\right)^{\frac{1}{2}(\beta-\kappa)} \mathcal{J}_{\beta-\kappa}\{4\pi\sqrt{(xt)}\}\,dt.$$

The sum (4.47) may be evaluated by means of (3.23) and (1.17). The result is, if s is not an even integer,

$$(4.48)\quad \frac{(-)^{h+\kappa}}{\sin\frac{1}{2}s\pi} \sum_{m=-p}^{[\frac{1}{2}\lambda+\frac{1}{2}]} \frac{(2\pi x)^{\lambda+1-2m}\,\zeta(2m+h-\kappa-1)\,\zeta(s+2m+h-\kappa-1)}{(2\pi)^{s+2m+h-\kappa-1}\,\Gamma(\lambda+2-2m)\,\Gamma(-s-2m-h+\kappa+2)};$$

and the value of (4.46) is

$$(4.49)\quad \frac{(-)^{h-\kappa-1+\theta}}{\sin\frac{1}{2}s\pi} \frac{\Gamma(s+1)\,\zeta(s+1)}{\Gamma(s-\theta+1)}$$

$$\times \sum_{m=-p}^{[\frac{1}{2}\lambda+\frac{1}{2}]} \frac{(2\pi)^{-s-2m-h+\kappa+1}\,(2\pi x)^{\lambda+1-2m}}{(h-\kappa+2m-2)\,\Gamma(-2m+\lambda+2)\,\Gamma(-s+\theta-h+\kappa-2m+2)}.$$

The only reason for choosing h to be an integer is the presence of the factor $(-1)^h$ in the definition of $\mathcal{J}_\beta(z)$. Ignoring this factor, the proof that

$$(4.51)\qquad (4.41) = (4.45)+(4.48)+(4.49)$$

is valid for complex h, if $R(h)$ is sufficiently large. But, if we interpret

* It is certainly sufficient to choose $h > \kappa + \sigma + 2$.

(4.48)+(4.49) when $h=\kappa-2m+2$ $(-p\leqslant m\leqslant 0)$ as the limit as h tends to this value, both sides of (4.51) are analytic functions of h in the half-plane $R(h)\geqslant\kappa+1$. Hence (4.51) is true when $h=\kappa+1$, *i.e.* when $\beta=\alpha$. The upshot of all this is that (4.26) and (4.34) are together equal to

$$(4.52)+(4.53)\quad (2\pi)^\lambda R_\lambda(x)-\frac{1}{(2\pi)^\theta}\int_1^\infty\left(\frac{x}{t}\right)^{\frac{1}{2}(\rho-\theta)}{}_{\rho-\theta}\{4\pi\sqrt{(xt)}\}\,\Phi_{-\theta-1}(t)\,dt,$$

where R_λ is defined in (1.151) and (4.53) is understood to mean the value of (4.49) when $h=\kappa+1$.

By the functional equation for the ζ-function, already used many times, (4.35) is equal to

$$(4.54)\qquad (2\pi)^\lambda x^\rho\,\Gamma(s+1)\,\zeta(s+1)/\Gamma(\rho+1).$$

Again, (4.31) is equal to

$$2\pi\sum_{n\leqslant N}\sigma_s(n)\int_n^N\left(\frac{x}{t}\right)^{\frac{1}{2}\rho+\frac{1}{2}}G_{\rho+1}\{4\pi\sqrt{(xt)}\}\,dt$$

$$(4.55)\qquad =\sum_{n\leqslant N}{}'\,\sigma_s(n)\left(\frac{x}{n}\right)^{\frac{1}{2}\rho}G_\rho\{4\pi\sqrt{(nx)}\}$$

$$(4.56)\qquad -D_0(N)(x/N)^{\frac{1}{2}\rho}\,G_\rho\{4\pi\sqrt{(Nx)}\}.$$

The term $k=0$ of (4.33) combined with (4.56) and added to (4.32) changes the lower bound of summation in (4.32) to $k=0$; and since $\Phi_{-1}(N)=-\Delta_{-1}(N)$ the term $k=1$ of (4.33) may be transferred to (4.32), changing the lower bound to $k=-1$. If we now integrate (4.21) once by parts, we obtain (4.11) together with a term which cancels the term $k=\kappa$ in (4.32). The modified forms of (4.32) and (4.33) are equal, respectively, to (4.12) and (4.13); and (4.38) is equal to (4.14). Hence Theorem 2 (with $\lambda>-\frac{1}{2}$) will follow from (4.52), (4.54), and (4.55) when we have proved

LEMMA 5. *The sum of the three integrals* (4.36), (4.37), *and* (4.53) *is equal to*

$$(4.57)\qquad -\tfrac{1}{2}(2\pi x)^\lambda\,\zeta(-s)/\Gamma(\lambda+1).$$

It is evident that the integrand in (4.53) is a polynomial in t and t^{-1} of the form

$$\sum_{m=-p}^{0}c_m t^{-2m}+\sum_{m=1}^{[\frac{1}{2}\lambda+\frac{1}{4}]}c_m t^{-2m}=\Sigma_1+\Sigma_2,$$

say. Hence (4.53) is equal to

$$\frac{1}{(2\pi)^\theta}\left(\int_1^\infty \Sigma_2\,dt - \int_0^1 \Sigma_1\,dt\right);$$

and, if we put

$$(4.61)\quad \mathbf{G}_{\rho-\theta}(z) = g_{\rho-\theta}(z) + \frac{(-)^\theta}{\sin\frac{1}{2}s\pi}\sum_{m=0}^{p}\frac{(\frac{1}{2}z)^{-s+\theta+\lambda+4m+1}}{\Gamma(2m+\lambda+2)\,\Gamma(-s+\theta+2m+1)}$$

[*cf.* (3.23)], it is clear that the sum of (4.36), (4.37), and (4.53) is equal to

$$(4.62)\qquad -\frac{1}{(2\pi)^\theta}\int_0^\infty \left(\frac{x}{t}\right)^{\frac{1}{2}\rho-\frac{1}{2}\theta}\mathbf{G}_{\rho-\theta}\{4\pi\sqrt{(xt)}\}\,\Phi_{-\theta-1}(t)\,dt.$$

We have, then, to show that, when $\sigma \geqslant 0$ and s is not an even integer, (4.62) = (4.57). We employ the method of fractional integration and, as in the proof of Lemma 4, we assume initially that $\lambda \geqslant 0$. We note that (4.62) may be written in the form

$$(4.63)\qquad \chi(s,x)\int_0^\infty t^{\frac{1}{2}\rho-\frac{1}{2}\theta}\,\mathbf{G}_{\rho-\theta}(2\sqrt{t})\,\frac{dt}{t^{\lambda+1}},$$

where

$$(4.631)\quad \chi(s,x) = -\frac{\Gamma(s+1)\,\zeta(s+1)}{(2\pi)^{s+1}\,\Gamma(s-\theta+1)}(2\pi x)^\lambda = \frac{(2\pi x)^\lambda\,\zeta(-s)}{2\sin\frac{1}{2}s\pi\,\Gamma(s-\theta+1)}.$$

The asymptotic formula (3.15) for $G_{\rho+k}(z)$ is true with no restriction on the integer k if $\lambda(\geqslant 0)$ is an integer. Hence, as in Lemma 4, if

$$\rho' = [\lambda]+s+1, \quad \delta = \lambda-[\lambda],$$

the integral in (4.63) is equal to

$$\frac{1}{\Gamma(\delta)}\int_0^\infty\frac{dt}{t^{\lambda+1}}\int_0^t (t-\tau)^{\delta-1}\tau^{\frac{1}{2}\rho'-\frac{1}{2}\theta}\,\mathbf{G}_{\rho'-\theta}(2\sqrt{\tau})\,d\tau$$

$$= \frac{1}{\Gamma(\delta)}\int_0^1 (1-\tau)^{\delta-1}\tau^{\frac{1}{2}\rho'-\frac{1}{2}\theta}\,d\tau\int_0^\infty \mathbf{G}_{\rho'-\theta}\{2\sqrt{(t\tau)}\}\,\frac{dt}{t^{\frac{1}{2}([\lambda]+\theta+1-s}}$$

$$(4.64)\qquad = \frac{2^{[\lambda]+\theta-s}}{\Gamma(\delta)}\int_0^1 (1-\tau)^{\delta-1}\tau^{[\lambda]}\,d\tau\int_0^\infty \mathbf{G}_{\rho'-\theta}(t)\,\frac{dt}{t^{[\lambda]+\theta-s}}$$

$$= -\frac{\Gamma(s-\theta+1)\sin\frac{1}{2}s\pi}{\Gamma(\delta)\,\Gamma([\lambda]+1)}\int_0^1 (1-\tau)^{\delta-1}\tau^{[\lambda]}\,d\tau$$

$$(4.65)\qquad = -\sin\tfrac{1}{2}s\pi\,\Gamma(s-\theta+1)/\Gamma(\lambda+1),$$

and the product of (4.65) and $\chi(s,x)$ is (4.57). Some care is needed in

the evaluation of the inner integral in (4.64); the materials are supplied by Watson*.

When $-1<\lambda<1$, $\mathbf{G}_\alpha(z)=G_\alpha(z)$, and the argument by which Lemma 4 was extended to cover the strip (3.42) is applicable here, so that Lemma 5 is proved. The lemma might also have been proved by Dixon and Ferrar's process of "increasing the argument by $\frac{1}{2}\pi$".

It is evident that we may immediately remove the restriction that $\lambda>-\frac{1}{2}$. We proceed to remove the further restrictions, under which Theorem 2 has so far been proved, namely that $\sigma\geqslant 0$ and that s is not an even integer (or zero). For this purpose we appeal once more to the principle of analytic continuation.

If θ ($\geqslant 1$), λ ($\geqslant 0$), and κ ($1\leqslant\kappa\leqslant\theta+\lambda+1$) are given integers, then both sides of the identity of Theorem 2 are analytic functions of s in the strip (2.291).

If λ (>-1) is not an integer, and η is given by (1.152), then both sides of the identity of Theorem 2 are analytic functions of s in the strip

$$\theta-\eta-\tfrac{1}{2}\{3-(-1)^\theta\}<\sigma<\kappa-\tfrac{1}{2}, \tag{4.71}$$

provided that (4.71) is included in (2.291). In this case also the theorem is true for the whole strip (2.291). This may be seen as follows.

(i) The left-hand boundary of the strip (4.71) is derived from the condition that $l=[\frac{1}{2}-\frac{1}{2}\theta]$ when $k=-\theta$ in (1.24). If $\theta>1$, the strip within which the theorem is proved when $\theta-1$ is written for θ, namely

$$\theta-\eta-\tfrac{1}{2}\{5+(-1)^\theta\}<\sigma<\mathrm{Min}\,(\theta+\lambda-\tfrac{1}{2},\ \kappa-\tfrac{1}{2}), \tag{4.72}$$

certainly includes a part of (4.71), of breadth at least $\frac{1}{2}$; hence the theorem is valid in the strip formed by combining (4.71) and (4.72). If $\theta>2$, we may in the same way write $\theta-1$ for θ in (4.72); and step by step the region of validity of the theorem may be extended to the strip

$$-1-\eta<\sigma<\kappa-\tfrac{1}{2}, \tag{4.73}$$

and only the case when $\sigma\leqslant-\eta-1$ and $\theta=1$ needs special consideration.

(ii) When $\sigma\leqslant-\eta-1$ and s is not an even integer, $z^{-\frac{1}{2}\rho}P_\rho(2\sqrt{z})$, as defined by (3.161) with $\alpha=\rho$, and so with $k=0$, is a non-evanescent

* 16, 388, 391, and 393.

polynomial. By partial summation, using (1.13) and (1.16), we have

$$(4.81)\qquad \sum_{n=1}^{N}{}' \sigma_s(n)\chi_\rho(n) = \{\Delta_0(N)+\Phi_0(N)\}\chi_\rho(N)$$

$$(4.82)\qquad +2\pi\int_0^N \{\Delta_0(t)+\Phi_0(t)\}\chi_{\rho+1}(t)\,dt,$$

where

$$(4.83)\qquad \chi_a(t) = (x/t)^{\frac{1}{2}a} P_a\{4\pi\sqrt{(xt)}\}.$$

Let $t_s = \zeta(1+s)\,t^{1+s}/(1+s)$, so that

$$\Phi_0(t)-t_s = t\zeta(1-s)-\tfrac{1}{2}\zeta(-s).$$

Then we may rewrite (4.82) in the equivalent form

$$(4.84)\qquad 2\pi\int_0^N \{\Delta_0(t)+t_s\}\chi_{\rho+1}(t)\,dt+2\pi\int_N^\infty t_s\chi_{\rho+1}(t)\,dt$$
$$-\tfrac{1}{2}\zeta(-s)\chi_\rho(0)-\Phi_0(N)\chi_\rho(N)+\Delta_{-1}(N)\chi_{\rho-1}(N)/(2\pi)$$
$$-\frac{s}{2\pi}\zeta(1+s)\int_N^\infty t^{s-1}\chi_{\rho-1}(t)\,dt;$$

and (4.84) may be evaluated by integrating by parts until the polynomial $\chi_{\rho+k}(t)$ disappears; since

$$2[-\tfrac{1}{2}\sigma]-2[\tfrac{1}{2}\lambda+\tfrac{1}{2}] \leqslant -\sigma < \kappa-\tfrac{1}{2},$$

this will occur for a value of $k \leqslant \kappa-1$. We thus find that

$$(4.85)\qquad \begin{cases} (2\pi)^\lambda x^\rho \dfrac{\Gamma(1+s)\,\zeta(1+s)}{\Gamma(1+\rho)} - \displaystyle\sum_{n=1}^{N}{}' \sigma_s(n)\chi_\rho(n) \\ \qquad = \dfrac{s}{2\pi}\zeta(1+s)\displaystyle\int_N^\infty t^{s-1}\chi_{\rho-1}(t)\,dt - \sum_{k=-1}^{\kappa-1}(2\pi)^k\Delta_k(N)\chi_{\rho+k}(N). \end{cases}$$

In the strip $-\eta-1<\sigma<0$, in which $\theta=1$, and therefore (4.13) is null, both sides of the identity in Theorem 2 are analytic functions of s; but on each side there are infinite discontinuities as $s\to-\eta-1, -\eta-3, \ldots$; if λ is not an integer, $\Phi_\lambda(x)$ is given by (1.15) and $l=[\frac{1}{2}k+\frac{1}{2}]$. The terms which give rise to these discontinuities are precisely those which occur on the two sides of (4.85); and reference to the definitions of $\Delta_\lambda(x)$ and $G_{\rho+k}\{4\pi\sqrt{(xt)}\}$ when* $\sigma\leqslant-\eta-1$ will show that the difference between (4.10) and the sum of (4.12) and (4.14) is analytic in the strip $\frac{1}{2}-\kappa<\sigma<0$.

This completes the proof of Theorem 2.

* That is to say, when $\Phi_\lambda(x)$ is given by (1.153), and l by (1.24).

5. *An integral equation for* $\Delta_\lambda(x)$.

LEMMA 6. *If* $\alpha=\rho+\kappa$, $\gamma=s+\kappa$, $\lambda>-1$, $N>A$, $Nx>A$, $\xi=[x+\frac{1}{2}]$, *and* $J_{\alpha,\gamma}$ *is defined by* (3.31), *then*

$$(5.11)\qquad (4.11)=\sum_{n=1}^{\infty}\sigma_s(n)\,J_{\alpha,\gamma}=\Xi+O\{N^{-\frac{1}{2}\lambda-\frac{1}{2}}x^{\frac{1}{2}(\lambda+\sigma+|\sigma|+1)+\epsilon}\},$$

for every $\epsilon>0$, *where*

(i) $\Xi=0$, *if* $\xi=0$ (*i.e. if* $x<\frac{1}{2}$) *or if* $x=\xi$ *and* $\lambda=0$,

$$(5.12)\quad \text{(ii)}\quad \Xi=-\frac{x^{\frac{1}{2}\lambda}}{\pi}\left(\frac{x}{\xi}\right)^{\frac{1}{2}(s+\kappa+\frac{3}{2})}\sigma_s(\xi)(4\pi|\sqrt{\xi}-\sqrt{x}|)^{\lambda}\times\int_{4\pi\sqrt{N}|\sqrt{\xi}-\sqrt{x}|}^{\infty}\sin\{t\,\mathrm{sgn}\,(\xi-x)+\tfrac{1}{2}\lambda\pi\}\,\frac{dt}{t^{\lambda+1}},$$

if $x\neq\xi$ $(\geqslant 1)$,

$$(5.13)\quad \text{(iii)}\quad \Xi=-\frac{\sin\frac{1}{2}\lambda\pi}{\lambda\pi}\left(\frac{x}{N}\right)^{\frac{1}{2}\lambda}\sigma_s(x),\quad \textit{if } x=\xi \textit{ and } \lambda>0.$$

If $|\sigma|>1$, x^{ϵ} *in* (5.11) *may be replaced by* $\log x$.

By (3.33), Ξ corresponds to the term $n=\xi$. If we exclude this term from (5.11), we have $|x-n|\geqslant\frac{1}{2}$. Since

$$|\sigma_s(n)|=\Big|\sum_{d|n}d^s\Big|\leqslant\sigma_\sigma(n)=O\{n^{\frac{1}{2}(\sigma+\,\sigma|)+\epsilon}\},$$

where $\epsilon=0$ if $|\sigma|>1$, and $\epsilon>0$ if $|\sigma|\leqslant 1$, we have, by (3.32–3.34), if $B=B(s,\sigma,\lambda)$ is an appropriate positive constant,

$$\begin{aligned}\Big|\sum_{n=1}^{\infty}\sigma_s(n)\,J_{\alpha,\gamma}-\Xi\Big|&<B\,\frac{x^{\frac{1}{2}(\lambda+\kappa+\sigma-\frac{1}{2})}}{N^{\frac{1}{2}\lambda+\frac{1}{2}}}\Big\{\sum_{n<Ax}n^{-\frac{1}{2}(\kappa+\frac{1}{2}-|\sigma|)+\epsilon}\\&\qquad+x^{\frac{1}{2}}\sum_{n>Ax}n^{-\frac{1}{2}(\kappa+\frac{3}{2}-|\sigma|)+\epsilon}+x^{-\frac{1}{2}(\kappa-\frac{1}{2}-|\sigma|)+\epsilon}\\&\qquad\times\Big(\sum_{Ax<n\leqslant x-\frac{1}{2}}\frac{1}{\sqrt{x}-\sqrt{n}}+\sum_{x+\frac{1}{2}<n<Ax}\frac{1}{\sqrt{n}-\sqrt{x}}\Big)\Big\}\\&=O\{N^{-\frac{1}{2}\lambda-\frac{1}{2}}x^{\frac{1}{2}(\lambda+\sigma+|\sigma|+1)+\epsilon}|\log x|\},\end{aligned}$$

provided that $\epsilon<\frac{1}{2}(\kappa-|\sigma|-\frac{1}{2})$; $\log x$ is, of course, without significance unless $\epsilon=0$, *i.e.* $|\sigma|>1$.

Lemma 6 will be required in the proof of Theorem 3. For the present it is sufficient to observe that it justifies change of order of summation

and integration in (4.11) unless x is an integer ($x = \xi$) and $0 > \lambda > -1$, in which case Ξ is meaningless.

LEMMA 7. *If* $\Delta(x) = \Delta_s^{(0)}(x)$, $x > A$, *then*

$$\Delta(x) = O\{x^{\frac{1}{2}(\sigma+|\sigma|)}\} \quad \text{if} \quad |\sigma| > 1, \tag{5.21}$$

$$\Delta(x) = O\{x^{\frac{1}{2}(\sigma+|\sigma|)}|\log x|\} \quad \text{if} \quad |\sigma| = 1, \tag{5.22}$$

$$\Delta(x) = O\{x^{\frac{1}{2}(1+\sigma)}\} \quad \text{if} \quad |\sigma| < 1. \tag{5.23}$$

Though these inequalities are elementary (the first two almost trivial) and must be well known, I have come across none but special cases in the literature. The cases $s = 0$ and $s = 1$ are due to Dirichlet*, $s = -1$ is due to Wigert†. Walfisz‡ has shown that, in (5.22), $\log x$ may be replaced by $\log x/\log\log x$ when $s = \pm 1$.

(i) If $\sigma < -1$, we have

$$D_s^{(0)}(x) = \sum_{k \leqslant x} k^s \left\{\frac{x}{k} - \left(\frac{x}{k} - \left[\frac{x}{k}\right]\right) - \tfrac{1}{2}\right\} - \tfrac{1}{2}\sigma_s(x) = x\zeta(1-s) + O(1).$$

If $\sigma = -1$, the remainder term is $O(\log x)$.

(ii) If $\sigma > 1$, $D_s^{(0)}(x)$ is equal to

$$\sum_{k \leqslant x} \sum_{h \leqslant x/k} h^s = \sum_{k \leqslant x} \left\{\frac{1}{s+1}\left(\frac{x}{k}\right)^{s+1} + O\left(\frac{x}{k}\right)^{\sigma}\right\} - \tfrac{1}{2}\sigma_s(x)$$

$$= \frac{x^{1+s}}{1+s}\zeta(1+s) + O(x^{\sigma}).$$

If $\sigma = 1$, the remainder term is $O(x \log x)$.

(iii) If $-1 < \sigma < 1$, we have, by Dirichlet's well-known process,

$$D_s^{(0)}(x) = \sum_{h \leqslant \sqrt{x}} h^s \sum_{k \leqslant x/h} 1 + \sum_{k \leqslant \sqrt{x}} \sum_{h \leqslant x/k} h^s - \sum_{h \leqslant \sqrt{x}} h^s \sum_{k \leqslant \sqrt{x}} 1 - \tfrac{1}{2}\sigma(x)$$

$$= x\left\{\zeta(1-s) + \frac{x^{\frac{1}{2}s}}{s}\right\} + O(x^{\frac{1}{2}+\frac{1}{2}\sigma}) + \sum_{k \leqslant \sqrt{x}} \left\{\frac{1}{s+1}\left(\frac{x}{k}\right)^{s+1} + O\left(\frac{x}{k}\right)^{\sigma}\right\}$$

$$- \sqrt{x}\{(\sqrt{x})^{s+1}/(s+1)\}$$

$$= x\zeta(1-s) + \frac{x^{1+\frac{1}{2}s}}{s(s+1)} + \frac{x^{1+s}}{1+s}\zeta(1+s) - \frac{x^{1+s}}{1+s}\frac{(\sqrt{x})^{-s}}{s} + O(x^{\frac{1}{2}+\frac{1}{2}\sigma}).$$

---

* 3, 56 and 59. Much more is known when $s = 0$; for references see 20.

† 17.

‡ 14.

LEMMA 8. *If $\phi(u)$ is L-integrable and bounded over $(0, x)$ and if, when $\gamma > 0$ and $x > 0$,*

$$\phi^{(\gamma)}(x) = \frac{1}{\Gamma(\gamma)} \int_0^x \phi(u)(x-u)^{\gamma-1}\, du;$$

and if, further,

$$|\phi(x)| < V, \quad |\phi^{(\gamma)}(x)| < W, \quad 0 < \gamma' < \gamma, \tag{5.31}$$

where V and W are non-decreasing functions of x, then

$$|\phi^{(\gamma')}(x)| < \gamma'' V^{1-(\gamma'/\gamma)} W^{(\gamma'/\gamma)} \quad (\gamma'' > 0), \tag{5.32}$$

where γ'' depends only on γ and γ'.

This is a known theorem*.

By Theorem 1 we have

$$\Delta_\kappa(x) = O\{x^{\frac{1}{2}(\kappa+\sigma+\frac{1}{2})}\}; \tag{5.33}$$

and it is readily verified, by means of (3.15) and Lemmas 7 and 8, that, if λ is an integer greater than $|\sigma| - \frac{1}{2}$, then every term of (4.12) is $o(1)$ as $N \to \infty$, (4.13) is null, and (4.14) is absolutely convergent. Thus, making $N \to \infty$ in Theorem 2, we have proved Theorem 4 (i) and (ii) for integral λ. Hence, when we change the order of summation and integration in (4.11), the result is

$$(2\pi)^\kappa \int_N^\infty \left(\frac{x}{t}\right)^{\frac{1}{2}(\rho+\kappa)} G_{\rho+\kappa}\{4\pi\sqrt{(xt)}\}\, \Delta_{\kappa-1}(t)\, dt. \tag{5.41}$$

If λ is not an integer and

$$\sigma < \lambda + \tfrac{1}{2} - 2\eta, \tag{5.42}$$

the above argument about the convergence of $\Sigma(\lambda, \infty)$ breaks down at one point; by (3.192) the term corresponding to $k = -1$ in (4.12) is then not $o(1)$ unless† $|\sigma| < \eta$; that is to say, if

$$\tfrac{1}{2}\lambda - \tfrac{1}{4} > [\tfrac{1}{2}\lambda + \tfrac{1}{2}],$$

or, in other words, if

$$2m + \tfrac{1}{2} < \lambda < 2m + 1 \quad (m = 0, 1, 2, \ldots),$$

so that $\eta = 2m+1$, then the series for $\Delta_\lambda(x)$ does not converge if

$$\lambda + \tfrac{1}{2} > -\sigma \geqslant 2m + 1.$$

---

* Marcel Riesz, **12**.

† In addition to $\lambda > |\sigma| - \frac{1}{2}$.

If all we know about λ is that $\lambda > -1$, we cannot improve upon Theorem 2; but for larger values of λ we may integrate (5.41) by parts, thus cancelling the terms $k = \kappa-1$, $\kappa-2$, etc., of (4.12) so long as the terms which are cancelled tend to zero as $N \to \infty$. To determine for what values of k this is true we may use

$$\Delta_\lambda(x) = O\{x^{\frac{1}{2}(\lambda+\sigma+\frac{1}{2})}\}, \quad \text{if} \quad \lambda > |\sigma|+\tfrac{1}{2}, \tag{5.51}$$

in place of (5.33), since it is impossible now to satisfy (5.42). From (5.51), with the help of (5.21–5.32), we immediately derive

LEMMA 9. *If* $0 < \lambda \leqslant |\sigma|+\frac{1}{2}$, $x > A$, *and* $\epsilon > 0$, *then*

$$\Delta_\lambda(x) = O\left[\exp\left\{\left(\tfrac{1}{2}r+\tfrac{1}{2}\sigma+\lambda\,\frac{|\sigma|+1-r}{2|\sigma|+1}+\epsilon\right)\log x\right\}\right], \tag{5.52}$$

where $r = \mathrm{Max}(|\sigma|, 1)$.

In (5.31–5.32) we take* $\phi(x) = \Delta(x) = \Delta_0(x)$, $\gamma = |\sigma|+\frac{1}{2}+\epsilon$, $\gamma' = \lambda$,

$$V = x^{\frac{1}{2}(r+\sigma)}|\log x|, \quad W = x^{\frac{1}{2}(1+\sigma+|\sigma|+\epsilon)}.$$

It is now easily verifiable that (5.41) may be integrated by parts until κ in (5.41) and in (4.12) is replaced by Λ, where

$$\Lambda = \mathrm{Max}\left([|\sigma|+\tfrac{1}{2}-\lambda-(\lambda+1)\{(|\sigma|+1-r)/(r-\tfrac{1}{2})\}],\ 1\right), \tag{5.53}$$

if either λ is an integer or $\sigma \geqslant -\eta-1$. If $\sigma < -\eta-1$ and λ is not an integer, Λ is the greater of the two integers† in (5.53) and

$$[|\sigma|+\tfrac{1}{2}-\eta-\{(\eta+1)/(2|\sigma|)\}]. \tag{5.54}$$

In particular, when λ ($\geqslant 0$) is an integer, we may put $\Lambda = \lambda+1$ if

$$\lambda+1 > \tfrac{1}{2}|\sigma|-\tfrac{1}{8}|\sigma|^{-1}. \tag{5.55}$$

When this condition is satisfied we may derive an integral equation for $\Delta_\lambda(x)$ corresponding to that first given by Hardy‡ for the case $s = \lambda = 0$. In (5.41) and (4.12) we replace κ by $\lambda+1$, and in Theorem 2, thus modified, we take $N < 1$, and therefore $\Delta_k(N) = -\Phi_k(N)$.

---

* If $\sigma < -1$, $\Delta(x)$ is not bounded as $x \to 0$, but the difficulty is trivial; we define $\phi(x)$ by

$$\phi(x) = \Delta(x) \quad (x \geqslant 1),$$

$$\phi(x) = \Delta(x) + \{x^{s+1}/(s+1)\}\,\zeta(s+1) \quad (0 < x < 1).$$

† I have consistently used square brackets as a symbol for "greatest integer not exeeeding", but where there was no danger of confusion I have not hesitated to employ them as ordinary brackets when it was convenient to do so.

‡ 7.

LEMMA 10. *If $x>0$, $0<N<1$, and λ is an integer satisfying (5.55), then*

$$(5.61)\quad (2\pi)^{\lambda}\Delta_{\lambda}(x)=(2\pi)^{\lambda+1}\int_{N}^{\infty}\left(\frac{x}{t}\right)^{\frac{1}{2}(\rho+\lambda+1)}G_{\rho+\lambda+1}\{4\pi\sqrt{(xt)}\}\Delta_{\lambda}(t)\,dt$$

$$(5.62)\quad +\sum_{k=-\theta}^{\lambda}(2\pi)^{k}\left(\frac{x}{N}\right)^{\frac{1}{2}(\rho+k)}G_{\rho+k}\{4\pi\sqrt{(Nx)}\}\Phi_{k}(N)$$

$$(4.14)\quad +\frac{1}{(2\pi)^{\theta}}\int_{N}^{\infty}\left(\frac{x}{t}\right)^{\frac{1}{2}(\rho-\theta)}G_{\rho-\theta}\{4\pi\sqrt{(xt)}\}\Phi_{-\theta-1}(t)\,dt.$$

Let $\sigma\geqslant 0$, and let

$$(5.63)\quad \mathbf{g}_{\rho+k}(z)=G_{\rho+k}(z)-\mathbf{G}_{\rho+k}(z),$$

where $\mathbf{G}_{\rho+k}(z)$ is defined by (4.61), if $\theta=-k$. It is then easy to see that the integral obtained by writing $\mathbf{g}$ for G in (5.61) is absolutely convergent, and that it may be integrated by parts any number of times, provided that (after λ integrations) we put $\Delta_{0}(t)=D_{0}(t)-\Phi_{0}(t)$ and proceed with $\Phi_{0}(t)$ only. Thus, in Lemma 10, we may replace G by $\mathbf{G}$, provided that we add to the right-hand side

$$2\pi\int_{N}^{\infty}\left(\frac{x}{t}\right)^{\frac{1}{2}\rho+\frac{1}{2}}\mathbf{g}_{\rho+1}\{4\pi\sqrt{(xt)}\}D_{0}(t)\,dt$$

$$=\sum_{n=1}^{\infty}\sigma_{s}(n)\left(\frac{x}{n}\right)^{\frac{1}{2}\rho}\mathbf{g}_{\rho}\{4\pi\sqrt{(nx)}\}=-(2\pi)^{\lambda}S_{\lambda}(x),$$

where $S_{\lambda}(x)$ is given by (4.27), with $\kappa=\lambda$. Let

$$\psi_{\lambda}(x)=\Delta_{\lambda}(x)+S_{\lambda}(x)-\tfrac{1}{2}(x^{\lambda}/\lambda!)\,\zeta(-s)$$

$$(5.64)\quad =D_{\lambda}(x)-\frac{x^{\lambda+s+1}}{\Gamma(\lambda+s+2)}\Gamma(1+s)\,\zeta(1+s)-\frac{x^{\lambda+1}}{(\lambda+1)!}\zeta(1-s),$$

then, by integration by parts and by the equality of (4.57) and (4.62), we have

$$(5.65)\quad (2\pi)^{\lambda}\psi_{\lambda}(x)=(2\pi)^{\lambda+1}\int_{N}^{\infty}\left(\frac{x}{t}\right)^{\frac{1}{2}(\rho+\lambda+1)}\mathbf{G}_{\rho+\lambda+1}\{4\pi\sqrt{(xt)}\}\psi_{\lambda}(t)\,dt$$

$$(5.66)\quad +\sum_{k=-\theta}^{\lambda}(2\pi)^{k}\left(\frac{x}{N}\right)^{\frac{1}{2}(\rho+k)}\mathbf{G}_{\rho+k}\{4\pi\sqrt{(Nx)}\}\phi_{k}(N)$$

$$-\frac{1}{(2\pi)^{\theta}}\int_{0}^{N}\left(\frac{x}{t}\right)^{\frac{1}{2}(\rho-\theta)}\mathbf{G}_{\rho-\theta}\{4\pi\sqrt{(xt)}\}\Phi_{-\theta-1}(t)\,dt,$$

where

$$\phi_{k}(N)=\frac{N^{s+k+1}}{\Gamma(s+k+2)}\Gamma(1+s)\,\zeta(1+s)+\frac{N^{k+1}}{\Gamma(k+2)}\zeta(1-s).$$

If $0 \leqslant \sigma < 1$, we may make $N \to 0$, and we obtain the integral equation which results from (5.65) on putting $N = 0$. But if $\sigma \geqslant 1$, the integral does not converge, and (5.66) does not tend to zero, as $N \to 0$. In this case we may cancel the troublesome term of $\phi_k(N)$, namely that in N^{k+1}, by adding $t^{\lambda+1}\zeta(1-s)/(\lambda+1)!$ to $\psi_\lambda(t)$. We thus have

THEOREM A. *If* $\sigma \geqslant 0$, λ $(\geqslant 0)$ *is an integer satisfying* (5.55), *and* $\psi_\lambda(x)$ *is given by* (5.64), *then*

$$\psi_\lambda(x) = 2\pi \int_0^\infty \left(\frac{x}{t}\right)^{\frac{1}{2}(\rho+\lambda+1)} \mathbf{G}_{\rho+\lambda+1}\{4\pi\sqrt{(xt)}\}\, \Psi_\lambda(t)\, dt,$$

where $\Psi_\lambda(t) = \psi_\lambda(t)$ *if* $\sigma < 1$, *and**

$$\Psi_\lambda(t) = \psi_\lambda(t) + t^{\lambda+1}\zeta(1-s)/(\lambda+1)!, \quad \text{if} \quad \sigma \geqslant 1.$$

When σ is negative there is little difficulty in deducing from Lemma 10 the analogous

THEOREM A′. (i) *If* $0 > \sigma > -1$ (*and therefore, by Theorem A, if* $|\sigma| < 1$) *and*

$$\psi_0(x) = \Delta_0(x) - \tfrac{1}{2}\zeta(-s),$$

then

$$\psi_0(x) = 2\pi \int_0^\infty \left(\frac{x}{t}\right)^{\frac{1}{2}\rho+\frac{1}{2}} G_{\rho+1}\{4\pi\sqrt{(xt)}\}\, \psi_0(t)\, dt.$$

(ii) *If* $\sigma < -\tfrac{1}{2}$, λ *is the* LEAST *integer satisfying* (5.55), *and*

$$\psi_0(x) = \Delta_0(x),$$

$$\psi_\lambda(x) = \Delta_\lambda(x) + \{x^\rho/\Gamma(\rho+1)\}\,\Gamma(s+1)\,\zeta(s+1) \quad (\lambda \geqslant 1),$$

then

$$\psi_\lambda(x) = 2\pi \int_0^\infty \left(\frac{x}{t}\right)^{\frac{1}{2}(\rho+\lambda+1)} G_{\rho+\lambda+1}\{4\pi\sqrt{(xt)}\}\, \Psi_\lambda(t)\, dt,$$

where

$$\Psi_0(t) = \Delta_0(t) + \{x^{s+1}/(s+1)\}\,\zeta(s+1) - \tfrac{1}{2}\zeta(-s),$$

$$\Psi_\lambda(t) = \psi_\lambda(t) - \zeta(-\lambda-s)\,\zeta(-\lambda)/\lambda! \quad (\lambda \geqslant 1).$$

6. *Proof of Theorem* 3.

Almost the whole of the ground has now been cleared for the proof of the theorem of this section; I require, in addition to what has been done,

* It is a very simple matter to verify the somewhat unexpected result of Theorem A in the particular case $s = 1$, $\lambda = 0$.

only an improvement on Lemma 9 for the case when $|\sigma| < \frac{1}{2}$. In this case $\kappa = 1$ and the second term of (3.192) is not significant for $\lambda > -1$, $k \geqslant -\theta = -1$. Thus, if $\Sigma(\lambda, N)$ is defined by (4.10), we have, by Theorem 2, (5.23), and Lemma 6, if $N > A$, $x > A$,

$$\Sigma(\lambda, N) = (5.11) + O\{x^{\frac{1}{2}(\lambda+\sigma+\frac{1}{2})} N^{-\frac{1}{2}(\lambda+\frac{1}{2})}\}$$

$$+ O\{x^{\frac{1}{2}(\lambda+\sigma-\frac{1}{2})} N^{\frac{1}{2}|\sigma|-\frac{1}{2}(\lambda+\frac{1}{2})} |\log N|\}. \tag{6.10}$$

In particular, if* $N < Ax$, $-1 < \lambda \leqslant |\sigma| + \frac{1}{2}$,

$$\Delta_\lambda(x) = x^{-\frac{1}{4}+\frac{1}{2}\sigma} \left(\frac{x}{N}\right)^{\frac{1}{2}\lambda+\frac{1}{2}} \{O(x^{\frac{1}{4}+\frac{1}{2}|\sigma|+\epsilon}) + O(N^{\frac{1}{2}+\frac{1}{2}|\sigma|+\epsilon})\}.$$

Choosing $N = x^{\{(|\sigma|+\frac{1}{2})/(|\sigma|+\frac{3}{2})\}}$, we have

LEMMA 11. *If* $|\sigma| < \frac{1}{2}$, $x > A$, $-1 < \lambda \leqslant |\sigma| + \frac{1}{2}$, *then*

$$\Delta_\lambda(x) = O\{x^{\frac{1}{2}(\sigma+|\sigma|)+\{(\lambda+1)/(2|\sigma|+3)\}+\epsilon}\} \quad (\epsilon > 0), \tag{6.11}$$

provided that, if $\lambda < 0$, x *is not an integer. The inequality* (6.11) *is uniform if* $\lambda \geqslant 0$; *if* $\lambda < 0$, *it is uniform in the set* I *of intervals* (1.31).

The index of x in (5.52) exceeds that of x in (6.11) by a positive multiple of

$$(\tfrac{1}{2} - |\sigma|)(|\sigma| + \tfrac{1}{2} - \lambda),$$

if $|\sigma| < 1$, so that (5.52) is the better inequality if $|\sigma| > \frac{1}{2}$.

We are now in a position to state

THEOREM 3. *If* $\Sigma(\lambda, N)$ *is given by* (4.10), Ξ *is as defined in Lemma* 6, $\lambda > -1$, $N > A$, $x > A$, η *is given by* (1.152), *and either* (i) $\lambda (\geqslant 0)$ *is an integer, or* (ii)

$$\sigma > \lambda + \tfrac{1}{2} - 2\eta \tag{6.21}$$

(*in particular, if* $\sigma > -\frac{1}{2}$), *then* $\Sigma(\lambda, N) - \Xi$ *is*

$$O\{x^{\frac{1}{2}(\lambda+\sigma+|\sigma|+1)+\epsilon} N^{-\frac{1}{2}\lambda-\frac{1}{2}}\} \tag{6.22}$$

$$+ O\{x^{\frac{1}{2}(\lambda+\sigma-\frac{1}{2})} N^{\frac{1}{2}(|\sigma|-\lambda-\frac{1}{2})} |\log N|\}, \tag{6.23}\dagger$$

uniformly in $A < x < A$ *for every* $\epsilon > 0$. *If neither* (i) *nor* (ii) *is satisfied,*

* See (5.51) for greater values of λ. If $\lambda < 0$, x must not be an integer. In (6.10) the factor $\log N$ arises only when $s = 0$.

† The factor $\log N$ may be omitted except when $s = 0$.

(6.23) *is to be replaced by**

$$O\{x^{\lambda-\eta}N^{|\sigma|-\eta}(1+|\log Nx|)\}. \tag{6.24}$$

If x is in the set I of intervals (1.31), Ξ *is included in* (6.22); *if, for a given* $\vartheta(>0)$, x *is not in* I, *then*

$$\Xi = O\{x^{\frac{1}{2}(\lambda+\sigma+|\sigma|)+\epsilon}N^{-\frac{1}{2}\lambda}\}+\phi(x), \tag{6.25}$$

where $\phi(x)=0$, *if* $\lambda \geqslant 0$, *and*

$$\phi(x) = O\{x^{\frac{1}{2}(\sigma+|\sigma|)+\epsilon}|x-\xi|^{-|\lambda|}\}, \quad \text{if} \quad \lambda<0 \quad (x\neq\xi). \tag{6.26}$$

An evident corollary of Theorem 3 is perhaps of sufficient interest to be stated separately.

THEOREM 3*a*. *Suppose that* (6.23) *dominates* (6.24), *that* $\sigma \leqslant 0$, *that* $\lambda \leqslant |\sigma|-\frac{1}{2}$ [*so that* $\Sigma(\lambda, \infty)$ *does not converge*], *and that*

$$0<c<1/(|\sigma|-\lambda-\tfrac{1}{2}).$$

Let $N=Ax^{1+c}$; *then, as* $x\to\infty$,

$$\Sigma(\lambda, N)-\Xi = O(x^{-\epsilon}) \quad \text{for some } \epsilon>0.$$

In particular, if $c=1/(|\sigma|+\frac{1}{2})$, *then*

$$\Sigma(\lambda, N)-\Xi = O\{x^{\epsilon-(\lambda+1)/(2|\sigma|+1)}\},$$

for every $\epsilon>0$.

It may be observed that (6.23) is included in (6.22) if

$$N<Ax^{\{(|\sigma|+\frac{3}{2})/(|\sigma|+\frac{1}{2})\}};$$

and if also $N>Ax$, then Ξ is the dominant term of $\Sigma(\lambda, N)$. When (6.24) is significant it dominates $\Sigma(\lambda, N)$, if x is in I and N is sufficiently large.

To take the last part of the theorem first, (6.25) follows from (5.12) if $\lambda \geqslant 0$; (6.26) follows from (5.12), if we rewrite the integral there as

$$\int_0^\infty - \int_0^{4\pi\sqrt{N}|\sqrt{x}-\sqrt{\xi}|}.$$

Theorem 3 will now follow from Theorem 2 and Lemma 6, if we show that (4.12–4.14) together satisfy one or other of the inequalities implied by (6.23) and (6.24), for (6.21) is a necessary and sufficient condition that

* The expression $\log(Nx)$ may be omitted except when $s=-2n$ or $s=0$.

(6.23) should dominate (6.24). It will appear that (6.23) and (6.24) both arise from the single term corresponding to $k=-1$ in (4.12).

It is a simple matter to show, from (3.192), that (4.14) is of the same order of magnitude as the term corresponding to $k=\theta+1$ would be, if it was included in (4.13). If therefore, for the present, we put

$$\Delta_k(N)=N^{s+k+1} \quad \text{when} \quad k\leqslant -2,$$

it is sufficient to consider, in place of (4.12–4.14), the single sum,

$$\sum_{k=-\theta-1}^{\kappa-1}\left|\left(\frac{x}{N}\right)^{\frac{1}{2}(\rho+k)} G_{\rho+k}\{\sqrt{(Nx)}\}\Delta_k(N)\right|. \tag{6.31}$$

In (6.31) the terms which arise from the part (3.191) of the asymptotic expansion of $G_{\rho+k}$ are of the order

$$x^{\lambda-\eta}N^{-(\eta+\sigma+k+1)}|\log(Nx)|\,|\Delta_k(N)|, \tag{6.32}$$

where $k=-\theta-1, \ldots, \kappa-1$; those which arise from (3.19) are of the order

$$x^{\frac{1}{2}(\sigma+\lambda+k+\frac{1}{2})}N^{-\frac{1}{2}(\sigma+\lambda+k+\frac{3}{2})}|\Delta_k(N)| \quad (k=-\theta-1, \ldots, \kappa-1). \tag{6.33}$$

Since* $|\Delta_{-1}(N)|=\Omega\{N^{\frac{1}{2}(\sigma+|\sigma|)}\}$, it is immediately evident that, in both (6.32) and (6.33), the terms $k=-\theta-1, \ldots, -2$ are dominated by the term $k=-1$.

The ratio of (6.32) to (6.33) is

$$(Nx)^{\frac{1}{2}(\lambda-\sigma-k-\frac{1}{2})-\eta}|\log(Nx)|.$$

Hence a necessary and sufficient condition for the predominance of (6.33) is†

$$\sigma-\lambda+k+\tfrac{1}{2}+2\eta>0. \tag{6.34}$$

It will be shown presently that the dominant term in (6.32) is that corresponding to $k=-1$. With this value of k, (6.34) is (6.21).

(i) If $|\sigma|<\frac{1}{2}$, we have $\kappa=\theta=1$, and $\Delta_0(N)$ is controlled by (6.11). In this case (6.33), when $k=0$, is

$$O\{x^{\frac{1}{2}(\lambda+\sigma+\frac{1}{2})}N^{\frac{1}{2}|\sigma|+(2|\sigma|+3)^{-1}-\frac{1}{2}\lambda-\frac{3}{4}+\epsilon}\},$$

* $\Delta_{-1}(N)=\Omega(\log N)$ when $s=0$.

† The negation of (3.193).

and this is included in (6.23) or (6.22) according as

$$x^{1+\vartheta} \leqq AN^{\{(|\sigma|+\frac{1}{2})/(|\sigma|+\frac{3}{2})\}},$$

for a sufficiently small positive ϑ.

(ii) If $|\sigma| \geqslant \frac{1}{2}$, we employ Lemma 9. The terms of (6.32) corresponding to $k = 0, 1, \ldots, \kappa-1$ are

$$O\{x^{\lambda-\eta} N^{\frac{1}{2}(r-\sigma)-\eta-1-k\{(|\sigma|+r)/(2|\sigma|+1)\}+\epsilon}\}$$
$$= O\{x^{\lambda-\eta} N^{\frac{1}{2}(|\sigma|-\sigma)-\eta} |\log(Nx)|\},$$

corresponding to $k = -1$, as already stated.

The terms of (6.33) corresponding to $k = 0, 1, \ldots, \kappa-1$ are together

$$(6.35)+(6.36) \qquad O\{x^{\frac{1}{2}(\lambda+\sigma+\frac{1}{2})} N^{\frac{1}{2}(r-\frac{1}{2})-\frac{1}{2}\lambda-\frac{1}{2}+\epsilon}\}\{1+(x/N^{(r-\frac{1}{2})/(|\sigma|+\frac{1}{2})})^{\frac{1}{2}\kappa-\frac{1}{2}}\}.$$

In the particular case when $|\sigma| = \frac{1}{2}$, we have $\kappa = 2$, $r = 1$, and the index of N in (6.36) is $-\frac{1}{2}\lambda-\frac{1}{2}+\epsilon$; hence, in this case,

$$(6.35) = O\{(6.22)\} \quad \text{and} \quad (6.36) = O\{(6.22)\} \quad \text{if} \quad N < Ax^{2+\vartheta},$$

$$(6.35) = O\{(6.23)\} \quad \text{and} \quad (6.36) = O\{(6.23)\} \quad \text{if} \quad N > Ax^{2+\vartheta},$$

for a sufficiently small positive ϑ. When $|\sigma| > \frac{1}{2}$ it is readily verified that

$$(6.35) = O\{(6.36)\} = O\{(6.22)\} \quad \text{if} \quad N < Ax^{\{(|\sigma|+\frac{1}{2})/(r-\frac{1}{2})\}},$$

$$(6.36) = O\{(6.35)\} = O\{(6.23)\} \quad \text{if} \quad N > Ax^{\{(|\sigma|+\frac{1}{2})/(r-\frac{1}{2})\}}.$$

This completes the proof of Theorem 3.

7. *Proof of Theorem* 4.

The proof of Theorem 4 follows closely the lines of the proof of the corresponding theorem in the theory of the lattice points of an n-dimensional ellipsoid*. I require a further simple lemma.

LEMMA 12. *If $\phi'(t)$ is bounded in $(0, 1)$, $u > A$, $0 \leqslant \beta < 1-A$, and $0 < \vartheta < 1$, then*

$$(7.11) \qquad \frac{1}{\Gamma(\vartheta)} \int_\beta^1 (1-t)^{\vartheta-1} \phi(t) \cos(ut-c)\, dt$$
$$= \phi(1) u^{-\vartheta} \cos(u-c-\tfrac{1}{2}\pi\vartheta) + O(1/u),$$

uniformly with respect to ϑ and β.

* Wilton, 18, Theorems D and D''.

Let $\chi(t) = \int_t^\infty v^{\vartheta-1}\cos(u-c-uv)\,dv$, then $|\chi(t)| < At^{\vartheta-1}/u$, and, therefore,

$$\int_\beta^1 (1-t)^{\vartheta-1}\phi(t)\cos(ut-c)\,dt = -\int_0^{1-\beta}\phi(1-t)\chi'(t)\,dt$$
$$= \phi(1)\chi(0) - \phi(\beta)\chi(1-\beta) - \int_0^{1-\beta}\phi'(1-t)\chi(t)\,dt,$$

which proves the lemma.

Clauses (i) and (ii) of Theorem 4 have already been proved. I now attack clause (iii), taking, as the definition of the sum* (R, n, q) of $\Sigma(\lambda, \infty)$,

$$\sum_{(R,n,q)}(\lambda, \infty) = \lim_{\omega\to\infty} q\omega^{-q}\int_0^\omega (\omega-\tau)^{q-1}\Sigma(\lambda, \tau)\,d\tau = \lim_{\omega\to\infty} q\omega^{-q}\int_A^\omega \ldots.$$

Since (4.11), or its equivalent (5.41), and (4.14) converge to zero as $N\to\infty$ (and are therefore summable to the sum 0, if $q > 0$), it follows from Theorem 2 that we have to examine the two limits,

$$(7.21)\qquad \lim_{\omega\to\infty} q\omega^{-q}\int_A^\omega (\omega-\tau)^{q-1}\left(\frac{x}{\tau}\right)^{\frac{1}{2}(\rho+k)} G_{\rho+k}\{4\pi\sqrt{(x\tau)}\}\Delta_k(\tau)\,d\tau,$$

for $k = -1, 0, \ldots, \kappa-1$, and

$$(7.22)\qquad \lim_{\omega\to\infty} q\omega^{-q}\int_A^\omega (\omega-\tau)^{q-1}\left(\frac{x}{\tau}\right)^{\frac{1}{2}(\rho-k)} G_{\rho-k}\{4\pi\sqrt{(x\tau)}\}\tau^{s-k+1}\,d\tau,$$

for $k = 2, \ldots, \theta$. If $\theta = 1$, (7.22) is null; if $\theta > 1$, it is equal to the limit, as $\omega\to\infty$, of a constant multiple of

$$(7.23)\qquad q\omega^{-\lambda}\int_{A/\omega}^1 (1-\tau)^{q-1}\tau^{s-k+1}\frac{d^m}{d\tau^m}\left[\left(\frac{\omega x}{\tau}\right)^{\frac{1}{2}(\rho-k-m)} G_{\rho-k-m}\{4\pi\sqrt{(\omega x\tau)}\}\right]d\tau,$$

where $$2 \leqslant k \leqslant k+m \leqslant \theta = [\sigma+\tfrac{1}{2}-\lambda] \quad (\sigma \geqslant \tfrac{3}{2}+\lambda).$$

When we integrate (7.23) by parts, the integrated part at every stage is $O(1/\omega)$ so long as $q > \theta-2$ $(\geqslant m)$. Hence, with an error $O(1/\omega)$, (7.23) is equal to a constant multiple of

$$(7.24)\qquad \omega^{-\lambda}\int_{A/\omega}^1 \frac{d^m}{d\tau^m}\{\tau^{s-k+1}(1-\tau)^{q-1}\}\left(\frac{\omega x}{\tau}\right)^{\frac{1}{2}(\rho-\theta)} G_{\rho-\theta}\{4\pi\sqrt{(\omega x\tau)}\}\,d\tau,$$

if we choose $m = \theta-k$. By Lemma 12 the term arising from (3.19) in

* Hardy and Riesz, **8**, 21.

the asymptotic expansion of $G_{\rho-\theta}$ is easily seen to be $o(1)$ provided that

$$q-m > \sigma+\tfrac{1}{2}-\lambda-\theta \ (\geqslant 0). \tag{7.25}$$

The term arising from (3.191) is of order

$$\omega^{-\eta}\int_{A/\omega}^{1}\frac{d^m}{d\tau^m}\{\tau^{s-k+1}(1-\tau)^{q-1}\}(x\tau)^{\lambda-\eta}\tau^{\theta-\rho}\,d\tau = O\left(\frac{1}{\omega}\right). \tag{7.26}$$

In (7.25) the greatest value of m is $\theta-2$; hence (7.22) is $o(1)$ if

$$q > \sigma-\tfrac{3}{2}-\lambda \ (\geqslant 0).$$

All this is on the supposition that $\theta \geqslant 2$.

It remains to consider (7.21). Here also, when we integrate by parts, the integrated part at each stage is $O(1/\omega)$. The term corresponding to $k=-1$ must be considered separately. One part of $\Delta_{-1}(\tau)$, namely

$$-\tau^s\,\zeta(1+s),$$

may be included in (7.22) by allowing k to take the value unity. This term makes $m=\theta-1$ in (7.25) and so requires us to take

$$q > \sigma-\tfrac{1}{2}-\lambda, \quad q \geqslant 0 \tag{7.271}$$

to ensure summability. The other part* of $\Delta_{-1}(\tau)$, namely the constant $-\zeta(1-s)$, may be included in (7.22), if we put $k=1$ and ignore the factor τ^{s-k+1}. This changes the factor $\omega^{-\lambda}$ of (7.23) to $\omega^{-\lambda-s}$, and the effect is to change σ into $-\sigma$ in (7.25); *i.e.* the vanishing of the limit in the case of this term requires that

$$q > -\sigma-\tfrac{1}{2}-\lambda, \quad q \geqslant 0. \tag{7.272}$$

In the same way, instead of (7.26), we have a term of order

$$\frac{\log\omega}{\omega^{\eta+\sigma}}\int_{A/\omega}^{1}(1-\tau)^{q-\theta}\tau^{\theta-\eta-\sigma-1}\,d\tau = O\left(\frac{\log\omega}{\omega^{\eta+\sigma}}\right)+O\left(\frac{\log\omega}{\omega}\right), \tag{7.28}$$

since $\theta \geqslant 1$. Moreover, for all values of $q>0$, this term corresponding to (7.28) is $\Omega(\omega^{-\sigma-\eta})$, if $\sigma \leqslant -\eta$ (and therefore $\theta=1$).

---

* In the particular case when $s=0$ the two parts of $\Delta_{-1}(\tau)$ cannot be separated; they are together equal to $-\log\tau-2C$, where C is Euler's constant; thus (7.271) is a sufficient condition for this special case, and $\Sigma(\lambda,\infty)$ is convergent if $\lambda > -\frac{1}{2}$.

Proceeding as in the case of (7.23), we see that, with an error $O(1/\omega)$ at each stage of the integration by parts, the remaining terms of (7.21) are constant multiples of

$$(7.31)\quad \omega^{-(\rho+k+m)}\int_{A/\omega}^{1}\frac{d^m}{d\tau^m}\left[(1-\tau)^{q-1}\left(\frac{\omega x}{\tau}\right)^{\frac{1}{2}(\rho+k)}G_{\rho+k}\{4\pi\sqrt{(\omega x\tau)}\}\right]\Delta_{k+m}(\omega\tau)\,d\tau \qquad (k=0,\ 1,\ \ldots,\ \kappa-1).$$

When the differentiation is carried out in (7.31) we obtain a set of terms which are constant multiples of

$$(7.32)\quad \omega^{-(\rho+k+m)}\int_{A/\omega}^{1}(1-\tau)^{q-m+j-1}\left(\frac{\omega x}{\tau}\right)^{\frac{1}{2}(\rho+k+j)}G_{\rho+k+j}\{4\pi\sqrt{(\omega x\tau)}\}\times\Delta_{k+m}(\omega\tau)\,d\tau$$
$$=O(\omega^{-\frac{1}{2}-\frac{1}{2}\lambda})+O(\omega^{-1})=o(1),$$

by (3.192), if $m+k=\kappa$, $j=0,\ 1,\ \ldots,\ m$. (It is, of course, supposed that $m<q$.) And, if $k+m<\kappa$, the term arising from (3.191) in the asymptotic expansion of $G_{\rho+k+j}$ in (7.32) is

$$(7.321)\qquad O(\omega^{-\sigma-\eta-1+\epsilon})+O(\omega^{-1}).$$

Again, if $m=[q]$, $q>|\sigma|-\frac{1}{2}-\lambda$, $k+m<\kappa$, and $j=0$, then, by Lemma 9, the term arising from (3.19) in (7.32) is

$$O\{\omega^{\frac{1}{2}(|\sigma|-\frac{1}{2}-\lambda-[q])+\epsilon}\}=o(1),$$

if $|\sigma|\geqslant 1$. But if $|\sigma|<1$, it will be found that, in order to ensure that $(7.32)=o(1)$ when $j=0$, we must take $\lambda>-\frac{1}{2}$*.

In the special case when $|\sigma|-\frac{1}{2}-\lambda$ is an integer the term corresponding to $j=0$ vanishes when $q=|\sigma|-\frac{1}{2}-\lambda$. This remark is required in the proof of clause (iv) of Theorem 4.

Thus if $m=[|\sigma|-\frac{1}{2}-\lambda]$ we need consider only the case $j>0$ in (7.32); we then integrate again by parts, either j or $\kappa-k-m$ times, whichever is the less. If $j<\kappa-k-m$, we may (after j integrations by parts) reject the one term which contains a negative power of $1-\tau$, since this has been shown to be $o(1)$. Repeating this process sufficiently often,

* Lemma 11 allows us to improve slightly on this when $|\sigma|<\frac{1}{2}$. It is then sufficient to take

$$\lambda>-\tfrac{1}{2}-\frac{|\sigma|+1}{|\sigma|+\frac{3}{2}}(\tfrac{1}{2}-|\sigma|).$$

In particular, when $\sigma=0$, and $-\frac{1}{2}\geqslant\lambda>-\frac{5}{6}$, $\mathfrak{Z}(\lambda,\infty)$ is summable $(R,n,q>-\lambda-\frac{1}{2})$. A further extension is possible when $s=0$.

we see that (7.21) vanishes when $k = 0, 1, \ldots, \kappa-1$, if*

$$q \geqslant m = [|\sigma|-\tfrac{1}{2}-\lambda].$$

This completes the proof of Theorem 4 (iii).

By (7.321) and the remark following (7.28), the series $\Sigma(\lambda, \infty)$ is not summable for any q if λ is not an integer and $\sigma \leqslant -\eta$, a condition which is impossible if $\lambda \geqslant |\sigma|$. The non-summability of the series in this case may also be seen directly from Theorem 3; for (6.24) arises from a single non-zero constant† multiple of $N^{-s-\eta}\log(Nx)$ or $N^{-s-\eta}$, according as s is or is not an even integer such that $s \leqslant -\eta-1$; and when $\sigma \leqslant -\eta$ this term is not summable for any value of q.

It remains to show that $\Sigma(\lambda, \infty)$ is not summable $(R, n, |\sigma|-\frac{1}{2}-\lambda)$. After what has been said this follows at once from Lemma 12; for the typical mean corresponding to one of the two parts of $\Delta_{-1}(\tau)$, viz. $-\tau^s\zeta(1+s)$ if $\sigma > 0$ and $-\zeta(1-s)$ if $\sigma < 0$, is a non-zero multiple of

$$\omega^{\frac{1}{2}it\,\mathrm{sgn}\,\sigma}\sin\{4\pi\sqrt{(\omega x)}-\tfrac{1}{2}|\sigma|\pi\}, \tag{7.41}$$

where $s = \sigma+it$; and (7.41) oscillates as $\omega \to \infty$. In case $s = 0$, the sine, in (7.41), is multiplied by $\log\omega$, and the mean oscillates infinitely.

Finally, the uniformity of convergence and summability when $\lambda > 0$, or when $\lambda \leqslant 0$ and x is in I, follows from (5.11), for the only possible failure of uniformity lies in the behaviour of the single term Ξ as $x \to \xi$.

In the particular case when $\lambda = 0$ and s is real the sum exhibits the Gibbs phenomenon of the series; for, if the limits refer to the integral

$$q\omega^{-q}\int_0^{\omega}(\omega-N)^{q-1}\Xi dN,$$

where Ξ is defined by (5.12) with $\lambda = 0$, we have

$$\lim_{\omega\to\infty}\lim_{x\to\xi+0} = \tfrac{1}{2}\sigma_s(\xi), \qquad \lim_{x\to\xi+0}\lim_{\omega\to\infty} = 0,$$

$$\lim_{\omega\to\infty,\ x\to\xi+0} = \frac{\sigma_s(\xi)}{\pi}\left\{\frac{\pi}{2} - \operatorname*{Max}_{u>0}\int_0^1(1-t^2)^q\frac{\sin ut}{t}dt\right\},$$

a negative multiple of $\sigma_s(\xi)$ if q is sufficiently small‡.

* This assumes the theorem of consistency, viz., that summability (R, n, q) implies summability (R, n, q'), if $q' > q$ ($\geqslant 0$ in our case).

† That is to say, independent of N.

‡ *Cf.* Gronwall, *Annals of Math.* (2), 31 (1930), 233–240.

References.

1. T. J. I'A. Bromwich, *Theory of infinite series* (ed. 1, London, 1908).
2. G. Lejeune Dirichlet, "Applications de l'analyse infinitésimale à la théorie des nombres", *Journal für Math.*, 19 (1839), 324–369, and 21 (1840), 1–12 and 134–155. (*Werke*, 1, 413–496.)
3. ———, "Über die Bestimmung der mittleren Werthe in der Zahlentheorie", *Verh. der K. Preuss. Akad. der Wiss.* (1849), 69–83. (*Werke*, 2, 51–66.)
4. A. L. Dixon and W. L. Ferrar, "Infinite integrals in the theory of Bessel functions", *Quart. J. of Math.* (Oxford), 1 (1930), 122–145.
5. ———, "Lattice-point summation formulae", *ibid.*, 2 (1931), 31–54.
6. G. H. Hardy, "The lattice points of a circle", *Proc. Royal Soc.* (A), 107 (1925), 623–635; *ibid.*, 120 (1928), 369.
7. ——— "On some points in the integral calculus, LXV", *Messenger of Math.*, 57 (1927), 113–120.
8. ——— and M. Riesz, *General theory of Dirichlet series* (Cambridge, 1915).
9. E. W. Hobson, *Theory of functions of a real variable*, 2 (Cambridge, 1926).
10. E. Landau, "Über das Konvergenzgebiet einer mit der Riemannschen Zetafunktion zusammenhängenden Reihe", *Math. Annalen*, 97 (1927), 251–290.
11. A. Oppenheim, "Some identities in the theory of numbers", *Proc. London Math. Soc.* (2), 26 (1927), 295–350.
12. M. Riesz, "Sur un théorème de la moyenne et ses applications à la théorie des fonctions", *Acta Reg. Univ. Hungaricae Franc. Jos.*, 1 (1923), 114–126.
13. A. Walfisz, "Über das Piltzsche Teilerproblem in algebraischen Zahlkörpern", *Math. Zeitschrift*, 22 (1925), 153–188.
14. ———, "Teilerprobleme", *ibid.*, 26 (1927), 66–88.
15. ———, "Über das Piltzsche Teilerproblem in algebraischen Zahlkörpern II", *ibid.*, 26 (1927), 487–494.
16. G. N. Watson, *Theory of Bessel functions* (Cambridge, 1922).
17. S. Wigert, "Sur quelques fonctions arithmétiques", *Acta Math.*, 37 (1914), 113–140.
18. J. R. Wilton, "A series of Bessel functions connected with the theory of lattice points", *Proc. London Math. Soc.* (2), 29 (1928), 168–188.
19. ———, "A series of Bessel functions connected with the lattice points of an n-dimensional ellipsoid", *Proc. Royal Soc.* (A), 120 (1928), 358–369.
20. ———, "On Dirichlet's divisor problem", *Proc. Royal Soc.* (A), 134 (1931), 192–202.

ÜBER EINIGE ASYMPTOTISCHE ENTWICKLUNGEN DER LEGENDRESCHEN FUNKTIONEN

Von G. SZEGÖ.

[Received 6 June, 1932.—Read 16 June, 1932.]

1. *Einleitung.*

Man kennt eine beträchtliche Anzahl von Näherungsformeln für die Legendreschen Polynome $P_n(z)$, die für grosse Werte von n in verschiedenen Gebieten der komplexen z-Ebene Gültigkeit haben. Um eine Übersicht und einen Vergleich dieser Formeln zu ermöglichen, schicken wir zunächst die wichtigsten unter ihnen voraus.

(*a*) Die Laplacesche Formel (1)*:

$$P_n(\cos\vartheta) = \sqrt{\left(\frac{2}{\pi n \sin\vartheta}\right)} \cos\left[(n+\tfrac{1}{2})\vartheta - \tfrac{1}{4}\pi\right] + O(n^{-\frac{3}{2}}),$$

gilt für $0 < \vartheta < \pi$, u. zw. gleichmässig in jedem Intervalle $\epsilon \leqslant \vartheta \leqslant \pi - \epsilon$, wobei $0 < \epsilon < \frac{1}{2}\pi$ ist.

(*b*) Die Heinesche Formel (2):

$$P_n(\cos\vartheta) = \sqrt{\left(\frac{2}{\pi n \sin\vartheta}\right)} \left\{\left(1 - \frac{1}{4n}\right) \cos\left[(n+\tfrac{1}{2})\vartheta - \tfrac{1}{4}\pi\right] + \frac{1}{8n} \operatorname{ctg}\vartheta \sin\left[(n+\tfrac{1}{2})\vartheta - \tfrac{1}{4}\pi\right]\right\} + O(n^{-\frac{5}{2}}),$$

stellt eine Verschärfung der Formel (*a*) dar; es ist darin wiederum

$$\epsilon \leqslant \vartheta \leqslant \pi - \epsilon, \quad 0 < \epsilon < \tfrac{1}{2}\pi.$$

---

* Die fett gedruckten Zahlen verweisen auf das Literaturverzeichnis am Schluss der vorliegenden Arbeit.

(*c*) Es gilt nach Darboux (3)*:

$$P_n(\cos\vartheta) = 2g_n \left\{ \frac{\cos[(n+\frac{1}{2})\vartheta - \frac{1}{4}\pi]}{(2\sin\vartheta)^{\frac{1}{2}}} - g_1 \frac{1}{2n-1} \frac{\cos[(n-\frac{1}{2})\vartheta + \frac{1}{4}\pi]}{(2\sin\vartheta)^{\frac{3}{2}}} \right.$$

$$\left. + g_2 \frac{1\cdot 3}{(2n-1)(2n-3)} \frac{\cos[(n-\frac{3}{2})\vartheta + \frac{3}{4}\pi]}{(2\sin\vartheta)^{\frac{5}{2}}} - \dots \right\};$$

$$g_n = \frac{1\cdot 3 \dots (2n-1)}{2\cdot 4 \dots 2n}.$$

Bricht man die Reihe in der geschweiften Klammer bei dem p-ten Gliede ab, so ist der Gesamtfehler gleich $O(n^{-p-\frac{1}{2}})$; es ist darin wiederum

$$\epsilon \leqq \vartheta \leqq \pi - \epsilon, \quad 0 < \epsilon < \tfrac{1}{2}\pi.$$

(*d*) Die Stieltjessche Formel (4):

$$P_n(\cos\vartheta) = \frac{4}{\pi} \frac{1}{(2n+1)g_n} \left\{ \frac{\cos[(n+\frac{1}{2})\vartheta - \frac{1}{4}\pi]}{(2\sin\vartheta)^{\frac{1}{2}}} + g_1 \frac{1}{2n+3} \frac{\cos[(n+\frac{3}{2})\vartheta - \frac{3}{4}\pi]}{(2\sin\vartheta)^{\frac{3}{2}}} \right.$$

$$\left. + g_2 \frac{1\cdot 3}{(2n+3)(2n+5)} \frac{\cos[(n+\frac{5}{2})\vartheta - \frac{5}{4}\pi]}{(2\sin\vartheta)^{\frac{5}{2}}} + \dots \right\}$$

liefert eine konvergente Darstellung von $P_n(\cos\vartheta)$, wenn $\frac{1}{6}\pi < \vartheta < \frac{5}{6}\pi$ ist†; sie ist ferner für alle ϑ aus dem Intervalle $0 < \vartheta < \pi$ asymptotisch richtig in dem folgenden schärferen Sinne: Bricht man die Reihe in der geschweiften Klammer bei dem p-ten Gliede ab, so ist der Gesamtfehler dem absoluten Betrage nach kleiner als

$$\frac{4}{\pi} \frac{1}{(2n+1)g_n} \cdot g_p \frac{1\cdot 3 \dots (2p-1)}{(2n+3)(2n+5)\dots(2n+2p+1)} \frac{2}{(2\sin\vartheta)^{p+\frac{1}{2}}}.$$

Die weiter zu nennenden Formeln von Watson und Hilb unterscheiden sich von den bisherigen darin, dass sie in einer gewissen Umgebung von $\vartheta = 0$ mit einem *gleichmässig* gegen Null konvergierenden Rest gelten. Das ist ein wesentlicher Vorzug bei den meisten Anwendungen, insbesondere bei der Abschätzung von Integralen, die $P_n(\cos\vartheta)$ enthalten und sich über eine Umgebung von $\vartheta = 0$ erstrecken. (Vgl. die Anwendungen in § 4, 2, 3.)

---

* Es finden sich übrigens bei Darboux a.a.O. [vgl. die Formel (30)] zwei sinnstörende Druckfehler: (*a*) es fehlt auf der rechten Seite der Faktor 2; (*b*) in dem dritten Gliede ist $2n-5$ durch $2n-1$ zu ersetzen.

† Auch die Darbouxsche Entwicklung ist übrigens konvergent, wenn $\frac{1}{6}\pi < \vartheta < \frac{5}{6}\pi$ ist.

(*e*) Die Watsonsche Formel (**6**) besagt, dass für $0 \leqslant \vartheta < \frac{1}{2}\pi$,

$$P_n(\cos\vartheta)+2i\pi^{-1}Q_n(\cos\vartheta)$$
$$= \frac{1}{\sqrt{\cos\vartheta}}\, e^{i(n+\frac{1}{2})(\operatorname{tg}\vartheta-\vartheta)}\{J_0[(n+\tfrac{1}{2})\operatorname{tg}\vartheta]$$
$$-iY_0[(n+\tfrac{1}{2})\operatorname{tg}\vartheta]\}+\frac{4\lambda}{\pi(n+\frac{1}{2})}\,\frac{1}{\sqrt{\cos\vartheta}}$$

ist, wobei Q_n, J_0, Y_0 die übliche Bedeutung haben (vgl. §§ 2, 5) und $|\lambda|<1$ gilt*.

(*f*) Die Hilbsche Formel lautet (**7**):

$$P_n(\cos\vartheta) = \sqrt{\left(\frac{\vartheta}{\sin\vartheta}\right)}\, J_0[(n+\tfrac{1}{2})\vartheta]$$
$$+\frac{1}{8n}\left(\operatorname{ctg}\vartheta-\frac{1}{\vartheta}\right)\sqrt{\left(\frac{2}{\pi n\sin\vartheta}\right)}\sin[(n+\tfrac{1}{2})\vartheta-\tfrac{1}{4}\pi]+O(n^{-\frac{3}{16}}),$$

wobei die Restabschätzung gleichmässig für $0 \leqslant \vartheta \leqslant \frac{1}{2}\pi$ gültig ist. Hieraus folgt insbesondere gleichmässig für $0 \leqslant \vartheta \leqslant \frac{1}{2}\pi$,

$$P_n(\cos\vartheta) = \sqrt{\left(\frac{\vartheta}{\sin\vartheta}\right)}\, J_0[(n+\tfrac{1}{2})\vartheta]+O(n^{-\frac{3}{2}}).$$

Ferner haben bekanntlich Heine (**2**), Darboux (**3**) und andere asymptotische Ausdrücke für $P_n(z)$ angegeben, welche in der längs des reellen Intervalles $-1, +1$ aufgeschlitzten komplexen z-Ebene gültig sind. Die Aufzählung dieser Formeln würde hier zu weit führen. Das gleiche gilt für die grosse Anzahl von bekannten Näherungsformeln für die Legendresche Funktion zweiter Art $Q_n(z)$ (**2**, **4**), ferner für die zugeordneten Funktionen $P_n^m(z)$ und $Q_n^m(z)$ (**5**). Analoge Formeln gelten schliesslich für die verallgemeinerten Legendreschen Funktionen mit beliebigem (nicht ganzzahligem) n (**5**).

Die Formeln (*a*)–(*d*) ergeben sich aus elementaren Eigenschaften der Legendreschen Polynome, insbesondere (*c*) aus der erzeugenden Reihe (nach der sog. Darbouxschen Methode), und (*d*) durch komplexe Integration. Die Watsonsche Formel (*e*) ergibt sich ebenfalls durch komplexe Integration, als Anwendung der sog. Sattelpunktsmethode. Zur Herleitung von (*f*) bedient sich Hilb der Differentialgleichung der Legendreschen Polynome.

Im folgenden stellen wir uns die Aufgabe, die Hilbschen Formeln (*f*) (namentlich die zweite) zu einer asymptotischen Entwicklung zu erweitern,

* Vgl. die letzte Fussnote in § 8.

in welcher der p-te Rest, d. h. der Fehler beim Abbrechen nach dem p-ten Gliede, die Form $O(n^{-p-\frac{1}{2}})$ erhält; hier bezeichnet p eine beliebig grosse, jedoch feste positive Zahl. Einen besonderen Wert legen wir dabei auf die Gleichmässigkeit der Restabschätzung in der Umgebung von $\vartheta = 0$. Eine solche Entwicklung lässt sich in der Tat auf eine höchst einfache Weise gewinnen. Es ergibt sich so u. a. ein neuer Beweis für die Hilbschen Formeln (f), für die erste sogar mit der schärferen Restabschätzung $O(n^{-2})$*.

Unser Hauptresultat beruht auf der bekannten Mehlerschen Formel

$$(M) \qquad P_n(\cos\vartheta) = \frac{2}{\pi}\int_0^{\vartheta} \frac{\cos(n+\frac{1}{2})t}{\sqrt{\{2(\cos t - \cos\vartheta)\}}}\,dt \quad (0 < \vartheta < \pi),$$

und lautet folgendermassen:

SATZ I. *Es sei* $0 \leqq \vartheta < \vartheta_0 = 2(\sqrt{2}-1)\pi = 0{\cdot}828\ldots\pi$. *Man hat*

$$(1) \qquad P_n(\cos\vartheta) = \sum_{\nu=0}^{\infty} a_\nu(\vartheta)\,\frac{J_\nu[(n+\frac{1}{2})\vartheta]}{(n+\frac{1}{2})^\nu},$$

wobei die von n *unabhängigen Koeffizienten* $a_\nu(\vartheta)$ $(\nu = 0, 1, 2, \ldots)$ *gewisse, leicht angebbare, für* $0 \leqq \vartheta < \pi$ *reguläre, elementare Funktionen sind. Diese Entwicklung konvergiert gleichmässig in jedem Intervalle*

$$0 \leqq \vartheta \leqq \vartheta_0 - \epsilon, \quad 0 < \epsilon < \vartheta_0.$$

Setzt man ferner mit $p \geqq 1$,

$$(2) \qquad P_n(\cos\vartheta) = \sum_{\nu=0}^{p-1} a_\nu(\vartheta)\,\frac{J_\nu[(n+\frac{1}{2})\vartheta]}{(n+\frac{1}{2})^\nu} + R_p,$$

so gilt gleichmässig für $0 \leqq \vartheta \leqq \vartheta_0 - \epsilon$,

$$(3) \qquad R_p = O(n^{-p-\frac{1}{2}}).$$

Die Reihe (1) konvergiert also in dem genannten Intervalle sowohl im gewöhnlichen als auch im asymptotischen Sinne. Es ist übrigens

$$(4) \qquad \begin{cases} a_0(\vartheta) = \sqrt{\left(\dfrac{\vartheta}{\sin\vartheta}\right)}, \\ a_1(\vartheta) = \sqrt{\left(\dfrac{\vartheta}{\sin\vartheta}\right)}\,\dfrac{\vartheta\cos\vartheta - \sin\vartheta}{8\vartheta\sin\vartheta}, \\ a_2(\vartheta) = \sqrt{\left(\dfrac{\vartheta}{\sin\vartheta}\right)}\,\dfrac{-15\sin^2\vartheta + 6\vartheta\sin\vartheta\cos\vartheta + \vartheta^2(9-\sin^2\vartheta)}{128\vartheta^2\sin^2\vartheta}, \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \end{cases}$$

* Vgl. § 4.1.

Aus dem in § 3 folgenden Beweise ergibt sich ohne Schwierigkeit eine Methode zur effektiven Berechnung der Funktionen $a(\vartheta)$; er liefert ferner die Erkenntnis, dass $\vartheta = 0$ eine genau ν-fache Nullstelle von $a_\nu(\vartheta)$ ist.

Zum Beweise von Satz I und der nachfolgenden weiteren Sätze ähnlicher Art brauchen wir einige Definitionen und Sätze aus der Theorie der Besselschen Funktionen, welche wir in § 2 vorausschicken. Darauf folgt in § 3 der Beweis von Satz I, sogar mit einer Verschärfung desselben. Als erste Anwendung wird sodann in § 4, 1 das Hilbsche Resultat (f) mit der schon erwähnten schärferen Restabschätzung hergeleitet. Weiter folgt in § 4, 2 ein neuer Beweis für einen Grenzwertsatz über die sog. Lebesgueschen Konstanten der Laplaceschen Reihe, der von Herrn Gronwall auf recht kompliziertem Wege mittels der Formel (d) bewiesen worden ist*. Schliesslich geben wir in § 4, 3 eine sehr kurze Herleitung eines schönen Satzes von Kogbetliantz-Bernstein, welcher die Verschärfung einer wichtigen, von Stieltjes herrührenden Abschätzung der Differenz $P_n(\cos\vartheta) - P_{n+2}(\cos\vartheta)$ bildet†. In den weiteren Paragraphen wenden wir uns der asymptotischen Entwicklung der Legendreschen Funktionen zweiter Art zu. Ausserdem wird eine Erweiterung dieser Entwicklungen auf beliebige komplexe z-Werte vorgenommen. Dies alles folgt aus dem Satz II in § 7, zu dessen Beweis vorbereitende Betrachtungen verschiedener Art (§§ 5, 6) nötig sind. Insbesondere ergibt sich aus diesen Erweiterungen, dass die asymptotische Entwicklung des eben formulierten Satzes I im ganzen Intervalle $0 \leqslant \vartheta < \pi$, u. zw. gleichmässig in $0 \leqslant \vartheta \leqslant \pi - \epsilon$, $0 < \epsilon < \pi$, Gültigkeit hat.

2. *Hilfssätze über Besselsche Funktionen.*

In den Bezeichnungen der Besselschen Funktionen folgen wir dem grossangelegten Werke von G. N. Watson, *A treatise on the theory of Bessel functions* (Cambridge University Press, 1922); im Laufe dieser Arbeit wird es mit W. zitiert. Im folgenden gelangen insbesondere drei Klassen von Besselschen Funktionen zur Anwendung:

$$(a)\ J_\nu(x); \qquad (b)\ Y_\nu(x); \qquad (c)\ K_\nu(x).$$

---

* T. H. Gronwall, "Über die Laplacesche Reihe", *Math. Annalen*, 74 (1913), 213–270, insbesondere 222–230; "On the degree of convergence of Laplace's series", *Trans. American Math. Soc.*, 15 (1914), 1–30, insbesondere 3–14.

† S. Bernstein, "Sur les polynomes orthogonaux relatifs à un segment fini", *Journal de Math.* (9), 10 (1931), 219–286, insbesondere 236–239. Der Satz ist implizite in einer von E. Kogbetliantz, *Journal de Math.* (9), 3 (1924), 107–187 angegebenen Abschätzung der ultrasphärischen Polynome (vgl. S. 125) enthalten.

Der Index ν durchläuft hier die Gesamtheit der nichtnegativen ganzen Zahlen; im Falle (a) werden wir auch die Werte $\nu = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$ zulassen.

1. Für die Funktionen (a) gilt die Integraldarstellung [vgl. W., S. 48, (2)]

$$(1) \qquad (2\vartheta)^\nu \frac{J_\nu(N\vartheta)}{N^\nu} = \frac{2}{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})} \int_0^\vartheta (\vartheta^2 - t^2)^{\nu-\frac{1}{2}} \cos Nt\, dt$$

$$(2\nu = 0, 1, 2, \ldots; \ \vartheta > 0, \ N > 0).$$

Das Verhalten von $J_\nu(x)$ für $x \to \infty$ wird durch die asymptotische Formel [W., S. 199, (1)]

$$(2) \qquad J_\nu(x) = \sqrt{\left(\frac{2}{\pi x}\right)} \cos(x - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi) + O(|x|^{-\frac{3}{2}})$$

gegeben; man hat hier $-\pi < \arg x < +\pi$ und $\sqrt{x}$ ist für $x > 0$ reell und positiv. Man schliesst aus (2)

$$(3) \qquad |J_\nu(x)| < \frac{A}{\sqrt{x}} \quad (x > 0),$$

wobei die positive Konstante A nur von ν abhängt.

2. Die Funktion $Y_\nu(x)$ definieren wir durch die Gleichung [vgl. W., S. 62, (3) und S. 64, (2)]

$$(4) \quad \pi Y_\nu(x) = 2\left(\gamma + \log\frac{x}{2}\right) J_\nu(x) - \sum_{\lambda=0}^{\nu-1} \frac{(\nu-1-\lambda)!}{\lambda!} \left(\frac{x}{2}\right)^{2\lambda-\nu} - \sum_{\lambda=0}^{\infty} \frac{(-1)^\lambda (\frac{1}{2}x)^{\nu+2\lambda}}{\lambda!\,(\nu+\lambda)!} (H_\lambda + H_{\nu+\lambda})$$

$$(\nu = 0, 1, 2, \ldots),$$

wobei $\gamma = 0{\cdot}577\ldots$ die Eulersche Konstante und

$$(5) \qquad H_0 = 0; \quad H_\lambda = \frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{\lambda} \quad (\lambda = 1, 2, 3, \ldots)$$

bedeutet. [Die erste Summe in (4) sei für $\nu = 0$ durch Null ersetzt.] Die Funktion $Y_\nu(x)$ ist eindeutig und regulär für $x \neq 0$, $-\pi < \arg x < +\pi$; es sei dabei $\log \frac{1}{2}x$ reell, wenn x reell und positiv ist. Für den so bestimmten Zweig von $Y_\nu(x)$ gilt eine ähnliche asymptotische Formel wie (2); es tritt nur sin an die Stelle von cos [W., S. 199, (2)].

3. Die Funktion $K_\nu(x)$ definieren wir durch die Gleichung [vgl. W., S. 73, 3.6, (1) und S. 78, (8)]

$$(6) \qquad K_\nu(x) = i^{\nu+1} \tfrac{1}{2}\pi \{J_\nu(ix) + iY_\nu(ix)\} \quad (\nu = 0, 1, 2, \ldots).$$

Durch Berücksichtigung von (4) ergibt sich auch hier ein für $x \neq 0$. $-\pi < \arg x < +\pi$, eindeutiger und regulärer Zweig; es sei dabei $\log \frac{1}{2}ix = \frac{1}{2}i\pi + \log \frac{1}{2}x$ gesetzt, wobei $\log \frac{1}{2}x$ reell ist, wenn x reell und positiv ausfällt. Für reelle und positive x ist dann $K_\nu(x)$ reell und positiv. Es gilt die Integraldarstellung [W., S. 172, (4)]

$$(7) \qquad K_\nu(x) = \frac{\Gamma(\frac{1}{2})}{\Gamma(\nu+\frac{1}{2})} \left(\frac{x}{2}\right)^\nu \int_1^\infty e^{-xt}(t^2-1)^{\nu-\frac{1}{2}}\,dt \quad (-\tfrac{1}{2}\pi < \arg x < +\tfrac{1}{2}\pi).$$

Man hat ferner für $x \to \infty$, $-\pi < \arg x < +\pi$ [W., S. 202, (1)],

$$(8) \qquad e^x K_\nu(x) \sim \sqrt{\left(\frac{\pi}{2x}\right)},$$

wobei $\sqrt{x}$ für $x > 0$ reell und positiv ist. Diese Näherungsformel gilt sogar (vgl. a.a.O.) für $-\frac{3}{2}\pi < \arg x < +\frac{3}{2}\pi$, also insbesondere auf der negativen reellen Achse, wobei das Vorzeichen von $\sqrt{x}$ an den beiden Ufern derselben verschieden ist.

Schliesslich noch eine Bemerkung. Es sei x nicht reell; in der längs der negativen reellen Achse aufgeschlitzten Ebene gilt dann wegen (6) und (4), wenn unter $K_\nu(-x)$ und $\log(-x)$ das aus $K_\nu(x)$ bzw. $\log x$ durch analytische Fortsetzung innerhalb dieses Schlitzgebietes hervorgehende Funktionselement verstanden wird,

$$(9) \qquad (-1)^\nu K_\nu(-x) - K_\nu(x) = i^{\nu+2} J_\nu(ix)\{\log(-x) - \log x\}.$$

Nun hat man aber $\log x - \log(-x) = \pm i\pi$, je nachdem ob $\Im x > 0$ bzw. $\Im x < 0$ ist. Es wird somit

$$(10) \qquad (-1)^\nu K_\nu(-x) - K_\nu(x) = \pm i^{\nu+1}\pi J_\nu(ix).$$

Bei der praktischen Anwendung des Satzes I und der weiteren Sätze ähnlicher Art können natürlich die bekannten Entwicklungen der Funktionen $J_\nu(x)$, $Y_\nu(x)$, $K_\nu(x)$ herangezogen werden [W., S. 199, (1), (2); S. 202, (1)]. Man erhält auf diese Weise asymptotische Abschätzungen der Legendreschen Funktionen durch gewisse elementare Funktionen.

3. *Beweis von Satz I.*

1. Unser Ausgangspunkt ist eine bekannte Entwicklung von

$$2(\cos t - \cos\vartheta)$$

bei festem ϑ als Funktion von t^2 in der Umgebung von ϑ^2. Aus dem Taylorschen Satze folgt nämlich leicht [vgl. W., S. 140, 5.22, (3)]

$$(1) \qquad 2(\cos t - \cos\vartheta) = \sqrt{\left(\frac{\pi}{2}\right)} \sum_{\nu=1}^{\infty} \frac{(\vartheta^2-t^2)^\nu}{\nu!} \frac{1}{2^{\nu-1}} \frac{J_{\nu-\frac{1}{2}}(\vartheta)}{\vartheta^{\nu-\frac{1}{2}}}.$$

Das erste Glied dieser Entwicklung ist

$$\sqrt{\left(\frac{\pi}{2}\right)}(\vartheta^2-t^2)\,\frac{J_{\frac{1}{2}}(\vartheta)}{\vartheta^{\frac{1}{2}}}=(\vartheta^2-t^2)\,\frac{\sin\vartheta}{\vartheta},$$

so dass

$$(1')\qquad 2(\cos t-\cos\vartheta)=(\vartheta^2-t^2)\,\frac{\sin\vartheta}{\vartheta}\left\{1+\sum_{\nu=2}^{\infty}(\vartheta^2-t^2)^{\nu-1}\phi_\nu(\vartheta)\right\}$$

wird, wobei

$$(2)\qquad \phi_\nu(\vartheta)=\frac{1}{\nu!}\,\frac{1}{2^{\nu-1}}\,\frac{J_{\nu-\frac{1}{2}}(\vartheta)}{\vartheta^{\nu-1}J_{\frac{1}{2}}(\vartheta)}$$

gesetzt worden ist. Diese Funktionen sind regulär für $0\leqq\vartheta<\pi$.

Die Entwicklung (1') ergibt nun

$$(3)\qquad \frac{1}{\sqrt{\{2(\cos t-\cos\vartheta)\}}}=\frac{1}{\sqrt{(\vartheta^2-t^2)}}\sqrt{\left(\frac{\vartheta}{\sin\vartheta}\right)}\left\{1+\sum_{\nu=2}^{\infty}(\vartheta^2-t^2)^{\nu-1}\phi_\nu(\vartheta)\right\}^{-\frac{1}{2}}$$

$$=\frac{1}{\sqrt{(\vartheta^2-t^2)}}\sqrt{\left(\frac{\vartheta}{\sin\vartheta}\right)}\sum_{\nu=0}^{\infty}(\vartheta^2-t^2)^{\nu}\psi_\nu(\vartheta),$$

wobei $\psi_0(\vartheta)=1$ ist und die $\psi_\nu(\vartheta)$ sich ganz und rational durch die $\phi_\nu(\vartheta)$ darstellen lassen. Die Entwicklung (3) gilt, solange $|\vartheta^2-t^2|<R$ ist, wobei $|z|<R$ den grössten Kreis um den Nullpunkt der komplexen z-Ebene bedeutet, in welchem

$$(4)\qquad \sqrt{\left(\frac{\vartheta^2-t^2}{2(\cos t-\cos\vartheta)}\right)},$$

als Funktion von $\vartheta^2-t^2=z$ (ϑ fest) regulär bleibt. Man hat offensichtlich

$$(5)\qquad R=|\vartheta^2-(2\pi-\vartheta)^2|=4\pi(\pi-\vartheta).$$

Wir wollen (3) in die Mehlersche Formel (M) in § 1 eintragen. Da darin die Variable t von 0 bis ϑ läuft, so muss

$$(6)\qquad \vartheta^2<4\pi(\pi-\vartheta),\quad \text{d.h.}\quad \vartheta<\vartheta_0=2(\sqrt{2}-1)\pi$$

vorausgesetzt werden. Man hat ferner für $0\leqq\vartheta\leqq\vartheta_0-\epsilon$ $(0<\epsilon<\vartheta_0)$, $\nu=0,\ 1,\ 2,\ \ldots,$

$$(7)\qquad |\psi_\nu(\vartheta)|<A\left(\frac{1}{R}+\delta\right)^{\nu}=A\left(\frac{1}{4\pi(\pi-\vartheta)}+\delta\right)^{\nu},$$

wobei $\delta>0$ beliebig ist und die positive Konstante A nur von ϵ und δ abhängt; die Entwicklung (3) konvergiert folglich für $0\leqq\vartheta\leqq\vartheta_0-\epsilon$, $0\leqq t\leqq\vartheta$ gleichmässig in ϑ und t.

Die Mehlersche Formel liefert jetzt

$$P_n(\cos\vartheta)=\frac{2}{\pi}\sqrt{\left(\frac{\vartheta}{\sin\vartheta}\right)}\int_0^{\vartheta}\cos(n+\tfrac{1}{2})t\,\left\{\sum_{\nu=0}^{\infty}(\vartheta^2-t^2)^{\nu-\frac{1}{2}}\psi_\nu(\vartheta)\right\}dt,$$

woraus durch gliedweise Integration und nach Anwendung der Formeln (1)

in § 2 die Entwicklung von Satz I hervorgeht. Man hat

$$(8)\qquad \begin{cases} a_0(\vartheta) = \sqrt{\left(\dfrac{\vartheta}{\sin\vartheta}\right)}\,; \\ a_\nu(\vartheta) = 1\,.\,3\,\ldots\,(2\nu-1)\sqrt{\left(\dfrac{\vartheta}{\sin\vartheta}\right)}\,\vartheta^\nu\psi_\nu(\vartheta) \quad (\nu = 1,\ 2,\ 3,\ \ldots)^*. \end{cases}$$

2. Wir schreiten jetzt zur Abschätzung des Restes R_p. Wir haben nachzuweisen, dass für $0 \leqslant \vartheta \leqslant \vartheta_0 - \epsilon$ gleichmässig

$$(9)\qquad \int_0^\vartheta \cos(n+\tfrac{1}{2})t \left\{\sum_{\nu=p}^{\infty} (\vartheta^2-t^2)^{\nu-\frac{1}{2}}\psi_\nu(\vartheta)\right\} dt = O(n^{-p-\frac{1}{2}})$$

gilt. Das Integral auf der linken Seite kann durch p-malige partielle Integration auf die folgende Gestalt gebracht werden:

$$(10)\qquad \pm\frac{1}{(n+\frac{1}{2})^p}\int_0^\vartheta \begin{matrix}\cos\\ \sin\end{matrix}(n+\tfrac{1}{2})t \left\{\sum_{\nu=p}^{\infty}\frac{d^p}{dt^p}(\vartheta^2-t^2)^{\nu-\frac{1}{2}}\psi_\nu(\vartheta)\right\} dt,$$

wo im Integranden cos oder sin steht, je nachdem ob p gerade oder ungerade ist. Die ausintegrierten Glieder fallen hierbei fort; für $t=\vartheta$ ist dies klar, für $t=0$ folgt es entweder aus dem Verschwinden von $\sin(n+\frac{1}{2})t$ oder aus dem der Ableitungen ungerader Ordnung von $(\vartheta^2-t^2)^{\nu-\frac{1}{2}}$. Die gliedweise Differentiation in (10) ist erlaubt; aus der Identität

$$\frac{d^m}{dt^m}(\vartheta^2-t^2)^{\nu-\frac{1}{2}} = \sum_{\lambda=0}^{m}\binom{m}{\lambda}\frac{d^\lambda}{dt^\lambda}(\vartheta-t)^{\nu-\frac{1}{2}}\frac{d^{m-\lambda}}{dt^{m-\lambda}}(\vartheta+t)^{\nu-\frac{1}{2}}$$

folgt nämlich die Abschätzung

$$\begin{aligned}\left|\frac{d^m}{dt^m}(\vartheta^2-t^2)^{\nu-\frac{1}{2}}\right| &\leqslant (\nu-\tfrac{1}{2})^m\sum_{\lambda=0}^{m}\binom{m}{\lambda}(\vartheta-t)^{\nu-\frac{1}{2}-\lambda}(\vartheta+t)^{\nu-\frac{1}{2}-(m-\lambda)} \\ &= (\nu-\tfrac{1}{2})^m(\vartheta^2-t^2)^{\nu-\frac{1}{2}}\left(\frac{1}{\vartheta-t}+\frac{1}{\vartheta+t}\right)^m \\ &= (\nu-\tfrac{1}{2})^m(\vartheta^2-t^2)^{\nu-\frac{1}{2}-m}(2\vartheta)^m \\ &\leqslant (2\nu-1)^m\vartheta^{2\nu-1-m}, \qquad (11)\end{aligned}$$

wenn nur $\nu > m$ ist. Für $\nu = m$ ist zu der oberen Schranke in (11) der Faktor $(1-t^2/\vartheta^2)^{-\frac{1}{2}}$ hinzuzufügen. Man erhält hieraus für $m=p$ und

---

* Eine geschlossene Darstellung der Funktionen $\psi_\nu(\vartheta)$ bzw. $a_\nu(\vartheta)$ zu finden, dürfte schwierig sein. Aus der erzeugenden Reihe (3) schliesst man

$$\psi_\nu(0) = (-1)^\nu(1-2^{1-2\nu})\frac{B_\nu}{(2\nu)!} \quad (\nu = 1,\ 2,\ 3,\ \ldots),$$

wobei die B_ν die Bernoullischen Zahlen sind. [Bezüglich der Bezeichnung der letzteren vgl. etwa Pólya-Szegö, *Aufgaben und Lehrsätze aus der Analysis*, 2 (Berlin, Springer, 1925), 146; oder Whittaker-Watson, *A course of modern analysis* (4. Auflage, Cambridge University Press, 1927), 125.]

mittels (7) die gleichmässige Konvergenz der Reihe in der geschweiften Klammer in (10). Damit ist für das fragliche Integral bereits die Abschätzung $O(n^{-p})$ gewonnen. Durch Abspaltung des ersten Gliedes der Summe in (9) und durch Berücksichtigung der Formel (1) in § 2 ergibt sich also

$$1 \cdot 3 \ldots (2p-1) \frac{\pi}{2} \frac{\vartheta^p J_p[(n+\frac{1}{2})\vartheta]}{(n+\frac{1}{2})^p} \psi_p(\vartheta) + O(n^{-p-1}).$$

Wegen § 2, (3) folgt hieraus die Behauptung unmittelbar. Damit ist der Satz I bewiesen.

3. Die vorhergehende Betrachtung behält auch für $p = 0$ ihren Sinn. Sie liefert dann die bekannte Stieltjessche Abschätzung*

$$P_n(\cos\vartheta) = \vartheta^{-\frac{1}{2}} O(n^{-\frac{1}{2}}), \tag{12}$$

welche für $0 < \vartheta \leqslant \frac{1}{2}\pi$ gleichmässig gültig ist.

4. Aus dem obigen Beweise geht leicht eine Verschärfung der Restabschätzung in Satz I hervor, die oft nützlich sein kann. (Vgl. § 4, 2.) Man erhält zunächst für den Betrag des Ausdruckes (10) die Schranke

$$(n+\tfrac{1}{2})^{-p} \sum_{\nu=p}^{\infty} (2\nu-1)^p \vartheta^{2\nu-p} |\psi_\nu(\vartheta)| = \vartheta^p O(n^{-p}).$$

(Hierbei ist die oben erwähnte Modifikation von (11) für $\nu = m = p$ zu berücksichtigen.) Die Schlussweise in 2 liefert somit die Schranke

$$\vartheta^{p-\frac{1}{2}} O(n^{-p-\frac{1}{2}}) + \vartheta^{p+1} O(n^{-p-1}),$$

woraus die schärfere Restabschätzung

$$R_p = \vartheta^{p-\frac{1}{2}} O(n^{-p-\frac{1}{2}}) \tag{13}$$

folgt.†

4. *Anwendungen.*

1. Wir wollen zunächst aus Satz I die erste Hilbsche Formel (f) mit der Restabschätzung $O(n^{-2})$ herleiten. Aus Satz I folgt für $p = 2$:

$$P_n(\cos\vartheta) = \sqrt{\left(\frac{\vartheta}{\sin\vartheta}\right)} J_0[(n+\tfrac{1}{2})\vartheta] - \frac{1}{8(n+\frac{1}{2})} \left(\operatorname{ctg}\vartheta - \frac{1}{\vartheta}\right) \sqrt{\left(\frac{\vartheta}{\sin\vartheta}\right)} J_1[(n+\tfrac{1}{2})\vartheta] + O(n^{-\frac{3}{2}}).$$

* Einen besonders einfachen Beweis für diese Ungleichung gab Herr L. Fejér, "Abschätzungen für die Legendreschen und verwandten Polynome", *Math. Zeitschrift*, 24 (1926), 285–298.

† Ist $n\vartheta$ beschränkt, so kann diese Restschranke noch mit $(n\vartheta)^{\frac{3}{2}}[(n\vartheta)^{p-1} + n^{-2}]$ multipliziert werden.

Es sei $(n+\frac{1}{2})\vartheta \leqslant 1$; dann ist gleichmässig für $0 \leqslant \vartheta \leqslant \frac{1}{2}\pi$,

$$\frac{1}{8(n+\frac{1}{2})}\left(\operatorname{ctg}\vartheta - \frac{1}{\vartheta}\right)\sqrt{\left(\frac{\vartheta}{\sin\vartheta}\right)}\, J_1[(n+\tfrac{1}{2})\vartheta] = O(n^{-1}.\vartheta.n\vartheta) = O(\vartheta^2) = O(n^{-2}),$$

und

$$\frac{1}{8n}\left(\operatorname{ctg}\vartheta - \frac{1}{\vartheta}\right)\sqrt{\left(\frac{2}{\pi n \sin\vartheta}\right)}\sin[(n+\tfrac{1}{2})\vartheta - \tfrac{1}{4}\pi] = O[n^{-1}.\vartheta.(n\vartheta)^{-\frac{1}{2}}] = O(\vartheta^{\frac{1}{2}} n^{-\frac{3}{2}}) = O(n^{-2}).$$

Es sei andererseits $(n+\frac{1}{2})\vartheta > 1$; dann hat man

$$\begin{aligned}
&\frac{1}{8(n+\frac{1}{2})}\left(\operatorname{ctg}\vartheta - \frac{1}{\vartheta}\right)\sqrt{\left(\frac{\vartheta}{\sin\vartheta}\right)}\, J_1[(n+\tfrac{1}{2})\vartheta] \\
&= \frac{1}{8(n+\frac{1}{2})}\left(\operatorname{ctg}\vartheta - \frac{1}{\vartheta}\right)\sqrt{\left(\frac{\vartheta}{\sin\vartheta}\right)}\left\{\sqrt{\left(\frac{2}{\pi(n+\frac{1}{2})\vartheta}\right)}\cos[(n+\tfrac{1}{2})\vartheta - \tfrac{3}{4}\pi] + O(n^{-\frac{3}{2}}\vartheta^{-\frac{3}{2}})\right\} \\
&= \frac{1}{8(n+\frac{1}{2})}\left(\operatorname{ctg}\vartheta - \frac{1}{\vartheta}\right)\sqrt{\left(\frac{2}{\pi(n+\frac{1}{2})\sin\vartheta}\right)}\sin[(n+\tfrac{1}{2})\vartheta - \tfrac{1}{4}\pi] + O(n^{-1}.\vartheta.n^{-\frac{3}{2}}\vartheta^{-\frac{3}{2}}) \\
&= \frac{1}{8n}\left(\operatorname{ctg}\vartheta - \frac{1}{\vartheta}\right)\sqrt{\left(\frac{2}{\pi n \sin\vartheta}\right)}\sin[(n+\tfrac{1}{2})\vartheta - \tfrac{1}{4}\pi] + O(n^{-2}),
\end{aligned}$$

woraus die Behauptung hervorgeht.

2. Weiter befassen wir uns mit den von Herrn Gronwall untersuchten Lebesgueschen Konstanten der Laplaceschen Reihe (vgl. § 1). Man kann diese durch die Formel

$$\rho_n = \frac{n+1}{2}\int_{-1}^{1}\left|\frac{P_n(x) - P_{n+1}(x)}{1-x}\right| dx \tag{1}$$

$$= \frac{n+1}{2}\int_0^{\pi} |P_n(\cos\vartheta) - P_{n+1}(\cos\vartheta)|\operatorname{ctg}\tfrac{1}{2}\vartheta\, d\vartheta$$

definieren. Zu ihrer asymptotischen Berechnung wählen wir eine feste positive Zahl ϵ, $\epsilon < \frac{1}{2}\pi$, und zerlegen wir das zweite Integral in (1) in drei Teile:

$$\rho_n = \frac{n+1}{2}\left\{\int_0^{\epsilon} + \int_{\epsilon}^{\pi-\epsilon} + \int_{\pi-\epsilon}^{\pi} |P_n(\cos\vartheta) - P_{n+1}(\cos\vartheta)|\operatorname{ctg}\tfrac{1}{2}\vartheta\, d\vartheta\right\} \tag{2}$$

$$= \rho_n^{(1)} + \rho_n^{(2)} + \rho_n^{(3)}.$$

Die Abschätzung von $\rho_n^{(1)}$ erfolgt nach Satz I unter Berücksichtigung von §3, (13) ($p=1$), folgendermassen:

$$\rho_n^{(1)} = \frac{n+1}{2}\int_0^{\epsilon} |J_0[(n+\tfrac{1}{2})\vartheta] - J_0[(n+\tfrac{3}{2})\vartheta]| \times \sqrt{\left(\frac{\vartheta}{\sin\vartheta}\right)} \operatorname{ctg}\tfrac{1}{2}\vartheta\, d\vartheta + \frac{n+1}{2}\int_0^{\epsilon} \vartheta^{\frac{1}{2}} \operatorname{ctg}\tfrac{1}{2}\vartheta\, d\vartheta \cdot O(n^{-\frac{3}{2}}).$$

Wegen $J_0'(x) = -J_1(x)$ erhält man also

$$\rho_n^{(1)} = \frac{n+1}{2}\int_0^{\epsilon} |J_1[(n+\mu)\vartheta]|\,\vartheta \sqrt{\left(\frac{\vartheta}{\sin\vartheta}\right)} \operatorname{ctg}\tfrac{1}{2}\vartheta\, d\vartheta + O(n^{-\frac{1}{2}}),$$

wobei $\mu = \mu(n, \vartheta)$ der Ungleichung $\frac{1}{2} < \mu < \frac{3}{2}$ genügt. Nach §2, (3) ist folglich

$$\rho_n^{(1)} \leqq \frac{n+1}{2}\frac{A}{\sqrt{n}}\int_0^{\epsilon} \frac{\vartheta \operatorname{ctg}\frac{1}{2}\vartheta}{\sqrt{\sin\vartheta}}\, d\vartheta + O(n^{-\frac{1}{2}}),$$

so dass

$$\limsup_{n\to\infty} \frac{\rho_n^{(1)}}{\sqrt{n}} \leqq \frac{A}{2}\int_0^{\epsilon} \frac{\vartheta \operatorname{ctg}\frac{1}{2}\vartheta}{\sqrt{\sin\vartheta}}\, d\vartheta$$

wird.

Für $\rho_n^{(2)}$ erhalten wir nach der Laplaceschen Formel [§1, (a)]

$$\rho_n^{(2)} = \frac{n+1}{2}\int_{\epsilon}^{\pi-\epsilon} \left| \sqrt{\left(\frac{2}{\pi n \sin\vartheta}\right)} \cos[(n+\tfrac{1}{2})\vartheta - \tfrac{1}{4}\pi] - \sqrt{\left(\frac{2}{\pi(n+1)\sin\vartheta}\right)} \cos[(n+\tfrac{3}{2})\vartheta - \tfrac{1}{4}\pi] \right| \operatorname{ctg}\tfrac{1}{2}\vartheta\, d\vartheta + O(n^{-\frac{1}{2}})$$

$$= \sqrt{\left(\frac{n}{2\pi}\right)}\int_{\epsilon}^{\pi-\epsilon} |\cos[(n+\tfrac{1}{2})\vartheta - \tfrac{1}{4}\pi] - \cos[(n+\tfrac{3}{2})\vartheta - \tfrac{1}{4}\pi]| \frac{\operatorname{ctg}\frac{1}{2}\vartheta}{\sqrt{\sin\vartheta}}\, d\vartheta + O(n^{-\frac{1}{2}})$$

$$= \sqrt{\left(\frac{2n}{\pi}\right)}\int_{\epsilon}^{\pi-\epsilon} |\sin[(n+1)\vartheta - \tfrac{1}{4}\pi]| \frac{\cos\frac{1}{2}\vartheta}{\sqrt{\sin\vartheta}}\, d\vartheta + O(n^{-\frac{1}{2}}).$$

Einer Bemerkung von Fejér zufolge* ergibt sich somit

$$\lim_{n\to\infty} \frac{\rho_n^{(2)}}{\sqrt{n}} = \left(\frac{2}{\pi}\right)^{\frac{3}{2}}\int_{\epsilon}^{\pi-\epsilon} \frac{\cos\frac{1}{2}\vartheta}{\sqrt{\sin\vartheta}}\, d\vartheta = \left(\frac{2}{\pi}\right)^{\frac{3}{2}}\int_0^{\pi} \frac{\cos\frac{1}{2}\vartheta}{\sqrt{\sin\vartheta}}\, d\vartheta + \eta(\epsilon)$$

$$= 2\sqrt{\left(\frac{2}{\pi}\right)} + \eta(\epsilon),$$

wobei $\lim_{\epsilon\to 0} \eta(\epsilon) = 0$ gilt.

* Vgl. Pólya-Szegö, *Aufgaben und Lehrsätze*, 1 (1925), Abschnitt II, Aufgabe 106, 60, 217.

Schliesslich wird $\rho_n^{(3)}$ in einfacher Weise mittels der Schwarzschen Ungleichung abgeschätzt:

$$\rho_n^{(3)} \leqq \frac{n+1}{2} \int_{\pi-\epsilon}^{\pi} \{|P_n(\cos\vartheta)|+|P_{n+1}(\cos\vartheta)|\} \operatorname{ctg}\tfrac{1}{2}\vartheta\, d\vartheta$$

$$\leqq (n+1) \left\{\frac{2}{2n+1}\right\}^{\frac{1}{2}} \left\{\int_{\pi-\epsilon}^{\pi} \frac{\operatorname{ctg}^2\frac{1}{2}\vartheta}{\sin\vartheta}\, d\vartheta\right\}^{\frac{1}{2}},$$

folglich $$\limsup_{n\to\infty} \frac{\rho_n^{(3)}}{\sqrt{n}} \leqq \left\{\int_{\pi-\epsilon}^{\pi} \frac{\operatorname{ctg}^2\frac{1}{2}\vartheta}{\sin\vartheta}\, d\vartheta\right\}^{\frac{1}{2}}.$$

Zusammenfassend erhalten wir

$$\limsup_{n\to\infty} \left|\frac{\rho_n}{\sqrt{n}} - 2\sqrt{\left(\frac{2}{\pi}\right)}\right|$$

$$\leqq \limsup_{n\to\infty} \frac{\rho_n^{(1)}}{\sqrt{n}} + \limsup_{n\to\infty} \left|\frac{\rho_n^{(2)}}{\sqrt{n}} - 2\sqrt{\left(\frac{2}{\pi}\right)}\right| + \limsup_{n\to\infty} \frac{\rho_n^{(3)}}{\sqrt{n}}$$

$$\leqq \frac{A}{2} \int_0^{\epsilon} \frac{\vartheta \operatorname{ctg}\frac{1}{2}\vartheta}{\sqrt{\sin\vartheta}}\, d\vartheta + \eta(\epsilon) + \left\{\int_{\pi-\epsilon}^{\pi} \frac{\operatorname{ctg}^2\frac{1}{2}\vartheta}{\sin\vartheta}\, d\vartheta\right\}^{\frac{1}{2}}.$$

Da der letzte Ausdruck mit ϵ gegen Null konvergiert, so folgt der Gronwallsche Satz*

$$\lim_{n\to\infty} \frac{\rho_n}{\sqrt{n}} = 2\sqrt{\left(\frac{2}{\pi}\right)}. \tag{3}$$

3. Die in der Einleitung erwähnte Ungleichung von Kogbetliantz-Bernstein lautet folgendermassen:

$$|P_n(\cos\vartheta) - P_{n+2}(\cos\vartheta)| < A\sqrt{\left(\frac{\vartheta}{n}\right)} \quad (0 \leqq \vartheta \leqq \tfrac{1}{2}\pi), \tag{4}$$

wobei A eine von n und ϑ unabhängige positive Konstante bedeutet†. Infolge der bekannten Formel

$$P_n(x) - P_{n+2}(x) = (2n+3) \int_x^1 P_{n+1}(t)\, dt$$

kann sie auch in der Gestalt

$$\int_0^{\vartheta} P_n(\cos t) \sin t\, dt = \vartheta^{\frac{1}{2}} O(n^{-\frac{1}{2}}) \tag{5}$$

geschrieben werden.

* Eine andere, in mancher Hinsicht noch einfachere Beweisanordnung für den Gronwallschen Satz lässt sich auf den unter 3 behandelten Satz begründen. Vgl. meine unten zitierte Note.

† Bei S. Bernstein erscheint sie in der folgenden äquivalenten Form:

$$(1-x^2)^{\frac{1}{4}} |P_n'(x)| < A\sqrt{n}.$$

Zum Beweis von (5) benützen wir den Spezialfall $p = 1$ von Satz I, d. h. die in § 1, (f) angeführte (zweite) Hilbsche Formel. Sie liefert für die linke Seite von (5) die Darstellung

$$\int_0^\vartheta \sqrt{\left(\frac{t}{\sin t}\right)}\, J_0[(n+\tfrac{1}{2})t] \sin t\, dt + O\left\{\int_0^\vartheta \sin t\, dt \,.\, n^{-\frac{3}{2}}\right\}$$
$$= \int_0^\vartheta \sqrt{\left(\frac{\sin t}{t}\right)}\, t\, J_0[(n+\tfrac{1}{2})t]\, dt + \vartheta^2 O(n^{-\frac{3}{2}}).$$

Infolge der bekannten Formel

$$xJ_0(x) = \frac{d}{dx} xJ_1(x)$$

erhalten wir für das letzte Integral durch partielle Integration

$$\sqrt{\left(\frac{\sin\vartheta}{\vartheta}\right)} \frac{\vartheta J_1[(n+\frac{1}{2})\vartheta]}{n+\frac{1}{2}} - \int_0^\vartheta \left\{\frac{d}{dt}\sqrt{\left(\frac{\sin t}{t}\right)}\right\} \frac{tJ_1[(n+\frac{1}{2})t]}{n+\frac{1}{2}}\, dt,$$

woraus die Behauptung mit Rücksicht auf § 2, (3) ohne weiteres hervorgeht*.

5. *Über die Legendreschen Funktionen zweiter Art.*

Liegt der Punkt z in der längs der reellen Strecke $-1 \leqslant z \leqslant +1$ aufgeschlitzten z-Ebene, so kann ein eindeutiger Zweig der Legendreschen Funktion zweiter Art $Q_n(z)$ durch die Forderung definiert werden, dass $Q_n(z)$ für reelle z, $z > 1$, reell sei. Die Abbildung

$$z = \cosh\xi$$

führt den halben Parallelstreifen

$$\Re\xi > 0, \quad -\pi < \Im\xi \leqslant \pi \tag{1}$$

in dieses Schlitzgebiet (ohne $z = \infty$) über. Das gleiche gilt, wenn man die Bedingung $\Re\xi > 0$ durch $\Re\xi < 0$ ersetzt. In einem beliebigen Punkte ξ des Halbstreifens (1) gilt dann die Darstellung†

$$Q_n(\cosh\xi) = \int_\xi^\infty \frac{e^{-(n+\frac{1}{2})x}}{\sqrt{\{2(\cosh x - \cosh\xi)\}}}\, dx, \tag{2}$$

wo die Integration längs der von ξ ausgehenden horizontalen, nach rechts gerichteten Halbgeraden zu erstrecken ist. Die Quadratwurzel im Nenner

* Weitere Beweise und Anwendungen der Ungleichung (4) finden sich in meiner Note: *Math. Annalen*, 108 (1933), 360–369.

† Vgl. z.B. Watson (**6**), 154, wo diese Formel aus der bekannten Darstellung von $Q_n(z)$ als Kurvenintegral hergeleitet wird.

wird dabei durch die Bedingung

$$-\tfrac{1}{2}\pi < \arg\sqrt{\{2(\cosh x - \cosh\xi)\}} \leqslant \tfrac{1}{2}\pi \tag{3}$$

eindeutig festgelegt.

Das Integral auf der rechten Seite von (2) behält im ganzen Parallelstreifen $-\pi \leqslant \Im\xi \leqslant \pi$, mit Ausnahme von $\xi = 0$, $\xi = \pm i\pi$, seinen Sinn. Dabei ist der Integrationsweg analog wie in (2) und die Quadratwurzel durch die Bedingung (3) festzulegen, wenn $\Im\xi \neq 0, \pm\pi$ oder wenn $\Im\xi = 0, \pm\pi$ und $\Re\xi > 0$ ist. (Im falle $\Im\xi = -\pi$, $\Re\xi > 0$ ist das fragliche Argument gleich $-\frac{1}{2}\pi$ zu setzen.) Man erhält so die *analytische Fortsetzung von* $Q_n(\cosh\xi)$ *in dem längs der negativen reellen Achse aufgeschlitzten Streifen* $-\pi < \Im\xi < +\pi$. *Wir bezeichnen sie im folgenden mit* $q_n(\xi)$. Für negative reelle ξ kann das Argument der Quadratwurzel im Intervall $\xi < x < -\xi$ gleich $+\frac{1}{2}\pi$ oder $-\frac{1}{2}\pi$ gesetzt werden; für $x > -\xi$ sei dieses Argument stets gleich Null. Dementsprechend stellt (1) den Grenzwert von $q_n(\xi)$ bei der Annäherung an das obere bzw. untere Schlitzufer dar. Wir schreiben dafür $q_n(\xi \pm i0)$. Ist ferner $\Im\xi = \pm\pi$, $\Re\xi < 0$, so ist das Argument der Quadratwurzel von ξ bis $-\xi \pm 2\pi i$ gleich Null und von $-\xi \pm 2\pi i$ bis ∞ gleich $\pm\frac{1}{2}\pi$ zu setzen. Man erhält sodann den Grenzwert von $q_n(\xi)$ bei der Annäherung an den oberen bzw. unteren Streifenrand*.

Aus (2) kann auch leicht eine Darstellung der Legendreschen Funktionen erster Art gewonnen werden. Bekanntlich ist†

$$Q_n(z) - \tfrac{1}{2}P_n(z)\log\frac{z+1}{z-1} \tag{4}$$

eine ganze rationale Funktion von z, so dass im Streifen $-\pi \leqslant \Im\xi \leqslant +\pi$ mit Ausnahme von $\xi = 0$, $\xi = \pm i\pi$

$$\pm i\pi P_n(\cosh\xi) = q_n(-\xi) - q_n(\xi) \tag{5}$$

gilt. Hier ist links das obere bzw. untere Zeichen zu nehmen, je nachdem ob $\Im\xi > 0$ oder $\Im\xi < 0$ ist.‡ Bei der Bestimmung der in diesen Integralen auftretenden Quadratwurzeln ist die obige Vorschrift (3) massgebend. Für reelle ξ wird (5) auch richtig, vorausgesetzt, dass man über die

* Man sieht dies alles am einfachsten ein, indem man die Bilder der Geraden $\Im\xi = \text{konst.}$ bei der Abbildung $z = \cosh\xi$ verfolgt; das sind homofokale Hyperbeln mit den Brennpunkten $z = -1$, $z = +1$.

† Vgl. z.B. Whittaker-Watson, *Modern analysis*, 319.

‡ Vgl. die obige Fussnote *; man verfolge etwa die Änderung von $\log\{(z+1)/(z-1)\}$, $z = \cosh\xi$, längs des Weges, welcher aus den geradlinigen Verbindungen der Punkte ξ, $\xi+\tau$, $\bar{\xi}+\tau$, $-\xi$, besteht, $\tau \to +\infty$.

vorkommenden Quadratwurzeln geeignet verfügt. Ist z.B. $\xi > 0$, so muss das fragliche Argument in dem Teil $-\xi < x < \xi$ des ersten Integrals $-\frac{1}{2}\pi$ (bzw. $+\frac{1}{2}\pi$) gewählt werden und auf der linken Seite von (5) steht das obere (bzw. untere) Zeichen; entsprechend, wenn $\xi < 0$ ist. Für reelle und positive ξ ist (5), wie leicht ersichtlich, mit der Darstellung

$$(5') \qquad P_n(\cosh\xi) = \frac{1}{\pi}\int_{-\xi}^{\xi} \frac{e^{-(n+\frac{1}{2})x}}{\sqrt{\{2(\cosh\xi - \cosh x)\}}}\, dx$$

gleichwertig, worin die Quadratwurzel reell und positiv ist. Für rein imaginäre ξ, $\xi = i\vartheta$, $0 < \vartheta < \pi$, geht (5) in die Mehlersche Formel § 1, (M) über.

6. *Hilfssätze.*

Um den Gang der weiteren Betrachtungen nicht unterbrechen zu müssen, schicken wir hier einige elementare Hilfssätze voraus.

Es sei $0 < \epsilon < \pi$ und G_ϵ bezeichne im folgenden das Gebiet

$$(1) \qquad -\pi < \Im\xi < +\pi, \quad \xi \neq 0, \quad |\xi \pm i\pi| > \epsilon.$$

HILFSSATZ I. Der feste Punkt ξ möge dem Gebiete G_ϵ oder dessen Rand angehören. Wir betrachten in der Umgebung von $x = \xi$ denjenigen Zweig der Funktion

$$(2) \qquad \sqrt{\left(\frac{\sinh\xi}{\xi}\right)}\,\sqrt{\left(\frac{x^2-\xi^2}{2(\cosh x - \cosh\xi)}\right)},$$

welcher für $x = \xi$ in $+1$ übergeht. Das ist eine eindeutige Funktion von x^2 in dem Bereiche $|x^2 - \xi^2| < R$, wobei $R = \mathrm{Min}|(\xi + 2\nu\pi i)^2 - \xi^2|$ bezeichnet, während ν die Gesamtheit der von Null verschiedenen ganzen Zahlen durchläuft.

Die Behauptung folgt unmittelbar durch Betrachtung der Stellen x, $x^2 \neq \xi^2$, an denen der Nenner von (2) verschwindet. Es ist offenbar

$$(3) \qquad R = 4\pi\,\mathrm{Min}(|\xi + i\pi|,\ |\xi - i\pi|) = 4\pi\mu(\xi).$$

Man hat übrigens $\mu(\xi) \geqq \epsilon$, also $R \geqq 4\pi\epsilon$. Es gilt ferner $|\xi|^{-1}\mu(\xi) \to 1$ für $\xi \to \infty$.

HILFSSATZ II. Es sei $0 < \gamma < 1$. Der absolute Betrag der Funktion (2) bleibt für

$$|x^2 - \xi^2| \leqq 4\pi\gamma\mu(\xi)$$

unterhalb einer festen, von ξ unabhängigen, Schranke $M = M(\gamma, \epsilon)$, wenn ξ die Menge (1) durchläuft.

Man kann zunächst $x^2 - \xi^2 = 4\pi\zeta\mu(\xi)$ setzen, wobei $|\zeta| = \gamma$ ist. Der Maximalbetrag von (2) für die bei festem ξ in Frage kommenden x-Werte

ist eine stetige Funktion von ξ; es genügt zu zeigen, dass er beschränkt bleibt, wenn ξ in G_ϵ gegen ∞ konvergiert.

Man hat $$x = \xi \sqrt{\left(1+4\pi\zeta \frac{\mu(\xi)}{\xi^2}\right)},$$

wo beide Zweige der Quadratwurzel zulässig sind. Es ist also für $\xi \to \infty$

$$\pm x = \xi + 2\pi\zeta \frac{\mu(\xi)}{\xi} + O(|\xi|^{-1}),$$

wo die Restabschätzung für $|\zeta| = \gamma$ gleichmässig gültig ist; d. h. man hat

$$|\pm x - \xi| \leqslant 2\pi\gamma + \eta(\xi),$$

wobei $\lim_{\xi\to\infty} \eta(\xi) = 0$ ist. Nun gilt

$$\left|\frac{x^2-\xi^2}{\xi}\right| = 4\pi|\zeta|\frac{\mu(\xi)}{|\xi|}.$$

Weiter liefert eine leichte Umformung

$$\frac{\sinh\xi}{\cosh x - \cosh\xi} = \frac{1}{[\cosh(x-\xi)-1]\operatorname{ctgh}\xi + \sinh(x-\xi)}$$

$$= \frac{1}{[\cosh(x+\xi)-1]\operatorname{ctgh}\xi - \sinh(x+\xi)}.$$

Die Häufungswerte von $\operatorname{ctgh}\xi$ sind aber für $\xi \to \infty$ gleich ± 1; ferner bleibt die Funktion

$$\frac{1}{\cosh z - 1 + \sinh z} = \frac{1}{e^z - 1}$$

für $|z| = 2\pi\gamma + \eta(\xi)$ beschränkt. Damit ist unsere Behauptung bewiesen.

HILFSSATZ III. Die Funktion

$$\frac{\sinh\xi}{\xi(\xi+i\pi)(\xi-i\pi)}$$

bleibt im Parallelstreifen $-\pi \leqslant \Im\xi \leqslant +\pi$ dem absoluten Betrage nach oberhalb einer festen positiven Zahl.

Die fragliche Funktion ist in diesem Parallelstreifen von Null verschieden; ferner konvergiert sie gegen ∞, wenn ξ innerhalb des Streifens gegen ∞ geht.

7. *Asymptotische Entwicklung von* $q_n(\xi)$.

Die in § 5 angegebenen Integrale können nach der Methode von § 3 asymptotisch entwickelt werden. Man erhält dadurch eine asymptotische

Darstellung von $Q_n(z)$, sowie auch eine von $P_n(z)$, welche in der ganzen komplexen z-Ebene gilt. Letztere verallgemeinert die in Satz I angegebene Entwicklung. Es sei jedoch schon hier bemerkt, dass die in diesem Paragraphen zu behandelnden Reihen nur im asymptotischen Sinne konvergieren.

1. Es sei $0 < \epsilon < \pi$ und ξ möge dem in §6, (1) definierten Gebiet G_ϵ oder dessen Rand angehören. Es sei jedoch $\xi \neq 0$. Wir stellen uns die Aufgabe, das Integral

$$(1) \qquad q_n(\xi) = \int_\xi^\infty \frac{e^{-(n+\frac{1}{2})x}}{\sqrt{\{2(\cosh x - \cosh \xi)\}}}\, dx,$$

wobei der Integrationsweg und die Quadratwurzel wie in §5 festgelegt sind, für grosse Werte von n abzuschätzen. Für negative reelle ξ gibt es bezüglich der Festlegung der Wurzel zwei Möglichkeiten, vgl. §5. Man erhält entsprechend $q_n(\xi \pm i0)$. Es sei $\delta = \delta(\epsilon)$ eine positive Zahl, über die noch geeignet verfügt wird. Man hat

$$(2) \quad e^{(n+\frac{1}{2})\xi} q_n(\xi) = \int_\xi^{\xi+\delta} \frac{e^{-(n+\frac{1}{2})(x-\xi)}}{\sqrt{\{2(\cosh x - \cosh \xi)\}}}\, dx + \int_{\xi+\delta}^\infty \frac{e^{-(n+\frac{1}{2})(x-\xi)}}{\sqrt{\{2(\cosh x - \cosh \xi)\}}}\, dx.$$

2. Das erste Integral behandeln wir ähnlich wie in §3. Es wird [vgl. §3, (1′)]

$$(3) \qquad 2(\cosh x - \cosh \xi) = (x^2 - \xi^2) \frac{\sinh \xi}{\xi} \left\{1 + \sum_{\nu=2}^\infty (x^2 - \xi^2)^{\nu-1} \phi_\nu(i\xi)\right\}$$

und

$$(4) \qquad \frac{1}{\sqrt{\{2(\cosh x - \cosh \xi)\}}} = \frac{1}{\sqrt{(x^2 - \xi^2)}} \sqrt{\left(\frac{\xi}{\sinh \xi}\right)} \sum_{\nu=0}^\infty (x^2 - \xi^2)^\nu \psi_\nu(i\xi).$$

Die Koeffizienten der Reihen (3) und (4) sind im Streifen $-\pi < \Im \xi < +\pi$ reguläre Funktionen von ξ. Die Reihe (4) konvergiert für $|x^2 - \xi^2| < R$, wobei R die Bedeutung in Hilfssatz I von §6 hat. Wir bestimmen δ so, dass für $0 \leqslant x - \xi \leqslant \delta$ diese Bedingung erfüllt ist, und dass überdies

$$(5) \qquad |x^2 - \xi^2| = |x - \xi|\,|x + \xi| \leqslant \delta(\delta + 2|\xi|) \leqslant 2\pi\mu(\xi)$$

gilt. Die letzte Ungleichung besagt, dass

$$\delta \leqslant \sqrt{\{|\xi|^2 + 2\pi\mu(\xi)\}} - |\xi|.$$

Die rechte Seite dieser Ungleichung konvergiert für $\xi \to \infty$ gegen π; sie bleibt somit in G_ϵ, worin ja $\mu(\xi) \geqslant \epsilon$ gilt, oberhalb einer nur von ϵ

abhängigen positiven Zahl, so dass die Bestimmung von $\delta = \delta(\epsilon)$ tatsächlich möglich ist. Man hat übrigens nach Hilfssatz II ($\gamma = \frac{2}{3}$)

$$|\psi_\nu(i\xi)| \leqslant \frac{M(\frac{2}{3}, \epsilon)}{(\frac{2}{3}R)^\nu} = \frac{M(\frac{2}{3}, \epsilon)}{[\frac{8}{3}\pi\mu(\xi)]^\nu} \quad (\nu = 0, 1, 2, \ldots). \tag{6}$$

In dem ersten Integral (2) besitzt somit die Reihe (4) (ohne den voranstehenden Faktor) die Majorante

$$\sum_{\nu=0}^{\infty} (\tfrac{3}{4})^\nu M(\tfrac{2}{3}, \epsilon).$$

Man erhält folglich nach gliedweise Integration

$$\int_\xi^{\xi+\delta} \frac{e^{-(n+\frac{1}{2})(x-\xi)}}{\sqrt{\{2(\cosh x - \cosh \xi)\}}}\, dx \tag{7}$$

$$= \sqrt{\left(\frac{\xi}{\sinh \xi}\right)} \sum_{\nu=0}^{\infty} \psi_\nu(i\xi) \int_\xi^{\xi+\delta} e^{-(n+\frac{1}{2})(x-\xi)} (x^2-\xi^2)^{\nu-\frac{1}{2}}\, dx.$$

Die Bestimmung der Quadratwurzel $\sqrt{(\xi/\sinh \xi)}(x^2-\xi^2)^{-\frac{1}{2}}$ erfolgt hier nach den Vorschriften von § 5, ferner § 6, Hilfssatz I. Man kann z. B. für $\sqrt{(\xi/\sinh \xi)}$ den in G_ϵ eindeutigen Zweig wählen, der für reelle und positive ξ reell und positiv ausfällt. In

$$(x^2-\xi^2)^{-\frac{1}{2}} = (x-\xi)^{-\frac{1}{2}}(x+\xi)^{-\frac{1}{2}}$$

ist der erste Faktor positiv. Das Argument des zweiten liegt zwischen $-\frac{1}{2}\pi$ und Null für $\Im\xi > 0$, zwischen Null und $+\frac{1}{2}\pi$ für $\Im\xi < 0$. Es sei ferner ξ reell, $\xi \neq 0$; dann ist $(x^2-\xi^2)^{-\frac{1}{2}}$ reell und positiv, wenn $\xi > 0$ oder $\xi < 0$, $x > -\xi$ ist und negativ bzw. positiv imaginär, wenn

$$\xi < 0, \quad \xi < x < -\xi$$

gilt.

3. Erstreckt man die Integration im ν-ten Gliede der Reihe (7) von ξ bis ∞ (statt $\xi+\delta$), so entsteht [vgl. § 2, (7)]

$$\int_\xi^\infty e^{-(n+\frac{1}{2})(x-\xi)} (x^2-\xi^2)^{\nu-\frac{1}{2}}\, dx = e^{(n+\frac{1}{2})\xi}\, \xi^{2\nu} \int_1^\infty e^{-(n+\frac{1}{2})\xi t} (t^2-1)^{\nu-\frac{1}{2}}\, dt$$

$$= 1 \cdot 3 \ldots (2\nu-1)\, e^{(n+\frac{1}{2})\xi}\, \frac{\xi^\nu K_\nu[(n+\frac{1}{2})\xi]}{(n+\frac{1}{2})^\nu}. \tag{8}$$

Diese Umformung ist nur für $\Re\xi > 0$ erlaubt. Durch analytische Fortsetzung erweist sich aber die Gültigkeit der Endformel in dem ganzen Gebiete G_ϵ. Der Integrationsweg im zweiten Integral ist für nichtreelle ξ zunächst nicht horizontal; nach dem Cauchyschen Integralsatze kann er aber horizontal gemacht werden.

Wir schreiten nun zur Abschätzung des Fehlers bei der erwähnten Abänderung der Integralgrenze. Dieser Fehler ist kleiner als

$$\int_{\xi+\delta}^{\infty} e^{-(n+\frac{1}{2})(x-\xi)} |x^2-\xi^2|^{\nu-\frac{1}{2}} dx < e^{-n\delta} \int_{\delta}^{+\infty} e^{-\frac{1}{2}t} |t(t+2\xi)|^{\nu-\frac{1}{2}} dt,$$

also wegen $|t+2\xi| \leqq t+2|\xi| \leqq (1+2|\xi|/\delta)\, t$ kleiner als

$$\left(1+\frac{2|\xi|}{\delta}\right)^{\nu-\frac{1}{2}} e^{-n\delta} \int_{\delta}^{+\infty} e^{-\frac{1}{2}t} t^{2\nu-1} dt,$$

vorausgesetzt, dass $\nu \geqq 1$ gilt. Für $\nu = 0$ ist diese Abschätzung etwas abzuändern; der Fehler ist dann kleiner als

$$e^{-n\delta} \int_{\delta}^{+\infty} e^{-\frac{1}{2}t} t^{-\frac{1}{2}} \left| t-2|\xi| \right|^{-\frac{1}{2}} dt.$$

Zur Abschätzung dieses Integrals sei D_1 derjenige Teil des Intervalles $(\delta, +\infty)$, in dem $\left|t-2|\xi|\right| \leqq 1$ ist und D_2 der komplementäre Teil. Der letzte Ausdruck ist dann kleiner als

$$e^{-n\delta} \int_{D_1} e^{-\frac{1}{2}t} t^{-\frac{1}{2}} \left| t-2|\xi| \right|^{-\frac{1}{2}} dt + e^{-n\delta} \int_{D_2} e^{-\frac{1}{2}t} t^{-\frac{1}{2}} dt$$

$$\leqq e^{-n\delta} \operatorname*{Max}_{t \geqq \delta} (e^{-\frac{1}{2}t} t^{-\frac{1}{2}}) \int_{-1}^{1} |t|^{-\frac{1}{2}} dt + e^{-n\delta} \int_{\delta}^{+\infty} e^{-\frac{1}{2}t} t^{-\frac{1}{2}} dt.$$

Der fragliche Fehler ist somit in allen Fällen kleiner als

$$C_\nu(\delta) \left(1+\frac{2|\xi|}{\delta}\right)^{\nu} e^{-n\delta},$$

wobei $C_\nu(\delta)$ eine nur von δ und ν abhängige positive Konstante ist. Man beachte jetzt (6). Bei festem ν beträgt folglich der Gesamtfehler im ν-ten Gliede der Reihe (7) [mit Berücksichtigung des beschränkten Faktors $\sqrt{(\xi/\sinh \xi)}$] $O(e^{-n\delta})$, u. zw. gleichmässig in G_ϵ.

4. Bricht man die Reihe (7) bei dem Gliede $\nu = p-1$ ab $(p \geqq 1)$, so wird der Rest

$$(9) \qquad R_p = \int_{\xi}^{\xi+\delta} e^{-(n+\frac{1}{2})(x-\xi)} \left\{ \sum_{\nu=p}^{\infty} (x^2-\xi^2)^{\nu-\frac{1}{2}} \psi_\nu(i\xi) \right\} dx.$$

Wir wollen zeigen, dass $R_p = O(n^{-p-\frac{1}{2}})$ gilt, u. zw. gleichmässig in G_ϵ. Nach p maliger partieller Integration und Weglassung der ausintegrierten Glieder folgt nämlich

$$(10) \qquad \frac{(-1)^p}{(n+\frac{1}{2})^p} \int_{\xi}^{\xi+\delta} e^{-(n+\frac{1}{2})(x-\xi)} \left\{ \sum_{\nu=p}^{\infty} \frac{d^p}{dx^p} (x^2-\xi^2)^{\nu-\frac{1}{2}} \psi_\nu(i\xi) \right\} dx.$$

Wir zeigen hier zunächst, dass die gliedweise Differentiation berechtigt ist, dass ferner die Weglassung der ausintegrierten Glieder einen Gesamtfehler von der Grössenordnung $O(e^{-n\delta})$ bedeutet, u. zw. gleichmässig in G_ϵ.

In der Tat erhält man analog wie in §3, 2 die Abschätzung

$$(11)\qquad \left|\frac{d^m}{dx^m}(x^2-\xi^2)^{\nu-\frac{1}{2}}\right| \leqq (2\nu-1)^m |x^2-\xi^2|^{\nu-\frac{1}{2}-m}\left(\frac{|x-\xi|+|x+\xi|}{2}\right)^m$$
$$\leqq (2\nu-1)^m |x^2-\xi^2|^{\nu-\frac{1}{2}-m}(\delta+|\xi|)^m.$$

Setzt man darin $m=p$, so folgt wegen (5) für die Reihe in der geschweiften Klammer in (10) die Majorante

$$\sum_{\nu=p}^{\infty}(2\nu-1)^p\,[2\pi\mu(\xi)]^{\nu-\frac{1}{2}-p}(\delta+|\xi|)^p\,\frac{M(\tfrac{2}{3},\,\epsilon)}{[\tfrac{8}{3}\pi\mu(\xi)]^\nu}$$

(im ersten Gliede $\nu=p$ ist $[2\pi\mu(\xi)]^{\nu-\frac{1}{2}-p}$ durch $|x^2-\xi^2|^{-\frac{1}{2}}$ zu ersetzen). Aus (11) ergibt sich auch unmittelbar (für $m=0,\,1,\,\ldots,\,p-1$) die oben erwähnte Abschätzung der ausintegrierten Glieder.

Damit ist schon für (9) die Abschätzung $O(n^{-p})$ bewiesen. Die für das erste Glied $\nu=p$ der Reihe erhaltene Schranke

$$\frac{1}{(n+\frac{1}{2})^p}\int_\xi^{\xi+\delta}|x^2-\xi^2|^{-\frac{1}{2}}\,dx \leqq \frac{1}{(n+\frac{1}{2})^p}\int_0^\delta t^{-\frac{1}{2}}\left|t-2|\xi|\right|^{-\frac{1}{2}}dt$$

ist dabei analog wie oben in 3 zu behandeln.

Weiter braucht man, wie in §3, nur das erste Glied von (9), d. h. wegen 3

$$(12)\qquad 1\cdot 3\ldots(2p-1)\,e^{(n+\frac{1}{2})\xi}\,\frac{\xi^p\,K_p[(n+\frac{1}{2})\,\xi]}{(n+\frac{1}{2})^p}\,\psi_p(i\xi)$$

zu betrachten. Man hat aber wegen §2, (4), (6), (8) für $\xi\neq 0$, $-\pi\leqq\arg\xi\leqq+\pi$,

$$|e^\xi K_p(\xi)| \leqq \begin{cases} A\,|\xi|^{-\frac{1}{2}}, & \text{wenn} \quad |\xi|\geqq 1,\\ A\,|\xi|^{-p}, & \text{wenn} \quad |\xi|\leqq 1,\end{cases}$$

ist, wobei A eine nur von p abhängige positive Konstante bedeutet. Nach Berücksichtigung von (6) erhält man also für (12) die Schranke

$$1\cdot 3\ldots(2p-1)\,\frac{A\,|\xi|^{p-\frac{1}{2}}}{(n+\frac{1}{2})^{p+\frac{1}{2}}}\,\frac{M(\frac{2}{3},\,\epsilon)}{[\frac{8}{3}\pi\mu(\xi)]^p},$$

bzw.
$$1\cdot 3\ldots(2p-1)\,\frac{A}{(n+\frac{1}{2})^{2p}}\,\frac{M(\frac{2}{3},\,\epsilon)}{[\frac{8}{3}\pi\mu(\xi)]^p},$$

je nachdem ob $(n+\frac{1}{2})|\xi|$ grösser oder kleiner als 1 ist. In beiden Fällen ist diese Schranke gleich $O(n^{-p-\frac{1}{2}})$, u. zw. gleichmässig in G_ϵ.

Diese Abschätzung des Restes (9) bleibt natürlich auch nach Multiplikation mit $\sqrt{(\xi/\sinh\xi)}$ erhalten.

5. Es bleibt schliesslich das zweite Integral in (2) abzuschätzen. Durch Berücksichtigung des Hilfssatzes III in § 6 erhält man dafür die Schranke

$$\tfrac{1}{2}e^{-(n+\frac{1}{2})\delta}\int_\delta^{+\infty}\frac{dt}{(\sinh\frac{1}{2}t)^{\frac{1}{2}}|\sinh(\frac{1}{2}t+\xi)|^{\frac{1}{2}}}$$

$$\leqslant \tfrac{1}{2}C\,e^{-(n+\frac{1}{2})\delta}\int_\delta^{+\infty}\frac{dt}{(\sinh\frac{1}{2}t)^{\frac{1}{2}}|(\frac{1}{2}t+\xi)(\frac{1}{2}t+\xi+i\pi)(\frac{1}{2}t+\xi-i\pi)|^{\frac{1}{2}}}.$$

Liegt ξ im Streifen $-\pi\leqslant\Im\xi\leqslant+\pi$, so sind mindestens zwei von den drei Zahlen

$$|\tfrac{1}{2}t+\xi|,\quad |\tfrac{1}{2}t+\xi+i\pi|,\quad |\tfrac{1}{2}t+\xi-i\pi|$$

gewiss nicht kleiner als $\frac{1}{2}\pi$. Es möge dies z. B. für die beiden letzten zutreffen. Dann ist das letzte Integral kleiner als

$$\frac{2}{\pi}\int_\delta^{+\infty}\frac{dt}{(\sinh\frac{1}{2}t)^{\frac{1}{2}}|\frac{1}{2}t+\Re\xi|^{\frac{1}{2}}}=\frac{2}{\pi}\int_{E_1}+\frac{2}{\pi}\int_{E_2},$$

wobei E_1 den Teil des reellen Intervalles $\delta\leqslant t\leqslant+\infty$ bezeichnet, in dem $|\frac{1}{2}t+\Re\xi|\leqslant 1$ gilt, während E_2 die komplementäre Menge ist. Der Beitrag von E_1 ist kleiner als

$$\frac{2}{\pi}\frac{1}{(\sinh\frac{1}{2}\delta)^{\frac{1}{2}}}\int_{-2}^{2}\frac{dt}{|\frac{1}{2}t|^{\frac{1}{2}}}=\frac{16}{\pi}\frac{1}{\sqrt{(\sinh\frac{1}{2}\delta)}},$$

während der von E_2 kleiner als

$$\frac{2}{\pi}\int_\delta^{+\infty}\frac{dt}{\sqrt{(\sinh\frac{1}{2}t)}}$$

wird.

Damit haben wir das folgende wichtige Gegenstück zu Satz I bewiesen:

SATZ II. *Es sei $0<\epsilon<\pi$ und die Funktion $q_n(\xi)$ wie in § 5 erklärt. Dann gilt in dem Gebiete G_ϵ und in dessen von Null verschiedenen Randpunkten*

$$q_n(\xi)\sim\sum_{\nu=0}^{\infty}A_\nu(\xi)\frac{K_\nu[(n+\frac{1}{2})\xi]}{(n+\frac{1}{2})^\nu},\tag{11}$$

wobei

$$A_0(\xi)=a_0(i\xi)=\sqrt{\left(\frac{\xi}{\sinh\xi}\right)};$$

$$A_\nu(\xi)=1\cdot3\ldots(2\nu-1)\sqrt{\left(\frac{\xi}{\sinh\xi}\right)}\,\xi^\nu\psi_\nu(i\xi)=i^{-\nu}a_\nu(i\xi)\tag{12}$$

$$(\nu=1,\,2,\,3,\,\ldots).$$

Die Funktionen $\psi_\nu(x)$ und $a_\nu(x)$ haben hier die gleiche Bedeutung wie in § 3. Bricht man die Reihe (11) bei dem Gliede $\nu=p-1$ ab, $p\geqslant1$, so ist der Rest

dem Betrage nach kleiner als

$$Ce^{-(n+\frac{1}{2})\Re\xi}n^{-p-\frac{1}{2}},$$

wobei die positive Konstante $C = C(\epsilon)$ ausser p nur von ϵ abhängt.

8. *Folgerungen aus Satz II.*

1. Für $\Re\xi > 0$, $-\pi < \Im\xi \leqslant +\pi$ erhält man aus Satz II eine asymptotische Entwicklung von $Q_n(z)$ in der längs der reellen Strecke

$$-1 \leqslant z \leqslant +1$$

aufgeschlitzten z-Ebene. Es seien noch zwei weitere wichtige Folgerungen dieses Satzes angeführt.

Für $\xi = \pm i\vartheta$, $0 < \vartheta < \pi$, ergibt sich aus Satz II eine asymptotische Darstellung der Grenzwerte von $Q_n(z)$ bei der Annäherung an $z = \cos\vartheta$, u. zw. von oben bzw. von unten. Wir schreiben für dieses Grenzwerte, wie üblich, $Q_n(\cos\vartheta \pm i0)$ und setzen

$$(1) \qquad Q_n(\cos\vartheta) = \frac{Q_n(\cos\vartheta+i0)+Q_n(\cos\vartheta-i0)}{2}.$$

Da ferner*

$$(2) \qquad -\frac{i\pi}{2}P_n(\cos\vartheta) = \frac{Q_n(\cos\vartheta+i0)-Q_n(\cos\vartheta-i0)}{2}$$

ist, so hat man

$$P_n(\cos\vartheta)-\frac{2i}{\pi}Q_n(\cos\vartheta) = -\frac{2i}{\pi}Q_n(\cos\vartheta-i0).$$

Folglich liefert Satz II für $\xi = -i\vartheta$ mit Rücksicht auf § 2, (6)

$$(3) \quad P_n(\cos\vartheta)-\frac{2i}{\pi}Q_n(\cos\vartheta) \sim -\frac{2i}{\pi}\sum_{\nu=0}^{\infty} i^{-\nu}a_\nu(\vartheta)\frac{K_\nu[-(n+\frac{1}{2})i\vartheta]}{(n+\frac{1}{2})^\nu}$$

$$\sim \sum_{\nu=0}^{\infty} a_\nu(\vartheta)\frac{J_\nu[(n+\frac{1}{2})\vartheta]+iY_\nu[(n+\frac{1}{2})\vartheta]}{(n+\frac{1}{2})^\nu}.$$

Hieraus geht wieder die asymptotische Entwicklung von Satz I hervor, u. zw. gleichmässig für $0 \leqslant \vartheta \leqslant \pi-\epsilon$ $(0 < \epsilon < \pi)$, d. h. mit einem grösseren Gültigkeitsbereich wie dort. Man hat ferner in demselben Intervalle

$$(4) \qquad -\frac{2}{\pi}Q_n(\cos\vartheta) \sim \sum_{\nu=0}^{\infty} a_\nu(\vartheta)\frac{Y_\nu[(n+\frac{1}{2})\vartheta]}{(n+\frac{1}{2})^\nu}.$$

Die Restabschätzung ist hier analog der in Satz I†.

---

* Vgl. z.B. Whittaker-Watson, a.a.O.

† Bei Herrn Watson (6) wird für unsere Funktion Y_0 die Bezeichnung $\pi^{-1}Y_0$ benutzt. In der zweiten Formel auf S. 157 ist dort $Q_n(\cos\eta)$ durch $2Q_n(\cos\eta)$ zu ersetzen. Entsprechend ist die Formel (2) in W., S. 158 [vgl. oben § 1, (e)] zu berichtigen.

Schliesslich kann noch aus Satz II eine asymptotische Entwicklung von $P_n(\cosh\xi)$ hergeleitet werden, welche die Entwicklung von Satz I verallgemeinert, und eine asymptotische Entwicklung von $P_n(z)$ in der längs des reellen Intervalls $-1 \leqslant z \leqslant +1$ aufgeschlitzten Ebene liefert.

Es sei $0 < \epsilon < \pi$ und ξ liege in G_ϵ oder auf dessen Rand. Aus § 5, (5) folgt durch Berücksichtigung von § 2, (10),

$$\pm i\pi P_n(\cosh\xi) \sim \sum_{\nu=0}^{\infty} \frac{A_\nu(-\xi)K_\nu[-(n+\frac{1}{2})\xi] - A_\nu(\xi)K_\nu[(n+\frac{1}{2})\xi]}{(n+\frac{1}{2})^\nu}$$

$$\sim \sum_{\nu=0}^{\infty} i^{-\nu} a_\nu(i\xi) \frac{(-1)^\nu K_\nu[-(n+\frac{1}{2})\xi] - K_\nu[(n+\frac{1}{2})\xi]}{(n+\frac{1}{2})^\nu}$$

$$\sim \pm i\pi \sum_{\nu=0}^{\infty} a_\nu(i\xi) \frac{J_\nu[(n+\frac{1}{2})i\xi]}{(n+\frac{1}{2})^\nu}.$$

Man hat somit in G_ϵ

$$(5) \qquad P_n(\cosh\xi) \sim \sum_{\nu=0}^{\infty} a_\nu(i\xi) \frac{J_\nu[(n+\frac{1}{2})i\xi]}{(n+\frac{1}{2})^\nu} = \sum_{\nu=0}^{\infty} i^\nu A_\nu(\xi) \frac{J_\nu[(n+\frac{1}{2})i\xi]}{(n+\frac{1}{2})^\nu}.$$

Für $\xi = i\vartheta$, $0 < \vartheta < \pi$, folgt hieraus wieder die Entwicklung von Satz I. Die Restschranke ist in diesem Falle

$$Ce^{(n+\frac{1}{2})|\Re\xi|} n^{-p-\frac{1}{2}},$$

wobei die positive Konstante $C = C(\epsilon)$ ausser p nur von ϵ abhängt.

Verzeichnis der wichtigsten Literatur.

1. P. S. Laplace, *Mécanique céleste*, 5 (1823), livre 11, supplément 1.
2. H. E. Heine, *Handbuch der Kugelfunktionen*, zweite Auflage (Berlin, 1878), 171–187.
3. G. Darboux, "Mémoire sur l'approximation des fonctions de très-grands nombres, et sur une classe étendue de développements en série", *Journal de Math.* (3), 4 (1878), 5–56, insbes. 39.
4. T. J. Stieltjes, "Sur la valeur asymptotique des polynomes de Legendre", *Comptes rendus*, 110 (1890), 1026–1027; "Sur les polynomes de Legendre", *Annales de la Faculté des Sc. de Toulouse* (1), 4 (1890), 1–17. Vgl. auch *Œuvres complètes*, 2 (Groningen, 1918), 234–252.
5. E. W. Hobson, "On a type of spherical harmonics of unrestricted degree, order, and argument", *Phil. Trans. Royal. Soc.*, 187 A (1896), 443–531, insbes. 484–491.
6. G. N. Watson, "Approximate formulae for Legendre functions", *Messenger of Math.*, 47 (1918), 151–160.
7. E. Hilb, "Über die Laplacesche Reihe", *Math. Zeitschrift*, 5 (1919), 17–25, insbes. 18–22.

ON LIDSTONE'S SERIES AND TWO-POINT EXPANSIONS OF ANALYTIC FUNCTIONS

By J. M. WHITTAKER.

[Received 5 May, 1932.—Read 19 May, 1932.]

Introduction.

1. Lidstone's series* expresses a polynomial $f(z)$ in terms of the values of the derivatives of even order† at the points 0, 1. It may be written

$$(1.1)\quad f(z)=\Lambda_0(z)f(1)+\Lambda_1(z)f''(1)+\Lambda_2(z)f^{(\mathrm{iv})}(1)+\ldots \\ -\Lambda_0(z-1)f(0)-\Lambda_1(z-1)f''(0)-\ldots,$$

the polynomials $\Lambda_n(z)$ being defined by the relations

$$(1.2)\quad \begin{cases} \text{(i)} & \Lambda_0(z)=z; \\ \text{(ii)} & \dfrac{d^2}{dz^2}\Lambda_n(z)=\Lambda_{n-1}(z) \quad (n\geqslant 1); \\ \text{(iii)} & \Lambda_n(0)=\Lambda_n(1)=0 \quad (n\geqslant 1). \end{cases}$$

It will appear that $\Lambda_n(z)$ is a Bernoullian polynomial.

Our first inquiry is to determine the class of functions for which (1.1) is valid. It is shown that this class includes all integral functions for which the inequality

$$(1.3)\quad \overline{\lim_{r\to\infty}}\,\frac{\log M(r)}{r}<\pi$$

is satisfied, $M(r)$ denoting the maximum modulus of the function on the circle $|z|=r$; and the example of $\sin\pi z$, for which the series vanishes

* Lidstone, **5**, Equation (8). The series is given as an example of a general theorem on linear operators which reduce the degree of a polynomial by two. Dr. Lidstone has pointed out to me that the use of the generating function $\sinh zt/\sinh t$ in § 2 is akin to D. C. Fraser's symbolic method of deriving Everett's interpolation formula. See Fraser, **1**, Lidstone, **4**.

† Including $f(z)$ itself, regarded as the derivative of zero order. This convention is adopted throughout.

identically, shows that the sign of equality cannot be admitted in (1.3). It follows that, if (1.3) is satisfied, and if also

$$f^{(2n)}(0) = f^{(2n)}(1) = 0 \quad (n \geqslant 0), \tag{1.4}$$

then $f(z)$ is identically zero.

2. These remarks suggest a general problem. Suppose that we are given two sequences of integers $p_1, p_2, \ldots$ and $q_1, q_2, \ldots$ and, corresponding to them, two sequences of complex numbers $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ and try to find a function $f(z)$ such that

$$f^{(p_n)}(1) = a_n, \quad f^{(q_n)}(0) = b_n \quad (n \geqslant 1). \tag{2.1}$$

In what circumstances will $f(z)$ be uniquely determined? The necessary conditions are found to be of two kinds. In the first place, we must make sure that there are, so to speak, just enough equations to determine $f(z)$. Thus, to be given the values of $f(z), f^{(3)}(z), f^{(6)}(z), \ldots$ at both points would not be enough; while if the values of $f(0), f'(0), f''(0), \ldots$ are assigned no further conditions can be imposed. It is shown that there will be "just enough" equations (2.1) to determine $f(z)$ if and only if the following two conditions are satisfied:

$$P(m)+Q(m) \geqslant m \quad (m \geqslant 1), \tag{2.2}$$

$$P(m_r)+Q(m_r) = m_r \quad \textit{for an infinite sequence } m_1, m_2, \ldots, \tag{2.3}$$

where $P(m)$ denotes the number of p's which are less than m and $Q(m)$ similarly the number of q's. The initial difficulty being thus disposed of, a further condition, of the same nature as (1.3), must be satisfied in order that (2.1) may determine $f(z)$ uniquely.

(2.2) and (2.3) are evidently satisfied if

$$P(m)+Q(m) = m \quad (m \geqslant 1), \tag{2.4}$$

i.e. if the p-sequence and the q-sequence are complementary. This case is worked out in greater detail and yields, for example, the following result.*

If every derivative of an integral function vanishes at one or other of the two points 0, 1, *and if*

$$\overline{\lim_{r\to\infty}} \frac{\log M(r)}{r} < 1, \tag{2.5}$$

then the function is identically zero.

* Readily deduced from the corollary to Theorem 7.

The significance of (2.5) is brought out by considering the function $\sin kz$, every derivative of which vanishes either at 0 or at $\pi/(2k)$, and for which

$$\frac{\log M(r)}{r} \to |k|.$$

If (2.2) and (2.3) are satisfied, the pair of sequences is said to be *complete*. It is shown that associated with such a pair there is a generalized Lidstone series

$$f^{(p_1)}(1)\,\pi_1(z)+f^{(p_2)}(1)\,\pi_2(z)+\ldots+f^{(q_1)}(0)\,\zeta_1(z)+\ldots$$

which, under certain conditions, converges to $f(z)$.

Lidstone's series.

3. It follows from (1.2) that $\Lambda_n(z)$ is of the form

$$\Lambda_n(z) = \sum_{r=0}^{n} \Lambda_r'(0)\,\frac{z^{2n+1-2r}}{(2n+1-2r)!}. \tag{3.1}$$

An explicit formula is

$$\Lambda_n(z) = \frac{1}{(2n)!}\left[\frac{d^{2n}}{dt^{2n}}\left(\frac{\sinh zt}{\sinh t}\right)\right]_{t=0}. \tag{3.2}$$

This is established by verifying that the expression on the right satisfies (1.2) and is suggested by observing that the function

$$v(z) = \sinh zt\,\operatorname{cosech} t \tag{3.3}$$

is such that

$$v^{(2n)}(0) = 0, \quad v^{(2n)}(1) = t^{2n} \quad (n \geqslant 0);$$

so that, if $v(z)$ can be expanded in the series (1.1), $\Lambda_n(z)$ must be the coefficient of t^{2n}. Thus, if $|t| < \pi$,

$$\sinh zt\,\operatorname{cosech} t = \sum_{n=0}^{\infty} t^{2n}\Lambda_n(z). \tag{3.4}$$

Now, by definition of the Bernoullian polynomials*,

$$\sinh zt\,\operatorname{cosech} t = \frac{e^{t(1+z)}-1}{e^{2t}-1} - \frac{e^{t(1-z)}-1}{e^{2t}-1}$$

$$= \sum_{n=1}^{\infty} \phi_n\left(\frac{1+z}{2}\right)\frac{(2t)^{n-1}}{n!} - \sum_{n=1}^{\infty} \phi_n\left(\frac{1-z}{2}\right)\frac{(2t)^{n-1}}{n!}.$$

* Whittaker and Watson, **6**, 126.

On comparing this with (3.4) it is seen that

$$\Lambda_n(z) = \frac{2^{2n}}{(2n+1)!}\left\{\phi_{2n+1}\left(\frac{1+z}{2}\right) - \phi_{2n+1}\left(\frac{1-z}{2}\right)\right\} \tag{3.5}$$

$$= \frac{2^{2n+1}}{(2n+1)!}\,\phi_{2n+1}\left(\frac{1+z}{2}\right) \quad (n \geqslant 1).$$

4. Various properties of $\Lambda_n(z)$ can be deduced from this result, but it is easier to start afresh.

Let C_r denote the circle in the t-plane of centre O, radius $(2r+1)\pi/2$, and let

$$\Lambda_{n,r}(z) = \frac{1}{2\pi i}\int_{C_r} t^{-2n-1}\sinh zt \operatorname{cosech} t\, dt, \tag{4.1}$$

so that

$$\Lambda_{n,0}(z) = \Lambda_n(z). \tag{4.2}$$

By Cauchy's theorem,

$$\Lambda_{n,r}(z) - \Lambda_n(z) = (-)^n \frac{2}{\pi^{2n+1}} \sum_{s=1}^{r} (-)^s \frac{\sin s\pi z}{s^{2n+1}}. \tag{4.3}$$

It can be shown that

$$\Lambda_{n,r}(x) \to 0 \quad \text{as} \quad r \to \infty \quad (-1 \leqslant x \leqslant 1), \tag{4.4}$$

so that we obtain the Fourier series, valid for $(-1 \leqslant x \leqslant 1)$,

$$\Lambda_n(x) = (-)^n \frac{2}{\pi^{2n+1}} \sum_{s=1}^{\infty} (-)^{s+1} \frac{\sin s\pi x}{s^{2n+1}}. \tag{4.5}$$

If $n \geqslant 1$, the series can be differentiated term by term, and

$$\Lambda_n'(0) = (-)^n \frac{2}{\pi^{2n}} \sum_{s=1}^{\infty} \frac{(-)^{s+1}}{s^{2n}} \quad (n \geqslant 1). \tag{4.6}$$

Again, $\operatorname{cosech} t$ is bounded on the circles C_r, so that

$$|\Lambda_{n,r}(z)| \leqslant \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{2}{(2r+1)\pi}\right)^{2n} K \exp\{(2r+1)\pi|z|/2\}\, d\theta$$

$$= \left(\frac{2}{(2r+1)\pi}\right)^{2n} K \exp\{(2r+1)\pi|z|/2\}. \tag{4.7}$$

5. By (4.3) and (4.7),

$$\Lambda_n(z) = (-)^n \frac{2}{\pi^{2n+1}} \sin \pi z + \Lambda_{n,1}(z), \tag{5.1}$$

where

$$|\Lambda_{n,1}(z)| \leqslant \left(\frac{2}{3\pi}\right)^{2n} K \exp\left(\frac{3\pi}{2}|z|\right). \tag{5.2}$$

The region of convergence of (1.1) can now be determined.

THEOREM 1. *If*

$$\sum_0^\infty (-)^n \frac{a_n}{\pi^{2n}}, \quad \sum_0^\infty (-)^n \frac{b_n}{\pi^{2n}} \tag{5.3}$$

are convergent, then

$$\sum_0^\infty a_n \Lambda_n(z) - \sum_0^\infty b_n \Lambda_n(z-1) \tag{5.4}$$

converges uniformly in every finite region of the plane, and its sum is an integral function $f(z)$. Moreover,

$$f^{(2n)}(1) = a_n, \quad f^{(2n)}(0) = b_n \quad (n \geqslant 0). \tag{5.5}$$

If one of the series (5.3) *is not convergent,* (5.4) *cannot converge for any non-integral value of z.*

If the series (5.3) are convergent, it follows immediately from (5.1) and (5.2) that (5.4), and the series obtained by differentiating it an even number of times, are uniformly convergent in any finite region of the plane. Term by term differentiation is therefore legitimate and (5.5) follows. The last part of the theorem is likewise an easy deduction from (5.1) and (5.2).

We prove next that any integral function for which (1.3) is satisfied can be expanded in a Lidstone series.

THEOREM 2. *If $f(z)$ is an integral function for which* (1.3) *is satisfied,*

$$f(z) = \sum_0^\infty f^{(2n)}(1) \Lambda_n(z) - \sum_0^\infty f^{(2n)}(0) \Lambda_n(z-1), \tag{5.6}$$

the series being uniformly convergent in any finite region of the plane.

The proof depends on the following lemma.

LEMMA 1. *Let $f(z)$ be an integral function for which*

$$\overline{\lim_{r\to\infty}}\frac{\log M(r)}{r}=k<\infty. \tag{5.7}$$

Then, if $l>k$,

$$|f^{(n)}(0)|<l^n \quad [n\geqslant n_0(l)]. \tag{5.8}$$

For $$|f^{(n)}(0)|\leqslant\frac{M(r)\,n!}{r^{n+1}}<\frac{e^{lr}\,n!}{r^{n+1}} \quad (r\geqslant r_0).$$

Taking $r=n/l$, this gives

$$|f^{(n)}(0)|<\frac{e^n\,n!}{n^{n+1}}\,l^{n+1}$$

$$<l^n \quad (n\geqslant n_0).$$

Now, formally,

$$\sum_{n=0}^{\infty}f^{(2n)}(1)\Lambda_n(z)-\sum_{n=0}^{\infty}f^{(2n)}(0)\Lambda_n(z-1)$$

$$=\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\frac{f^{(2n+r)}(0)}{r!}\Lambda_n(z)-\sum_{n=0}^{\infty}f^{(2n)}(0)\Lambda_n(z-1)$$

$$=f(0)\{\Lambda_0(z)-\Lambda_0(z-1)\}+f'(0)\Lambda_0(z)$$

$$+\frac{f''(0)}{2!}\{\Lambda_0(z)+2\Lambda_1(z)-2\Lambda_1(z-1)\}+\ldots$$

$$=f(0)+f'(0)z+\frac{f''(0)}{2!}z^2+\ldots$$

$$=f(z).$$

The rearrangement of the series is justifiable if

$$\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\frac{|f^{(2n+r)}(0)\Lambda_n(z)|}{r!}+\sum_{n=0}^{\infty}|f^{(2n)}(0)\Lambda_n(z-1)| \tag{5.9}$$

is convergent, and this is the case. For, by (5.1) and (5.2),

$$\Lambda_n(z)=O\left(\frac{1}{\pi^{2n}}\right)$$

in a finite region of the z-plane. Moreover, $l<\pi$ can be chosen so that (5.8) is satisfied. Thus the series (5.9) is dominated by

$$\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\frac{Kl^{2n+r}}{r!\,\pi^{2n}}+\sum_{n=0}^{\infty}\frac{Kl^{2n}}{\pi^{2n}}=K(e^l+1)\frac{\pi^2}{\pi^2-l^2}.$$

The next result is of a different type.

THEOREM 3. *If $f(z)$, $g(z)$ are any integral functions, there is an integral function $h(z)$ such that*

$$h^{(2n)}(1)=f^{(2n)}(1), \quad h^{(2n)}(0)=g^{(2n)}(0) \quad (n\geqslant 0). \tag{5.10}$$

This can be deduced from Guichard's theorem that every integral function has a "sum"*.

Write $$f_1(z)=f(z+1)+f(-z+1),$$

so that $f_1(z)$ is an even function. By Guichard's theorem, there is an integral function $H(z)$ such that

$$H(z+1)-H(z-1)=f_1(z).$$

Since $f_1(z)$ is even,

$$H(-z+1)-H(-z-1)=f_1(z).$$

Thus, if $$H_1(z)=\tfrac{1}{2}\{H(z)-H(-z)\},$$

then

$$\begin{aligned} H_1(z+1)-H_1(z-1) &= \tfrac{1}{2}\{H(z+1)-H(-z-1)-H(z-1)+H(-z+1)\} \\ &= f_1(z). \end{aligned}$$

Thus, by differentiating this relation,

$$H_1^{(2n)}(1)-H_1^{(2n)}(-1)=f_1^{(2n)}(0)=2f^{(2n)}(1),$$

and, since $H_1(z)$ is an odd function, this gives

$$H_1^{(2n)}(1)=f^{(2n)}(1), \quad H_1^{(2n)}(0)=0 \quad (n\geqslant 0).$$

In the same way a function $H_2(z)$ can be found such that

$$H_2^{(2n)}(1)=0, \quad H_2^{(2n)}(0)=g^{(2n)}(0) \quad (n\geqslant 0),$$

and the required integral function is defined by

$$h(z)=H_1(z)+H_2(z).$$

$h(z)$ is not, of course, unique, since any constant multiple of $\sin \pi z$ may be added to it.

6. The equations

$$f^{(2n)}(1)=a_n, \quad f^{(2n+1)}(0)=b_n \quad (n\geqslant 0) \tag{6.1}$$

can be studied in the same manner. The appropriate polynomials are

* Guichard, 2.

defined by

$$\cosh zt \operatorname{sech} t = \sum_{n=0}^{\infty} t^n M_n(z), \tag{6.2}$$

and it can be shown that

$$f(z) = \sum_{n=0}^{\infty} f^{(2n)}(1) M_n(z) - \sum_{n=0}^{\infty} f^{(2n+1)}(0) M'_{n+1}(1-z), \tag{6.3}$$

the series being uniformly convergent in any finite region of the plane, provided that

$$\overline{\lim_{r\to\infty}} \frac{\log M(r)}{r} < \frac{\pi}{2}. \tag{6.4}$$

The Fourier series, valid for $(-1 \leqslant x \leqslant 1)$, is

$$M_n(x) = (-)^n 2 \left(\frac{2}{\pi}\right)^{2n+1} \sum_{s=0}^{\infty} \frac{(-)^s \cos(2s+1)\tfrac{1}{2}\pi x}{(2s+1)^{2n+1}}. \tag{6.5}$$

The analogue of Theorem 3 can be established with the aid of the functions

$$M_{n,n}(z) = M_n(z) + (-)^{n+1} 2 \left(\frac{2}{\pi}\right)^{2n+1} \sum_{s=0}^{n-1} \frac{(-)^s \cos(2s+1)\tfrac{1}{2}\pi z}{(2s+1)^{2n+1}} \tag{6.6}$$

on the lines of Hurwitz's proof of Guichard's theorem.*

Standard sets of polynomials.

7. The polynomials $\Lambda_n(z)$, $-\Lambda_n(z-1)$ form what will be termed a *standard set* of polynomials in relation to the pair of sequences $(0, 2, 4, \ldots; 0, 2, 4, \ldots)$. Generally, let $p_1, p_2, \ldots$ and $q_1, q_2, \ldots$ be any two strictly increasing sequences of positive integers (including zero). A set of polynomials $\pi_n(z), \zeta_n(z)$ will be said to form a *standard set in relation to the pair of sequences* $(p_1, p_2, \ldots; q_1, q_2, \ldots)$, which will be written briefly $(p; q)$, if

$$\begin{cases} \pi_n^{(t)}(1) = 1, & t = p_n; \\ \qquad = 0, & t = p_r \quad (r \neq n); \\ \pi_n^{(t)}(0) = 0, & t = q_r \quad (r \geqslant 1). \end{cases} \tag{7.1}$$

$$\begin{cases} \zeta_n^{(t)}(0) = 1, & t = q_n; \\ \qquad = 0, & t = q_r \quad (r \neq n); \\ \zeta_n^{(t)}(1) = 0, & t = p_r \quad (r \geqslant 1). \end{cases} \tag{7.2}$$

* Hurwitz, 3.

A pair of sequences will be said to be *complete* if one and only one standard set of polynomials exists, *indeterminate* if more than one such set exists, and *redundant* if no standard set exists.

It is convenient to define two functions of the positive integral variable m:

$$(7.3) \qquad \begin{cases} P(m) = \text{number of } p\text{'s which are less than } m; \\ Q(m) = \quad ,, \quad q\text{'s} \quad ,, \quad ,, \quad . \end{cases}$$

The three cases mentioned above can be simply distinguished by means of these functions.

THEOREM 4. *In order that the pair of sequences* $(p\,;\ q)$ *may be complete, it is necessary and sufficient that*

$$(7.4) \qquad P(m)+Q(m) \geqslant m \quad (m \geqslant 1),$$

and

$$(7.5) \qquad P(m_r)+Q(m_r) = m_r, \quad \text{for an infinite sequence } m_1, m_2, \ldots.$$

In order that the pair may be indeterminate, it is necessary that

$$(7.6) \qquad P(m_r)+Q(m_r) < m_r, \quad \text{for an infinite sequence } m_1, m_2, \ldots.$$

The proof depends on the following lemma, of which a proof, as yet unpublished, has been found by Dr. A. C. Aitken*.

LEMMA 2. *If* $a, b, \ldots, a', b', \ldots$ *are positive integers satisfying the inequalities*

$$a < b < \ldots < k, \quad a' < b' < \ldots < k',$$

$$a' \leqslant a, \quad b' \leqslant b, \quad \ldots, \quad k' \leqslant k,$$

then

$$\begin{vmatrix} \{a, a'\} & \{b, a'\} & \ldots & \{k, a'\} \\ \{a, b'\} & \{b, b'\} & \ldots & \{k, b'\} \\ \ldots & \ldots & \ldots & \ldots \\ \{a, k'\} & \{b, k'\} & \ldots & \{k, k'\} \end{vmatrix} \geqslant 1,$$

where

$$\{a, a'\} = a(a-1)(a-2)\ldots(a-a'+1), \ \ldots,$$

and the determinant may be of any order.

* *Added in proof.* See Zia-uddin, *Proc. Edinburgh Math. Soc.*, 3 (1933), 296–299.

To establish the first part of Theorem 4, suppose in the first place that (7.4), (7.5) are satisfied. Let $u_1, u_2, \ldots$ be the sequence complementary to $q_1, q_2, \ldots$ with respect to 0, 1, 2, ..., and let

$$U(m) = \text{number of } u\text{'s which are less than } m$$
$$= m - Q(m).$$

(7.4) becomes $$P(m) \geqslant U(m) \quad (m \geqslant 1),$$

or

$$p_r \leqslant u_r \quad (r \geqslant 1). \tag{7.7}$$

To determine $\pi_n(z)$ for some fixed value of n, take $N \geqslant n$ to be any one of the integers $m_1, m_2, \ldots$, and assume that $\pi_n(z)$ is of degree not greater than N. If we write $P(N) = P$, $Q(N) = Q$, we see that $\pi_n(z)$ must be of the form

$$d_1 z^{u_1} + d_2 z^{u_2} + \ldots + d_P z^{u_P}.$$

The last condition of (7.1) is then automatically satisfied and the first two furnish $P = N - Q$ equations to determine the coefficients $d_1, d_2, \ldots$, namely

$$\left\{\begin{array}{l} d_1\{u_1, p_1\} + d_2\{u_2, p_1\} + \ldots + d_P\{u_P, p_1\} = 0, \\ d_1\{u_1, p_2\} + \ldots = 0, \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ d_1\{u_1, p_n\} + \ldots = 1, \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ d_1\{u_1, p_P\} + \ldots = 0. \end{array}\right. \tag{7.8}$$

The inequalities of Lemma 2 being satisfied in virtue of (7.7), these equations have a non-zero determinant and so a unique solution. Since N may be arbitrarily large, it follows that there is one and only one polynomial $\pi_n(z)$ satisfying (7.1). The existence of a unique set of polynomials $\zeta_n(z)$ is established in the same way, and the pair $(p; q)$ is therefore complete.

Next suppose that (7.4) is satisfied but that (7.5) is not, so that

$$P(m) + Q(m) > m \quad (m \geqslant M). \tag{7.9}$$

M may be taken to be the smallest integer for which this is true. Thus

$$P(M-1) + Q(M-1) = M - 1,$$
$$P(M) + Q(M) > M.$$

$M-1$ is therefore both a p and a q. Omit $M-1$, considered as a q, from the pair of sequences. Then we are left with a new pair for which (7.4) is satisfied. If this pair does not satisfy (7.5), (7.9) must be true with $M' > M$ in place of M. We can now omit another q, and so on. In this way a pair $(p\,;\ v)$ is obtained, $v_1, v_2, \ldots$ being a subsequence of $q_1, q_2, \ldots$, for which (7.4), (7.5) are satisfied. This pair is therefore complete.

Let q_n be one of the omitted q's. If a standard set of polynomials in relation to the pair $(p\,;\ q)$ exists, the polynomial $\zeta_n(z)$ is such that

$$\zeta_n^{(t)}(0) = 0, \quad t = v_1, v_2, \ldots,$$

$$\zeta_n^{(t)}(1) = 0, \quad t = p_1, p_1, \ldots.$$

Constant multiples of $\zeta_n(z)$ can therefore be added to the polynomials of the standard set attached to the pair $(p\,;\ v)$ without destroying their properties; and, since $\zeta_n(z)$ is not identically zero, the pair $(p\,;\ v)$ cannot be complete. The contradiction implies that $(p\,;\ q)$ is redundant.

If (7.4) is not satisfied, so that

$$P(M)+Q(M) < M$$

for some M, it is clear that a non-zero polynomial $\rho(z)$ of degree less than M can be determined so that

$$\rho^{(t)}(1) = 0, \quad t = p_1, p_2, \ldots, p_{P(M)},$$

$$\rho^{(t)}(0) = 0, \quad t = q_1, q_2, \ldots, q_{Q(M)}.$$

Thus the pair cannot be complete.

Again, if (7.6) is not satisfied, either (7.4) must be satisfied, a case dealt with already, or there must be an integer M such that

$$P(M)+Q(M) < M, \tag{7.10}$$

and also an integer n_0 such that

$$P(n)+Q(n) \geqslant n \quad (n \geqslant n_0).$$

By removing, in the manner described above, some of the q's which are greater than n_0-1, the pair of sequences can be transformed into a pair $(p\,;\ v)$ for which

$$\begin{cases} P(n)+V(n) \geqslant n & (n \geqslant n_0), \\ P(n_r)+V(n_r) = n_r & \text{for an infinite sequence } n_1, n_2, \ldots. \end{cases}$$

Let $w_1, w_2, \ldots$ be the sequence complementary to $v_1, v_2, \ldots$.

Now M may be supposed to be the smallest integer for which (7.10) is satisfied, and, since the first u's are the same as the first w's, (7.10) is

$$P(M) < W(M)$$

while

$$P(M-1) = W(M-1).$$

$M-1$ is therefore a w, say w_k, but not a p. Thus

$$p_k > w_k.$$

Let l be the largest integer which is such that

$$p_l > w_l. \tag{7.11}$$

That such a largest integer exists follows from the fact that

$$P(n) \geqslant W(n) \quad (n \geqslant n_0).$$

It will now be shown that the pair $(p\,;\ v)$ is redundant. If not, let $\pi_l(z)$ belong to a standard set associated with $(p\,;\ v)$ and let N, one of the sequence $n_1, n_2, \ldots$, be chosen so that $P = P(N) \geqslant l$, and also so that N is larger than the degree of $\pi_l(z)$. Then $\pi_l(z)$ is of the form

$$b_1 z^{w_1} + b_2 z^{w_2} + \ldots + b_P z^{w_P},$$

and the last $P-l+1$ of the equations similar to (7.8) reduce to

$$\left\{\begin{array}{l} b_{l+1}\{w_{l+1}, p_l\} + \ldots + b_P\{w_P, p_l\} = 1, \\ b_{l+1}\{w_{l+1}, p_{l+1}\} + \ldots + b_P\{w_P, p_{l+1}\} = 0, \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ b_{l+1}\{w_{l+1}, p_P\} + \ldots + b_P\{w_P, p_P\} = 0, \end{array}\right. \tag{7.12}$$

the coefficients of $b_1, \ldots, b_l$ being zero in virtue of (7.11). These equations are inconsistent. For, since l is the largest integer for which (7.11) is satisfied, the determinant of the last $P-l$ equations of (7.12) is not zero, and hence

$$b_{l+1} = b_{l+2} = \ldots = b_P = 0.$$

This contradicts the first equation of (7.12). Thus the pair $(p\,;\ v)$ must be redundant, and *a fortiori* the pair $(p\,;\ q)$ is redundant. This completes the proof of Theorem 4.

Series of standard polynomials.

8. THEOREM 5. *The polynomials of a standard set are linearly independent, and, if the associated pair of sequences is complete, an arbitrary*

polynomial $u(z)$ can be expressed in one and only one way as a linear combination of standard polynomials, namely

$$(8.1) \qquad u(z) = u^{(p_1)}(1)\,\pi_1(z) + u^{(p_2)}(1)\,\pi_2(z) + \ldots + u^{(q_1)}(0)\,\zeta_1(z) + \ldots.$$

The proof of the first statement is immediate. If $\pi_n(z)$, say, is a polynomial occurring in a linear relation connecting standard polynomials, a contradiction is reached on differentiating p_n times and setting $z = 1$.

If the pair $(p\,;\ q)$ is complete take, as above, $N = P+Q$. The N equations expressing $\pi_1(z), \ldots, \pi_P(z), \zeta_1(z), \ldots, \zeta_Q(z)$ as polynomials of degree $N-1$ can be regarded as a set of linear equations in the unknowns $z^0, z^1, \ldots, z^{N-1}$. The determinant of these equations cannot be zero, for this would imply that the standard polynomials were linearly dependent. Hence the equations can be solved. In this way any polynomial $u(z)$ can be expressed in the form

$$A_1\,\pi_1(z) + A_2\,\pi_2(z) + \ldots + B_1\,\zeta_1(z) + \ldots.$$

On differentiating $p_1, p_2, \ldots$ times and setting $z = 1, 0$, we obtain the values of the coefficients, as in (8.1).

Still assuming the pair $(p\,;\ q)$ to be complete, take next a function $f(z)$ regular in $|z-\frac{1}{2}| < R$ $(R > \frac{1}{2})$, and consider the series

$$(8.2) \qquad f^{(p_1)}(1)\,\pi_1(z) + f^{(p_2)}(1)\,\pi_2(z) + \ldots + f^{(q_1)}(0)\,\zeta_1(z) + \ldots$$

$$= \sum_{k=p_1}^{\infty} \frac{1}{(k-p_1)!\ 2^{k-p_1}}\, f^{(k)}(\tfrac{1}{2})\,\pi_1(z)$$

$$+ \ldots + \sum_{k=q_1}^{\infty} \frac{(-)^{k-q_1}}{(k-q_1)!\ 2^{k-q_1}}\, f^{(k)}(\tfrac{1}{2})\,\zeta_1(z) + \ldots.$$

The coefficient of $f^{(k)}(\frac{1}{2})$ is

$$\Sigma \frac{1}{(k-p_n)!\ 2^{k-p_n}}\,\pi_n(z) + \Sigma \frac{(-)^{k-q_n}}{(k-p_n)!\ 2^{k-q_n}}\,\zeta_n(z),$$

the summations being taken for all $p_n, q_n \leqslant k$, and by (8.1) this expression is equal to

$$\frac{1}{k!}\,(z-\tfrac{1}{2})^k.$$

Thus the series (8.2) may be formally rearranged as

$$\sum_{k=0}^{\infty} \frac{1}{k!}\,(z-\tfrac{1}{2})^k\, f^{(k)}(\tfrac{1}{2}) = f(z).$$

The rearrangement will be legitimate if

$$\sum_{k=0}^{\infty}\left\{\Sigma\frac{\alpha_n(R)}{(k-p_n)!\,2^{k-p_n}}+\Sigma\frac{\beta_n(R)}{(k-q_n)!\,2^{k-q_n}}\right\}|f^{(k)}(\tfrac{1}{2})| \tag{8.3}$$

converges, where

$$\alpha_n(R)=\max_{|z-\frac{1}{2}|=R}|\pi_n(z)|,\quad \beta_n(R)=\max_{|z-\frac{1}{2}|=R}|\zeta_n(z)|. \tag{8.4}$$

Writing

$$\lambda(k,\ R)=\text{coefficient of } |f^{(k)}(\tfrac{1}{2})| \text{ in (8.3)}, \tag{8.5}$$

we may sum up these results as follows.

THEOREM 6. *Associated with a complete pair* $(p\,;\,q)$ *there is a positive function* $\lambda(k,\ R)$ *with the following property. If* $f(z)$ *is regular in* $|z-\frac{1}{2}|<R$ $(R>\frac{1}{2})$, *and*

$$\sum_{k=0}^{\infty}\lambda(k,\ R)|f^{(k)}(\tfrac{1}{2})| \tag{8.6}$$

converges, then

$$f(z)=f^{(p_1)}(1)\,\pi_1(z)+\ldots+f^{(q_1)}(0)\,\zeta_1(z)+\ldots, \tag{8.7}$$

the series being absolutely and uniformly convergent in $|z-\frac{1}{2}|<R$.

To return to the problem of finding a function $f(z)$ for which

$$f^{(p_n)}(1)=a_n,\quad f^{(q_n)}(0)=b_n\quad (n\geqslant 1), \tag{8.8}$$

we see that, if $(p\,;\,q)$ is complete and a_n, b_n tend to zero sufficiently rapidly, a unique function exists satisfying (8.6) and (8.8); that, if $(p\,;\,q)$ is indeterminate and a_n, b_n tend to zero sufficiently rapidly, more than one solution of (8.8) exists; and that, if $(p\,;\,q)$ is redundant, a set of values a_n, b_n exists, tending to zero as rapidly as we please, and such that no function satisfying (8.6) and (8.8) exists.

In the first two cases the required function is

$$f(z)=\sum_{n=1}^{\infty}a_n\pi_n(z)+\sum_{n=1}^{\infty}b_n\zeta_n(z), \tag{8.9}$$

and a_n, b_n must be such that the function so defined satisfies (8.6) for some $R>\frac{1}{2}$. It is evident that if a_n, b_n are small enough this will be the case.

Analogues of Theorems 1 and 2 have now been established for the general case of complete pairs of sequences. Theorem 3 is in a different category. The method by which it was established does not seem to

generalize in any simple manner, though it may well be that the analogue of Theorem 3 is true. This result, if valid, would presumably run as follows.

If $(p\,;\ q)$ is a complete pair of sequences and $f(z)$, $g(z)$ are any integral functions, there is an integral function $h(z)$ such that

$$h^{(t)}(1) = f^{(t)}(1) \quad (t = p_1, p_2, \ldots),$$
$$h^{(t)}(0) = g^{(t)}(0) \quad (t = q_1, q_2, \ldots).$$

The case of complementary sequences.

9. An important case is that for which

$$P(m) + Q(m) = m \quad (m \geqslant 1). \tag{9.1}$$

The sequences are then complementary and form a complete pair.

It is convenient to define certain algebraic expressions in the following manner. Let $a, b, \ldots$ be a decreasing sequence of positive integers (or zero). Write

$$\begin{cases} [a, b] = \dfrac{1}{(a-b)!}, \\ [a, b, \ldots, k, l, m] = [a, b, \ldots, k, l][l, m] - [a, b, \ldots, k, m], \end{cases} \tag{9.2}$$

so that

$$[a, b, c] = \frac{1}{(a-b)!\,(b-c)!} - \frac{1}{(a-c)!},$$

$$[a, b, c, d] = \frac{1}{(a-b)!\,(b-c)!\,(c-d)!} - \frac{1}{(a-b)!\,(b-d)!} - \frac{1}{(a-c)!\,(c-d)!} + \frac{1}{(a-d)!},$$

... ...

With the aid of these expressions the standard set of polynomials can be constructed. The equations

$$\begin{cases} \Omega^{(n)}(z) = 1, \\ \Omega_n^{(t)}(1) = 0 \quad (t = p_1, p_2, \ldots, p_P), \\ \Omega_n^{(t)}(0) = 0 \quad (t \neq n, p_1, p_2, \ldots, p_P), \end{cases} \tag{9.3}$$

where $P = P(n)$, define a unique polynomial $\Omega_n(z)$ of the form

$$\Omega_n(z) = \frac{z^n}{n!} - c_P z^{p_P} + c_{P-1} z^{p_{P-1}} - \ldots + (-)^P c_1 z^{p_1}. \tag{9.4}$$

The coefficients are determined by the equations, for $r = 1, 2, \ldots, P$,

$$\frac{1}{(n-p_r)!} - \frac{p_P!\, c_P}{(p_P-p_r)!} + \frac{p_{P-1}!\, c_{P-1}}{(p_{P-1}-p_r)!} - \ldots + (-)^{P-r+1} p_r!\, c_r = 0,$$

of which the solution is

$$c_r = \frac{1}{p_r!}\,[n, p_P, p_{P-1}, \ldots, p_r] \quad (r = 1, 2, \ldots, P). \tag{9.5}$$

The polynomials $\Omega_n(z)$ form the standard set, since

$$\begin{aligned} \Omega_m(z) &= \pi_n(z) \quad (m = p_n) \\ &= \zeta_n(z) \quad (m = q_n). \end{aligned}$$

To determine their order of magnitude a further lemma is needed.

LEMMA 3. *Let $x_1, x_2, \ldots, x_k, n$ be integers satisfying the inequality*

$$0 \leqslant x_1 < x_2 < \ldots < x_k < n.$$

Then

$$x_1!\,(x_2-x_1)!\ldots(x_k-x_{k-1})!\,(n-x_k)! \geqslant \left(\frac{n}{k+1}\right)^{\frac{1}{2}n}. \tag{9.6}$$

Evidently it is enough to prove the result when $x_1 > 0$. Now

$$n! \geqslant n^{\frac{1}{2}n} \quad (n \geqslant 1),$$

so that

$$\begin{aligned} x_1!\,(x_2-x_1)!\ldots(n-x_k)! &\geqslant x_1^{\frac{1}{2}x_1}(x_2-x_1)^{\frac{1}{2}(x_2-x_1)}\ldots(n-x_k)^{\frac{1}{2}(n-x_k)} \\ &= F(x_1, x_2, \ldots, x_k) \quad \text{(say)}. \end{aligned}$$

The conditions for a stationary value, regarding F as a function of k real variables, are

$$\frac{\partial F}{\partial x_1} = \frac{\partial F}{\partial x_2} = \ldots = \frac{\partial F}{\partial x_k} = 0.$$

These give

$$\log x_1 = \log(x_2-x_1) = \ldots = \log(n-x_k),$$

whence

$$x_1 = x_2 = \ldots = x_k = \frac{n}{k+1},$$

and it is easy to see that the value of F so found is a minimum.

This leads to

$$|\Omega_n(z)| < 2^P (P+2)^{\frac{1}{2}n+1}(n+2)^{-1} n^{-\frac{1}{2}n} \max(1, |z|^n), \tag{9.7}$$

c_{P-t} is the sum of 2^t terms, each with not more than $t+2$ factorials in the denominator. Thus, by the last lemma,

$$|c_{P-t}| \leqslant 2^t(t+2)^{\frac{1}{2}n}\, n^{-\frac{1}{2}n}, \tag{9.8}$$

and

$$\begin{aligned}|\Omega_n(z)| &\leqslant n^{-\frac{1}{2}n}\{|z|^n + |z|^{p_P} 2^{\frac{1}{2}n} + 2|z|^{p_{P-1}} 3^{\frac{1}{2}n} + \ldots + 2^{P-1}|z|^{p_1}(P+1)^{\frac{1}{2}n}\} \\ &\leqslant 2^{P-1} n^{-\frac{1}{2}n}\{1 + 2^{\frac{1}{2}n} + 3^{\frac{1}{2}n} + \ldots + (P+1)^{\frac{1}{2}n}\} \max(1, |z|^n) \\ &< 2^{P-1} n^{-\frac{1}{2}n} \int_0^{P+2} x^{\frac{1}{2}n}\, dx \max(1, |z|^n),\end{aligned}$$

which is (9.7). Similarly, writing $Q = n - P$, we have

$$|\Omega_n(z)| < 2^Q (Q+2)^{\frac{1}{2}n+1} (n+2)^{-1} n^{-\frac{1}{2}n} \max(1, |z-1|^n). \tag{9.9}$$

Now define

$$\mu(n) = \min[P(n), Q(n)]. \tag{9.10}$$

Then (9.7) and (9.9) give

$$|\Omega_n(z)| < 2^\mu (\mu+2)^{\frac{1}{2}n+1} (n+2)^{-1} n^{-\frac{1}{2}n} \max[1, (\tfrac{1}{2}+R)^n], \tag{9.11}$$

where $R = |z - \frac{1}{2}|$, and, since

$$\mu(n) \leqslant \tfrac{1}{2}n, \tag{9.12}$$

this gives, for $R \geqslant \frac{1}{2}$,

$$\begin{aligned}|\Omega_n(z)| &< \tfrac{1}{2}\left(1 + \frac{2}{n}\right)^{\frac{1}{2}n} \left(1 + \frac{1}{\frac{1}{2}n+1}\right)^{\frac{1}{2}n+1} (\tfrac{1}{2}+R)^n \\ &< \tfrac{1}{2}e^2(\tfrac{1}{2}+R)^n. \end{aligned} \tag{9.13}$$

10. By (8.1),

$$\frac{z^n}{n!} = \Omega_n(z) + \frac{\delta_{n-1}}{1!}\Omega_{n-1}(z) + \frac{\delta_{n-2}}{2!}\Omega_{n-2}(z) + \ldots + \frac{\delta_0}{n!}\Omega_0(z), \tag{10.1}$$

where

$$\begin{cases} \delta_t = 1, & t = p_1, p_2, \ldots \\ \quad = 0, & \text{otherwise.} \end{cases} \tag{10.2}$$

Thus, if

$$f(z) = \sum_{n=0}^{\infty} \frac{f_n}{n!} z^n, \tag{10.3}$$

we have formally

$$f(z) = \sum_{n=0}^{\infty} f_n \Omega_n(z) + \delta_0 \Omega_0(z) \sum_{n=1}^{\infty} \frac{f_n}{n!} + \delta_1 \Omega_1(z) \sum_{2}^{\infty} \frac{f_n}{(n-1)!} + \dots$$

$$= \sum_{n=0}^{\infty} f_n \Omega_n(z) + \delta_0 \Omega_0(z)\{f(1)-f(0)\} + \delta_1 \Omega_1(z)\{f'(1)-f'(0)\} + \dots$$

$$= \sum_{n=0}^{\infty} \{\delta_n f^{(n)}(1) + (1-\delta_n) f^{(n)}(0)\} \Omega_n(z). \tag{10.4}$$

The rearrangement will be legitimate if

$$\sum_{n=0}^{\infty} |f_n| \sum_{m=0}^{n} \frac{|\Omega_m(z)|}{(n-m)!}$$

converges, and by (9.13) this series is less than

$$\tfrac{1}{2}e^2 \sum_{n=0}^{\infty} |f_n| \sum_{m=0}^{n} \frac{(\frac{1}{2}+R)^m}{(n-m)!} = \tfrac{1}{2}e^2 \sum_{n=0}^{\infty} |f_n| (\tfrac{1}{2}+R)^n \sum_{k=0}^{n} \frac{1}{k!\,(\frac{1}{2}+R)^k}$$

$$< \tfrac{1}{2}e^{2+(\frac{1}{2}+R)^{-1}} \sum_{n=0}^{\infty} |f_n| (\tfrac{1}{2}+R)^n,$$

and, if

$$\overline{\lim_{n\to\infty}} |f_n|^{1/n} = \gamma < 1, \tag{10.5}$$

this expression is finite for $R < \gamma^{-1} - \frac{1}{2}$. From this and Lemma 1 follows

THEOREM 7. *If $f(z)$ is an integral function satisfying the condition*

$$\overline{\lim_{r\to\infty}} \frac{\log M(r)}{r} = \gamma < 1, \tag{10.6}$$

then the series (10.4) *converges uniformly and absolutely to $f(z)$ in the circle $|z-\frac{1}{2}| < \gamma^{-1} - \frac{1}{2}$.*

A corollary is worth mentioning.

If $f(z)$ is an integral function satisfying the condition

$$\overline{\lim_{r\to\infty}} \frac{\log M(r)}{r} < 1, \tag{10.7}$$

and

$$f^{(n)}(0) f^{(n)}(1) = 0 \quad (n \geqslant 0), \tag{10.8}$$

then $f(z)$ is identically zero.

Consideration of the function $\sin \frac{1}{2}\pi z$ shows that the constant on the right of (10.7) cannot be replaced by any number larger than $\frac{1}{2}\pi$. This is very possibly the true value.

[*Added in proof.* Reference should have been made to the work of W. Gontcharoff, *Annales Sci. de l'Ecole Normale*, 47 (1930), 1–78, and S. Takenaka, *Proc. Physico. Math. Soc. Japan*, 13 (1931), 111–132, which partly anticipates the work of § 10. A weaker form of Theorem 7, the number 1 on the right of (10.6) being replaced by $1/e$, is a particular case of a general theorem proved by these writers.]

References.

1. D. C. Fraser, *Journal of the Institute of Actuaries*, 43 (1909), 238.
2. C. Guichard, *Annales Sci. de l'Ecole Normale*, 4 (1887), 361.
3. A. Hurwitz, *Acta Math.*, 20 (1897), 285; 22 (1899), 179.
4. G. J. Lidstone, *Proc. Edinburgh Math. Soc.*, 40 (1922), 26.
5. ———, *Proc. Edinburgh Math. Soc.*, 2 (1930), 16.
6. E. T. Whittaker and G. N. Watson, *Modern analysis* (1920).

ON THE REPRESENTATIONS OF A NUMBER AS A SUM OF SQUARES AND PRODUCTS

(*Second Paper.*)

By A. PAGE.

[Received 14 June, 1932.—Read 16 June, 1932.]

1. *Introduction.* In a recent paper* I have proved an asymptotic formula for the number of representations of an integer $n \geqslant 2$ as a sum of r squares and s products when $r+2s \geqslant 5$, $r > 0$, $s > 0$. The proof is a direct application of the well-known Hardy-Littlewood method. In the case $r+2s = 4$, this method gives an error term of order as great as the dominant term, and then an asymptotic formula may be obtained by an application of the Kloostermann refinement of the Hardy-Littlewood method. In this paper I investigate the problem by an elementary method which was first used by Ingham† to obtain an asymptotic formula for the number of representations of an integer as the sum of two products. Estermann‡ has used a refinement of this method to obtain a more precise result for the same problem.

The method may be applied to obtain elementary results for the number of representations of an integer as the sum of r squares and one product when r is even, and it will be shown that, in the case of four squares and one product, we are led to the same result as is given by the transcendental method.

I shall prove the following theorems.

THEOREM I. *Let n be a positive integer, and let $\nu(n)$ be the number of solutions of the equation*

$$x_1^2+x_2^2+yz = n$$

in integers x_1, x_2, y, and z, with $y > 0$, $z > 0$.

* *Proc. London Math. Soc.* (2), 36 (1933), 241–256.

† *Journal London Math. Soc.*, 2 (1927), 207–208

‡ *Journal London Math. Soc.*, 5 (1930), 131–137.

Let

$$E_r(n) = \sum_d (-1)^{\frac{1}{2}(d-1)} d^{-1} \log^r d \quad (r = 0,\ 1), \tag{1}$$

where d runs through the odd positive divisors of n;

$$E_r'(n) = \sum_d (-1)^{\frac{1}{2}(n/d-1)} d^{-1} \log^r d \quad (r = 0,\ 1), \tag{2}$$

where d runs through the positive divisors of n which are such that n/d is odd;

$$F_r(n) = 4E_r(n) + E_r'(n) \quad (r = 0,\ 1); \tag{3}$$

let

$$b_r = \sum_{2}^{q} (-1)^{\frac{1}{2}(q-1)} \mu(q)\, q^{-2} \log^r q \quad (r = 0,\ 1), \tag{4}$$

where $\mu(q)$ is Möbius's function, and let γ denote Euler's constant.

Then

$$\nu(n) = \tfrac{1}{4}\pi n \left[\{(\log n + 2\gamma - 1)\, b_0 - 2b_1\} F_0(n) - 2b_0 F_1(n) - 4b_0 \log 2\, E_0'(n)\right] + O(n). \tag{5}$$

THEOREM II. *Let n be a positive integer, and let $\nu(n)$ be the number of solutions of the equation*

$$\sum_{r=1}^{4} x_r^2 + yz = n$$

in integers x_r $(r = 1,\ 2,\ 3,\ 4)$, y, and z, with $y > 0$, $z > 0$.

Let

$$\sigma_s(n) = \sum_d d^s, \quad \sigma_s'(n) = \sum_d d^s \log d, \tag{6}$$

where d runs through the positive divisors of n; let $n = 2^a n_1$, where n_1 is odd; let γ denote Euler's constant, $c_0 = [\zeta(3)]^{-1}$, and c_1 the derivative of $[\zeta(s)]^{-1}$ for $s = 3$.

Then

$$\nu(n) = \tfrac{4}{7}\pi^2 n^2 \left[\{(\log n + 2\gamma - \tfrac{3}{2} - \tfrac{2}{7}\log 2)\, c_0 + 2c_1\}\, \sigma_{-2}(n) - 2c_0 \sigma_{-2}'(n)\right] + O(n^{\frac{3}{4}} \log n) \quad \text{if } n \text{ is odd}; \tag{7}$$

$$\nu(n) = \tfrac{2}{7}\pi^2 n^2 \left[\{(\log n + 2\gamma - \tfrac{3}{2})\, c_0 + 2c_1\}\{11\sigma_{-2}(n_1) - 7\sigma_{-2}(n)\} - 2c_0\{11\sigma_{-2}'(n_1) - 7\sigma_{-2}'(n)\} - \tfrac{3}{7}c_0 \log 2\, \sigma_{-2}(n_1)\right] + O(n^{\frac{3}{4}} \log n)$$

if n is even.

2. *Notation.* $d, m, q, r, s, t, u, v, y$, and z are positive integers, $d(m)$ is the number of divisors of m, and $\mu(q)$ is Möbius's function. I use Estermann's notation for sums, in which certain of the conditions of summation are replaced by numbers. These are defined here as follows:

1. $rs+yz=n$;
2. $r\equiv 1 \pmod 4$;
3. $r\equiv 3 \pmod 4$;
4. $y<z$;
5. $r<s$;
6. $r>s$;
7. $r^2+y^2<n$;
8. $u^2+v^2\leqslant nd^{-2}$;
9. $(u, v)=1$;
10. $vd\equiv 1 \pmod 4$;
11. $vd\equiv 3 \pmod 4$;
12. $r^2+s^2\leqslant nd^{-2}q^{-2}$;
13. $q\leqslant d^{-1}\sqrt{(\frac{1}{2}n)}$;
14. $r^2+s^2\leqslant a$;
15. $r\leqslant\sqrt{(a-1)}$;
16. $r\leqslant\sqrt{(\frac{1}{2}a)}$;
17. $r>\sqrt{(\frac{1}{2}a)}$;
18. $s\leqslant\sqrt{(a-1)}$;
19. $s^2+y^2<n$;
20. $y\leqslant\sqrt{n}, \quad s\leqslant n$;
21. $u\leqslant\sqrt{nd^{-1}}$;
22. $v\leqslant nd^{-1}$;
23. $r\leqslant\sqrt{nd^{-1}q^{-1}}$;
24. $s\leqslant nd^{-1}q^{-1}$.

3. *Proof of Theorem I.* It is known that

$$\sum_{\substack{x_1, x_2 \\ x_1^2+x_2^2=m}} 1 = 4 \sum_{\substack{r, s \\ rs=m \\ r\equiv 1 \pmod 4}} 1 - 4 \sum_{\substack{r, s \\ rs=m \\ r\equiv 3 \pmod 4}} 1,$$

and hence we can express $\nu(n)$ in the following form:

$$\begin{aligned}
(8) \qquad \nu(n) &= \sum_{\substack{m, t \\ m+t=n}} \sum_{\substack{x_1, x_2 \\ x_1^2+x_2^2=m}} 1 \sum_{\substack{y, z \\ yz=t}} 1 + \sum_{\substack{y, z \\ yz=n}} 1 \\
&= 4 \sum_{\substack{r, s, y, z \\ 1, 2}} 1 - 4 \sum_{\substack{r, s, y, z \\ 1, 3}} 1 + O(n) \\
&= 8 \sum_{\substack{r, s, y, z \\ 1, 2, 4}} 1 - 8 \sum_{\substack{r, s, y, z \\ 1, 3, 4}} 1 + O(n) + O\Big(\sum_{\substack{r, s, y \\ rs+y^2=n}} 1 \Big) \\
&= 8 \sum_{\substack{r, s, y, z \\ 1, 2, 4}} 1 - 8 \sum_{\substack{r, s, y, z \\ 1, 3, 4}} 1 + O(n).
\end{aligned}$$

Let

$$(9) \qquad Q_1 = \sum_{\substack{r, s, y, z \\ 1, 2, 4, 5}} 1;$$

$$(10) \qquad Q_2 = \sum_{\substack{r, s, y, z \\ 1, 2, 4, 6}} 1;$$

$$(11) \qquad Q_3 = \sum_{\substack{r, s, y, z \\ 1, 3, 4, 5}} 1;$$

$$(12) \qquad Q_4 = \sum_{\substack{r, s, y, z \\ 1, 3, 4, 6}} 1.$$

On expressing the sums occurring in (8) in terms of the Q's, we obtain

$$(13) \qquad \nu(n) = 8(Q_1+Q_2-Q_3-Q_4)+O(n).$$

3.1. In this paragraph I obtain an asymptotic formula for (Q_1-Q_3). First we may express Q_1, defined by (9), in the form

$$Q_1 = \sum_{\substack{r, y \\ 2, 7}} \sum_{\substack{s, z \\ 1, 4, 5}} 1.$$

On putting $(r, y) = d$, $y = ud$, $r = vd$, and

$$(14) \qquad x = u^{-1} v^{-1} d^{-1} (n - u^2 d^2 - v^2 d^2),$$

we easily see that

$$(15)\qquad \sum_{1,4,5}^{s,z} 1 = \begin{cases} 0 & \text{if } d \nmid n; \\ x+O(1) & \text{if } d \mid n \text{ and } u^2+v^2 \leqslant nd^{-2}. \end{cases}$$

Hence

$$(16)\qquad Q_1 = \sum_{d \mid n,\ 2 \nmid d}^{d} \sum_{8,9,10}^{u,v} x+O(n).$$

We perform a similar calculation for Q_3, defined by (11), and obtain

$$(17)\qquad Q_3 = \sum_{d \mid n,\ 2 \nmid d}^{d} \sum_{8,9,11}^{u,v} x+O(n).$$

It follows from (16) and (17) that

$$(18)\qquad Q_1-Q_3 = \sum_{d \mid n,\ 2 \nmid d}^{d} \sum_{2 \nmid v,\ 8,9}^{u,v} (-1)^{\frac{1}{2}(vd-1)} x+O(n)$$

$$= \sum_{d \mid n,\ 2 \nmid d}^{d} (-1)^{\frac{1}{2}(d-1)} f_1+O(n),$$

where

$$(19)\qquad f_1 = \sum_{2 \nmid v,\ 8,9}^{u,v} (-1)^{\frac{1}{2}(v-1)} x.$$

If $\mu(q)$ is Möbius's function, it is known that

$$(20)\qquad \sum_{qr=u,\ qs=v}^{q,r,s} \mu(q) = \begin{cases} 1 & \text{if } (u, v) = 1; \\ 0 & \text{if } (u, v) > 1; \end{cases}$$

and hence, by (19) and (14), we have

$$(21)\qquad f_1 = \sum_{2 \nmid qs,\ 12}^{r,s,q} (-1)^{\frac{1}{2}(qs-1)} q^{-2} \mu(q) r^{-1} s^{-1} d^{-1} (n-r^2 q^2 d^2-s^2 q^2 d^2)$$

$$= \sum_{2 \nmid q,\ 13}^{q} \Big\{ (-1)^{\frac{1}{2}(q-1)} q^{-2} \mu(q) \times \sum_{2 \nmid s,\ 12}^{r,s} (-1)^{\frac{1}{2}(s-1)} r^{-1} s^{-1} d^{-1} (n-r^2 q^2 d^2-s^2 q^2 d^2) \Big\}.$$

Now, if $a \geqslant 2$,

$$\sum_{2 \nmid s,\ 14}^{r,s} (-1)^{\frac{1}{2}(s-1)} r^{-1} s^{-1} = \sum_{15}^{r} r^{-1} \sum_{2 \nmid s,\ 14}^{s} (-1)^{\frac{1}{2}(s-1)} s^{-1}$$

$$= \sum_{15}^{r} r^{-1} [\tfrac{1}{4}\pi + O\{(a-r^2)^{-\frac{1}{2}}\}].$$

To estimate the error term in the last formula, we put

$$(22)\quad \sum_{15}^{r} r^{-1}(a-r^2)^{-\frac{1}{2}} \leqslant (\tfrac{1}{2}a)^{-\frac{1}{2}} \sum_{16}^{r} r^{-1} + (\tfrac{1}{2}a)^{-\frac{1}{2}} \sum_{15,\,17}^{r} (\sqrt{a}+r)^{-\frac{1}{2}} (\sqrt{a}-r)^{-\frac{1}{2}}$$

$$= O(a^{-\frac{1}{2}} \log a),$$

and hence

$$(23)\quad \sum_{2\,\dagger\, r+s,\,14}^{r,\,s} (-1)^{\frac{1}{2}(s-1)} r^{-1} s^{-1} = \tfrac{1}{4}\pi(\tfrac{1}{2}\log a + \gamma) + O(a^{-\frac{1}{2}} \log a).$$

Next we have

$$(24)\quad \sum_{2\,\dagger\, r+s,\,14}^{r,\,s} (-1)^{\frac{1}{2}(s-1)} r^{-1} s = \sum_{15}^{r} r^{-1} \sum_{2\,\dagger\, s,\,14}^{s} (-1)^{\frac{1}{2}(s-1)} s$$

$$= \sum_{15}^{r} r^{-1} [O(a^{\frac{1}{2}})] = O(a^{\frac{1}{2}} \log a).$$

Finally,

$$(25)\quad \sum_{2\,\dagger\, r+s,\,14}^{r,\,s} (-1)^{\frac{1}{2}(s-1)} r s^{-1} = \sum_{2\,\dagger\, s,\,18}^{s} (-1)^{\frac{1}{2}(s-1)} s^{-1} \sum_{14}^{r} r$$

$$= \sum_{2\,\dagger\, s,\,18}^{s} (-1)^{\frac{1}{2}(s-1)} s^{-1} [\tfrac{1}{2}(a-s^2) + O(a^{\frac{1}{2}})]$$

$$= \tfrac{1}{8}\pi a + O(a^{\frac{1}{2}} \log a).$$

We apply the formulae (23), (24), and (25) to estimate the inner sums occurring on the right-hand side of equation (21). This gives

$$f_1 = \tfrac{1}{8}\pi n d^{-1} \sum_{2\,\dagger\, q,\,13}^{q} (-1)^{\frac{1}{2}(q-1)} q^{-2} \mu(q) \{\log(nd^{-2}q^{-2}) + 2\gamma - 1\} + O(n^{\frac{1}{2}} \log^2 n).$$

The error term introduced by omitting the condition $q \leqslant d^{-1}\sqrt{(\frac{1}{2}n)}$ from the conditions of summation of this series is clearly

$$O(nd^{-1} \sum_{q > d^{-1}\sqrt{(\frac{1}{2}n)}}^{q} q^{-2} \log q) = O(n^{\frac{1}{2}} \log n),$$

and hence

$$(26)\quad f_1 = \tfrac{1}{8}\pi n d^{-1} \sum_{2\,\dagger\, q}^{q} q^{-2} \mu(q) (-1)^{\frac{1}{2}(q-1)} \{\log(nd^{-2}q^{-2}) + 2\gamma - 1\} + O(n^{\frac{1}{2}} \log^2 n)$$

$$= \tfrac{1}{8}\pi n d^{-1} [\{\log(nd^{-2}) + 2\gamma - 1\} b_0 - 2b_1] + O(n^{\frac{1}{2}} \log^2 n),$$

where b_0 and b_1 have been defined in the introduction.

Equations (18) and (26) lead to the following asymptotic formula for Q_1-Q_3:

$$Q_1-Q_3=\tfrac{1}{8}\pi n \sum_{\substack{d \\ d\ n,\ 2\nmid d}} (-1)^{\frac{1}{2}(d-1)}d^{-1}\left[\{\log(nd^{-2})+2\gamma-1\}b_0-2b_1\right]+O(n).$$

Using (1), we may write this equation in the form

$$(27)\qquad Q_1-Q_3=\tfrac{1}{8}\pi n\left[\{(\log n+2\gamma-1)b_0-2b_1\}E_0(n)-2b_0E_1(n)\right]+O(n).$$

3.2. We now find an asymptotic formula for Q_2-Q_4 by a method similar to that used for Q_1-Q_3 in the last paragraph.

By (10) and (12),

$$(28)\qquad Q_2=\sum_{\substack{s,y \\ 19}}\ \sum_{\substack{r,z \\ 1,2,4,6}} 1;\quad Q_4=\sum_{\substack{s,y \\ 19}}\ \sum_{\substack{r,z \\ 1,3,4,6}} 1.$$

Let $(s,y)=d$; $y=ud$; $s=vd$. Then a simple calculation shows that, if $d\mid n$ and $u^2+v^2\leqslant nd^{-2}$,

$$\sum_{\substack{r,z \\ 1,2,4,6}} 1=\begin{cases}\frac{1}{4}x+O(1) & \text{if } 2\nmid u;\\ \frac{1}{2}x+O(1) & \text{if } u\equiv 2 \pmod 4 \text{ and } 2d\nmid n;\\ x+O(1) & \text{if } 4\mid u,\ 2d\nmid n, \text{ and } v\equiv n/d \pmod 4;\\ 0 & \text{otherwise,}\end{cases}$$

and

$$\sum_{\substack{r,z \\ 1,3,4,6}} 1=\begin{cases}\frac{1}{4}x+O(1) & \text{if } 2\nmid u;\\ \frac{1}{2}x+O(1) & \text{if } u\equiv 2 \pmod 4 \text{ and } 2d\nmid n;\\ x+O(1) & \text{if } 4\mid u,\ 2d\nmid n, \text{ and } v\equiv n/d+2 \pmod 4;\\ 0 & \text{otherwise,}\end{cases}$$

where x has been defined in equation (14); if $d\nmid n$, both of these sums are empty. It follows from (28) that

$$(29)\qquad Q_2-Q_4=\sum_{\substack{d \\ d\mid n,\ 2d\nmid n}}\ \sum_{\substack{u,v \\ 4\mid u,\ 2\nmid v,\ 8,9}} (-1)^{\frac{1}{2}(n/d-v)}x+O(n)$$

$$=\sum_{\substack{d \\ d,n,\ 2d\nmid n}} (-1)^{\frac{1}{2}(n/d-1)}f_2+O(n),$$

where

$$(30)\quad f_2 = \sum_{4\mid u,\ 2\nmid v,\ 8,\ 9} (-1)^{\frac{1}{2}(v-1)} u^{-1} v^{-1} d^{-1}(n-u^2d^2-v^2d^2)$$

$$= \sum_{4\mid qr,\ 2\nmid qs,\ 12} (-1)^{\frac{1}{2}(qs-1)} q^{-2}\mu(q) r^{-1} s^{-1} d^{-1}(n-r^2q^2d^2-s^2q^2d^2)$$

$$= \sum_{2\nmid q,\ 13} \Big\{ (-1)^{\frac{1}{2}(q-1)}\mu(q)\, q^{-2} \times \sum_{4\mid r,\ 2\nmid s,\ 12} (-1)^{\frac{1}{2}(s-1)} r^{-1} s^{-1} d^{-1}(n-r^2q^2d^2-s^2q^2d^2)\Big\}.$$

Now

$$\sum_{4\mid r,\ 2\nmid s,\ 14} (-1)^{\frac{1}{2}(s-1)} r^{-1} s^{-1} = \sum_{4\mid r,\ 15} r^{-1} \sum_{2\nmid s,\ 14} (-1)^{\frac{1}{2}(s-1)} s^{-1}$$

$$= \sum_{4\nmid r,\ 15} r^{-1}\left[\tfrac{1}{4}\pi + O\{(a-r^2)^{-\frac{1}{2}}\}\right],$$

and, with (22), this gives

$$(31)\quad \sum_{4\mid r,\ 2\nmid s,\ 14} (-1)^{\frac{1}{2}(s-1)} r^{-1} s^{-1} = \tfrac{1}{16}\pi\left[\tfrac{1}{2}\log(\tfrac{1}{16}a)+\gamma\right]+O(a^{-\frac{1}{2}}\log a).$$

Next we have

$$(32)\quad \sum_{4\mid r,\ 2\nmid s,\ 14} (-1)^{\frac{1}{2}(s-1)} r^{-1} s = \sum_{4\mid r,\ 15} r^{-1} \sum_{2\nmid s,\ 14} (-1)^{\frac{1}{2}(s-1)} s$$

$$= \sum_{4\mid r,\ 15} r^{-1}\left[O(a^{\frac{1}{2}})\right] = O(a^{\frac{1}{2}}\log a).$$

Finally,

$$(33)\quad \sum_{4\mid r,\ 2\nmid s,\ 14} (-1)^{\frac{1}{2}(s-1)} r s^{-1} = \sum_{2\nmid s,\ 18} (-1)^{\frac{1}{2}(s-1)} s^{-1} \sum_{4\mid r,\ 14} r$$

$$= \sum_{2\nmid s,\ 18} (-1)^{\frac{1}{2}(s-1)} s^{-1}\left[\tfrac{1}{8}(a-s^2)+O(a^{\frac{1}{2}})\right]$$

$$= \tfrac{1}{32}\pi a + O(a^{\frac{1}{2}}\log a).$$

On applying the results of (31), (32), and (33) in (30), we obtain

$$f_2 = \tfrac{1}{32}\pi n d^{-1} \sum_{2\nmid q,\ 13} (-1)^{\frac{1}{2}(q-1)} q^{-2}\mu(q)\left\{\log(\tfrac{1}{16}nd^{-2}q^{-2})+2\gamma-1\right\}+O(n^{\frac{1}{2}}\log^2 n),$$

and in this formula we may replace the finite series by the infinite series

without affecting the error term as in (26). Thus

$$(34)\quad f_2 = \tfrac{1}{3}\tfrac{1}{2}\pi n d^{-1} \sum_{\substack{q \\ 2\nmid q}} (-1)^{\frac{1}{2}(q-1)} q^{-2} \mu(q) \{\log(\tfrac{1}{16} n d^{-2} q^{-2}) + 2\gamma - 1\} + O(n^{\frac{1}{2}} \log^2 n)$$

$$= \tfrac{1}{3}\tfrac{1}{2}\pi n d^{-1} [\{\log(\tfrac{1}{16} n d^{-2}) + 2\gamma - 1\} b_0 - 2b_1] + O(n^{\frac{1}{2}} \log^2 n).$$

On combining (29) with (34) and using (2), we have the following asymptotic formula for $(Q_2 - Q_4)$:

$$(35)\quad Q_2 - Q_4 = \tfrac{1}{3}\tfrac{1}{2}\pi n \sum_{\substack{d \\ d\,\mid\, n,\ 2d\,\nmid\, n}} (-1)^{\frac{1}{2}(n/d-1)} d^{-1} [\{\log(\tfrac{1}{16} n d^{-2}) + 2\gamma - 1\} b_0 - 2b_1] + O(n)$$

$$= \tfrac{1}{3}\tfrac{1}{2}\pi n [\{(\log n + 2\gamma - 1 - 4\log 2) b_0 - 2b_1\} E_0'(n) - 2b_0 E_1'(n)] + O(n).$$

The result (5) follows from the equations (13), (27), (35), and (3).

4. *Proof of Theorem II.* It is known that

$$\sum_{\substack{x_1, x_2, x_3, x_4 \\ x_1^2 + x_2^2 + x_3^2 + x_4^2 = m}} 1 = \begin{cases} 8 \sum_{\substack{r, s \\ rs = m}} r & \text{if } 2 \nmid m; \\ 24 \sum_{\substack{r, s \\ rs = m,\ 2\,\nmid\, r}} r & \text{if } 2 \mid m. \end{cases}$$

Hence we can express $\nu(n)$ in the form

$$(36)\quad \nu(n) = \sum_{\substack{m, t \\ m+t=n}} \sum_{\substack{x_1, x_2, x_3, x_4 \\ x_1^2 + x_2^2 + x_3^2 + x_4^2 = m}} 1 \sum_{\substack{y, z \\ yz = t}} 1 + \sum_{\substack{y, z \\ yz = n}} 1$$

$$= 8 \sum_{\substack{r, s, y, z \\ 2\,\nmid\, rs,\ 1}} r + 24 \sum_{\substack{r, s, y, z \\ 2\,\mid\, rs,\ 2\,\nmid\, r,\ 1}} r + O(n)$$

$$= 8 \sum_{\substack{r, s, y, z \\ 2\,\nmid\, r,\ 1}} r + 16 \sum_{\substack{r, s, y, z \\ 2\,\mid\, rs,\ 2\,\nmid\, r,\ 1}} r + O(n)$$

$$= 16 Q_1 + 32 Q_2 + O(n) + O\Big(\sum_{\substack{r, s, y \\ rs + y^2 = n}} r\Big),$$

where

$$(37)\quad Q_1 = \sum_{\substack{r, s, y, z \\ 2\,\nmid\, r,\ 1,\ 4}} r, \qquad Q_2 = \sum_{\substack{r, s, y, z \\ 2\,\mid\, rs,\ 2\,\nmid\, r,\ 1,\ 4}} r.$$

The error term in (36) is $O(n^{\frac{3}{4}} \log n)$. For

$$\sum_{\substack{r,\,s\\ s+y^2=n}} \sum_{y} r = \sum_{y \leqslant \sqrt{n}} \sum_{\substack{r,\,s\\ rs=n-y^2}} r,$$

and

$$\sum_{\substack{r,\,s\\ rs=m}} r = \sum_{s \mid m} (m/s) \leqslant m \sum_{s=1}^{m} s^{-1} = O(m \log m),$$

whence it follows that

$$\sum_{\substack{r,\,s,\,y\\ rs+y^2=n}} r = O(n \log n \sum_{y \leqslant \sqrt{n}} 1) = O(n^{\frac{3}{4}} \log n).$$

Thus we have

$$\nu(n) = 16Q_1 + 32Q_2 + O(n^{\frac{3}{4}} \log n). \tag{38}$$

4.1. We may express Q_1 and Q_2, defined by (37), in the form

$$Q_1 = \sum_{\substack{s,\,y\\ 2 \nmid s,\,20}} \sum_{\substack{r,\,z\\ 2 \nmid r,\,1,\,4}} r; \quad Q_2 = \sum_{\substack{s,\,y\\ 2 \mid s,\,20}} \sum_{\substack{r,\,z\\ 2 \nmid r,\,1,\,4}} r. \tag{39}$$

Let $(s, y) = d$, $y = ud$, $s = vd$, and

$$x = u^{-1} v^{-1} d^{-1} (n - u^2 d^2). \tag{40}$$

Then, in order that the equation $rs + yz = n$ may possess solutions in positive integers r and z with $z > y$, we must have $d \mid n$ and $u^2 \leqslant nd^{-2}$, when the number of solutions is $x + O(1)$. If these conditions are satisfied, a simple calculation shows that

$$\sum_{\substack{r,\,z\\ 2 \nmid r,\,1,\,4}} 1 = \begin{cases} \frac{1}{2}x + O(1) & \text{if } 2 \nmid u; \\ x + O(1) & \text{if } 2 \mid u \text{ and } 2d \nmid n; \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\sum_{\substack{r,\,z\\ 2 \nmid r,\,1,\,4}} r = \begin{cases} \frac{1}{4}ux^2 + O(ns^{-1}) & \text{if } 2 \nmid u; \\ \frac{1}{2}ux^2 + O(ns^{-1}) & \text{if } 2 \mid u \text{ and } 2d \nmid n; \\ 0 & \text{otherwise.} \end{cases}$$

Hence, by (39), we have

$$(41)\qquad Q_1 = \sum_{\substack{d \\ d\,|\,n,\ 2d\,\nmid\, n}} \sum_{\substack{u,v \\ 2\,\nmid\, u,\ 9,\ 21,\ 22}} \tfrac{1}{2}ux^2 + \sum_{\substack{d \\ d\,|\,n}} \sum_{\substack{u,v \\ 2\,\nmid\, u,\ 9,\ 21,\ 22}} \tfrac{1}{4}ux^2 + O\Big(n \sum_{\substack{s,y \\ 20}} s^{-1}\Big)$$

$$= \sum_{\substack{d \\ d\,|\,n,\ 2d\,\nmid\, n}} f_1 + \sum_{\substack{d \\ d\,|\,n}} f_2 + O(n^{\frac{3}{4}} \log n),$$

where

$$(42)\qquad f_1 = \sum_{\substack{u,v \\ 2\,|\,u,\ 9,\ 21,\ 22}} \tfrac{1}{2}ux^2,$$

and

$$(43)\qquad f_2 = \sum_{\substack{u,v \\ 2\,\nmid\, u,\ 9,\ 21,\ 22}} \tfrac{1}{4}ux^2.$$

We deal with Q_2 in the same way. By (39),

$$(44)\qquad Q_2 = \sum_{\substack{d \\ d\,|\,n,\ 2d\,\nmid\, n,\ 2\,|\,d}} \sum_{\substack{u,v \\ 2\,|\,u,\ 9,\ 21,\ 22}} \tfrac{1}{2}ux^2 + \sum_{\substack{d \\ d\,|\,n}} \sum_{\substack{u,v \\ 2\,\nmid\, u,\ 2\,|\,v,\ d \\ 9,\ 21,\ 22}} \tfrac{1}{4}ux^2 + O(n^{\frac{3}{4}} \log n)$$

$$= \sum_{\substack{d \\ d\,|\,n,\ 2d\,\nmid\, n,\ 2\,\nmid\, d}} \sum_{\substack{u,v \\ 2\,|\,u,\ 9,\ 21,\ 22}} \tfrac{1}{2}ux^2 + \sum_{\substack{d \\ d\,|\,n,\ 2\,|\,d}} \sum_{\substack{u,v \\ 2\,\nmid\, u,\ 9,\ 21,\ 22}} \tfrac{1}{4}ux^2$$

$$+ \sum_{\substack{d \\ d\,|\,n,\ 2\,\nmid\, d}} \sum_{\substack{u,v \\ 2\,|\,v,\ 9,\ 21,\ 22}} \tfrac{1}{4}ux^2 + O(n^{\frac{3}{4}} \log n)$$

$$= \sum_{\substack{d \\ d\,|\,n,\ 2d\,\nmid\, n,\ 2\,|\,d}} f_1 + \sum_{\substack{d \\ d\,|\,n,\ 2\,|\,d}} f_2 + \sum_{\substack{d \\ d\,|\,n,\ 2\,\nmid\, d}} f_3 + O(n^{\frac{3}{4}} \log n),$$

where f_1 and f_2 have been defined in equations (42) and (43), and f_3 is given by

$$(45)\qquad f_3 = \sum_{\substack{u,v \\ 2\,|\,v,\ 9,\ 21,\ 22}} \tfrac{1}{4}ux^2.$$

4.2. In this section we find asymptotic formulae for f_1, f_2, and f_3.

First consider f_1, defined by (42). On using (20) to eliminate the condition $(u, v) = 1$ from the conditions of summation, and substituting for x from (40), we obtain

$$(46)\qquad f_1 = \sum_{\substack{r,s,q \\ 2\,|\,qr,\ 2\,\nmid\, qs,\ 23,\ 24}} \tfrac{1}{2}\mu(q)\, q^{-3} s^{-2} \{n^2 d^{-2} r^{-1} + d^2 q^4 r^3 - 2nq^2 r\}$$

$$= \sum_{\substack{q \\ 2\,\nmid\, q}} \tfrac{1}{2}\mu(q)\, q^{-3} \sum_{\substack{r \\ 2\,r,\ 23}} \{n^2 d^{-2} r^{-1} + d^2 q^4 r^3 - 2nq^2 r\} \sum_{\substack{s \\ 2\,\nmid\, s,\ 24}} s^{-2}.$$

The sums over r and s occurring in (46) can be estimated quite simply. We have

$$\sum_{2\nmid s,\,24} s^{-2} = \tfrac{1}{8}\pi^2 + O(n^{-1}dq);$$

$$\sum_{2\nmid r,\,23} r^{-1} = \tfrac{1}{4}\{\log(\tfrac{1}{4}nd^{-2}q^{-2}) + 2\gamma\} + O(n^{-\frac{1}{2}}dq);$$

$$\sum_{2\nmid r,\,23} r = \tfrac{1}{4}nd^{-2}q^{-2} + O(n^{\frac{1}{2}}d^{-1}q^{-1});$$

$$\sum_{2\nmid r,\,23} r^3 = \tfrac{1}{8}n^2d^{-4}q^{-4} + O(n^{\frac{3}{2}}d^{-3}q^{-3}).$$

It follows that

$$\sum_{2\nmid r,\,23} \{n^2d^{-2}r^{-1} + d^2q^4r^3 - 2nq^2r\} \sum_{2\nmid s,\,24} s^{-2}$$

$$= [\tfrac{1}{4}n^2d^{-2}\{\log(\tfrac{1}{4}nd^{-2}q^{-2}) + 2\gamma - \tfrac{3}{2}\} + O(n^{\frac{3}{2}}d^{-1}q)][\tfrac{1}{8}\pi^2 + O(n^{-1}dq)],$$

and hence, if $q < nd^{-1}$,

$$\sum_{2\nmid r,\,23} \{n^2d^{-2}r^{-1} + d^2q^4r^3 - 2nq^2r\} \sum_{2\nmid s,\,24} s^{-2}$$

$$= \tfrac{1}{32}\pi^2n^2d^{-2}\{\log(\tfrac{1}{4}nd^{-2}q^{-2}) + 2\gamma - \tfrac{3}{2}\} + O(n^{\frac{3}{2}}d^{-1}q) + O(nd^{-1}q\log n).$$

This formula also holds if $q > nd^{-1}$, for then the left-hand side is zero and the first term on the right-hand side is of order

$$O(n^2d^{-2}\{nd^{-2}q^{-2}\}^{-\frac{1}{2}}) = O(n^{\frac{3}{2}}d^{-1}q).$$

Hence, on putting $\log n = O(n^{\frac{1}{2}})$ and using (46), we obtain

$$f_1 = \tfrac{1}{64}\pi^2n^2d^{-2} \sum_{2\nmid q} \mu(q)q^{-3}\{\log(\tfrac{1}{4}nd^{-2}q^{-2}) + 2\gamma - \tfrac{3}{2}\} + O(n^{\frac{3}{2}}d^{-1}).$$

Now

$$\sum_{2\nmid q} \mu(q)q^{-3} = \tfrac{8}{7}c_0; \qquad \sum_{2\nmid q} \mu(q)q^{-3}\log q = \tfrac{8}{49}c_0\log 2 - \tfrac{8}{7}c_1$$

(with the definitions of c_0 and c_1 given in the introduction), so that we may write f_1 in the form

$$(47) \quad f_1 = \tfrac{1}{64}\pi^2n^2d^{-2}[\{\log(\tfrac{1}{4}nd^{-2}) + 2\gamma - \tfrac{3}{2}\}\tfrac{8}{7}c_0 - \tfrac{16}{49}c_0\log 2 + \tfrac{16}{7}c_1] + O(n^{\frac{3}{2}}d^{-1}).$$

The investigation of f_2 and f_3 is similar to that of f_1. We have, by (43),

$$f_2 = \sum_{2\nmid q}^{q} \tfrac{1}{4}\mu(q)\,q^{-3} \sum_{2\nmid r,\,23}^{r} \{n^2 d^{-2} r^{-1} + d^2 q^4 r^3 - 2nq^2 r\} \sum_{24}^{s} s^{-2}. \tag{48}$$

The evaluation of the inner sums in this formula yields the following results:

$$\sum_{24}^{s} s^{-2} = \tfrac{1}{6}\pi^2 + O(n^{-1}dq);$$

$$\sum_{2\nmid r,\,23}^{r} r^{-1} = \tfrac{1}{4}\{\log(4nd^{-2}q^{-2}) + 2\gamma\} + O(n^{-\frac{1}{3}}dq); \tag{49}$$

$$\sum_{2\nmid r,\,23}^{r} r = \tfrac{1}{4}nd^{-2}q^{-2} + O(n^{\frac{1}{3}}d^{-1}q^{-1}); \tag{50}$$

$$\sum_{2\nmid r,\,23}^{r} r^3 = \tfrac{1}{8}n^2 d^{-4} q^{-4} + O(n^{\frac{2}{3}}d^{-3}q^{-3}). \tag{51}$$

Hence, by (48),

$$f_2 = \tfrac{1}{96}\pi^2 n^2 d^{-2}\,[\{\log(4nd^{-2}) + 2\gamma - \tfrac{3}{2}\}\tfrac{8}{7}c_0 - \tfrac{16}{49}c_0 \log 2 + \tfrac{16}{7}c_1] + O(n^{\frac{2}{3}}d^{-1}). \tag{52}$$

Finally we have, by (45),

$$f_3 = \sum_{2\nmid q}^{q} \tfrac{1}{4}\mu(q)\,q^{-3} \sum_{2\nmid r,\,23}^{r} \{n^2 d^{-2} r^{-1} + d^2 q^4 r^3 - 2nq^2 r\} \sum_{2\,s,\,24}^{s} s^{-2},$$

and, on using (49), (50), and (51) with the formula

$$\sum_{2\,s,\,24}^{s} s^{-2} = \tfrac{1}{24}\pi^2 + O(n^{-1}dq),$$

we obtain

$$f_3 = \tfrac{1}{384}\pi^2 n^2 d^{-2}\,[\{\log(4nd^{-2}) + 2\gamma - \tfrac{3}{2}\}\tfrac{8}{7}c_0 - \tfrac{16}{49}c_0 \log 2 + \tfrac{16}{7}c_1] + O(n^{\frac{2}{3}}d^{-1}). \tag{53}$$

4.3. Now consider the value of Q_1 given by (41). Substitute in this equation for f_1 and f_2 from equations (47) and (52); then, when the sum is taken over d, the error term will give rise to an error term in (41) of order

$$O\Big(n^{\frac{2}{3}} \sum_{d=1}^{n} d^{-1}\Big) = O(n^{\frac{2}{3}} \log n).$$

Thus, on putting

$$a = (\log n + 2\gamma - \tfrac{3}{2})\tfrac{8}{7}c_0 + \tfrac{96}{49}c_0 \log 2 + \tfrac{16}{7}c_1, \tag{54}$$

we may write

$$(55)\quad Q_1 = \tfrac{1}{64}\pi^2 n^2 (a - \tfrac{32}{7} c_0 \log 2) \sum_{d \mid n,\ 2d \nmid n} d^{-2} - \tfrac{1}{28}\pi^2 n^2 c_0 \sum_{d \mid n,\ 2d \nmid n} d^{-2} \log d + \tfrac{1}{96}\pi^2 n^2 a \sum_{d \mid n} d^{-2} - \tfrac{1}{42}\pi^2 n^2 c_0 \sum_{d \mid n} d^{-2} \log d + O(n^{\frac{3}{2}} \log n).$$

The corresponding result for Q_2 is obtained from formulae (44), (47), (52), and (53).

$$(56)\quad Q_2 = \tfrac{1}{64}\pi^2 n^2 (a - \tfrac{32}{7} c_0 \log 2) \sum_{d \mid n,\ 2d \nmid n,\ 2 \mid d} d^{-2} - \tfrac{1}{28}\pi^2 n^2 c_0 \sum_{d \mid n,\ 2d \nmid n,\ 2 \mid d} d^{-2} \log d + \tfrac{1}{96}\pi^2 n^2 a \sum_{d \mid n,\ 2 \mid d} d^{-2} - \tfrac{1}{42}\pi^2 n^2 c_0 \sum_{d \mid n,\ 2 \mid d} d^{-2} \log d + \tfrac{1}{384}\pi^2 n^2 a \sum_{d \mid n,\ 2 \nmid d} d^{-2} - \tfrac{1}{168}\pi^2 n^2 c_0 \sum_{d \mid n,\ 2 \nmid d} d^{-2} \log d + O(n^{\frac{3}{2}} \log n).$$

It only remains to express the sums occurring in (55) and (56) in terms of the functions $\sigma_s(n)$ and $\sigma_s'(n)$ defined in the introduction by means of formula (6).

First suppose that n is odd. Then we have the following results:

$$\sum_{d \mid n,\ 2d \nmid n} d^s = \sum_{d \mid n,\ 2 \nmid d} d^s = \sum_{d \mid n} d^s = \sigma_s(n),$$

$$\sum_{d \mid n,\ 2d \nmid n,\ 2 \mid d} d^s = \sum_{d \mid n,\ 2 \mid d} d^s = 0.$$

On differentiating the above formulae with respect to s, we have corresponding results for the series $\Sigma d^s \log d$. Hence, from (55), (56), and (38), it follows that

$$(57)\quad \nu(n) = \tfrac{1}{2}\pi^2 n^2 [(a - \tfrac{16}{7} c_0 \log 2)\sigma_{-2}(n) - \tfrac{16}{7} c_0 \sigma'_{-2}(n)] + O(n^{\frac{3}{2}} \log n), \quad \text{if } n \text{ is odd.}$$

Next, if n is even, we easily prove the following results:

$$\sum_{d \mid n,\ 2d \nmid n} d^s = \sum_{d \mid n,\ 2d \nmid n,\ 2 \mid d} d^s = 2^{-s}\sigma_s(n_1) + (1 - 2^{-s})\sigma_s(n)$$

(with the definition of n_1 given in the introduction);

$$\sum_{d\ n,\ 2\ d} d^s = \sigma_s(n) - \sigma_s(n_1); \qquad \sum_{d\ n,\ 2\text{-}\text{-}d} d^s = \sigma_s(n_1).$$

On differentiating the above formulae with respect to s, we have

$$\sum_{d\ n,\ 2d\text{-}\text{-}n} d^s \log d = \sum_{d\ n,\ 2d\text{-}\text{-}n,\ 2\ d} d^s \log d$$

$$= \sigma_s'(n) + 2^{-s}\left[\sigma_s'(n_1) - \sigma_s'(n) - \log 2\,\{\sigma_s(n_1) - \sigma_s(n)\}\right];$$

$$\sum_{d\ n,\ 2\ d} d^s \log d = \sigma_s'(n) - \sigma_s'(n_1); \qquad \sum_{d\ n,\ 2\text{-}\text{-}d} d^s \log d = \sigma_s'(n_1).$$

Hence, from (38), (55), and (56), it follows by a simple calculation that

$$\begin{aligned}(58)\quad \nu(n) = {} & \tfrac{1}{4}\pi^2 n^2 \left[(a - \tfrac{96}{49} c_0 \log 2)\{11\sigma_{-2}(n_1) - 7\sigma_{-2}(n)\}\right. \\ & - \tfrac{16}{7} c_0 \{11\sigma'_{-2}(n_1) - 7\sigma'_{-2}(n)\} - \left.\tfrac{288}{49} c_0 (\log 2)\,\sigma_{-2}(n_1)\right] \\ & + O(n^{\frac{3}{4}} \log n) \quad \text{if } n \text{ is even.}\end{aligned}$$

On substituting the value of a given by (54) in equations (57) and (58), we have the required formula (7).

ON EPSTEIN'S ZETA-FUNCTION

By E. C. Titchmarsh.

[Received 5 July, 1932.—Read 10 November, 1932.]

1. In many problems in the analytic theory of numbers it is necessary to obtain inequalities satisfied by sums of the form

$$\sum_n e^{if(n)}, \tag{1}$$

where $f(x)$ is a real function. An important method of obtaining such inequalities is due to van der Corput*.

In the present paper we consider sums of the form†

$$\sum_m \sum_n e^{if(m,n)}, \tag{2}$$

where $f(x, y)$ is a real function of x and y. Observing that

$$\left| \sum_m \sum_n e^{if(m,n)} \right| \leqslant \sum_m \left| \sum_n e^{if(m,n)} \right|, \tag{3}$$

and that the inner sum on the right-hand side is of the form (1), we can clearly deduce from van der Corput's results inequalities satisfied by sums of the form (2). However, in using (3) we usually lose something of importance. Here we avoid (3) by dealing with (2) directly, and extending each step of van der Corput's argument to functions of two variables.

The method seems to be of general importance, but in applications there are considerable difficulties of detail. As a fairly straightforward illustration, we consider the problem of the order of the function

$$Z(s) = \sum_{m=-\infty}^{\infty} {\sum_{n=-\infty}^{\infty}}' \frac{1}{(m^2+an^2)^s} \tag{4}$$

* For an account of van der Corput's method and references, see E. C. Titchmarsh, *Quart. J. of Math.* (Oxford), 2 (1931), 161–173. We shall have to use some of the lemmas of this paper, and shall refer to them simply as Lemma 1, etc.

† The factor 2π, which occurred in the exponent in the previous paper, is here inappropriate, and is omitted.

on the line $\sigma = \Re(s) = \frac{1}{2}$. Here a is a positive constant, fixed throughout the argument, and the term $m = n = 0$ is omitted from the sum. This function is a particular case of Epstein's zeta-functions.

In the case $a = 1$, we have

$$Z(s) = 4\zeta(s)\, L(s),$$

where $\zeta(s)$ is Riemann's zeta-function, and

$$L(s) = 1^{-s} - 3^{-s} + 5^{-s} + \ldots.$$

It is known that $\zeta(\frac{1}{2}+it) = O(t^{\frac{1}{6}} \log t)$, and a similar result can clearly be obtained for $L(\frac{1}{2}+it)$. Hence, in this case,

$$Z(\tfrac{1}{2}+it) = O(t^{\frac{1}{3}} \log^2 t).$$

Similar considerations apply to other particular cases, but are not applicable in general. Using the method referred to above, however, we prove that, for any value of a,

$$Z(\tfrac{1}{2}+it) = O(t^{\frac{1}{3}} \log^3 t). \tag{5}$$

We first prove a number of lemmas. We then use them to prove theorems on finite sums of the form

$$\Sigma\Sigma(m^2 + an^2)^{-\frac{1}{2}-it}.$$

Lastly, we apply these theorems to the problem of the order of $Z(\frac{1}{2}+it)$.

Throughout the paper A denotes a positive absolute constant, not necessarily always the same one. For each n, A_n always denotes the same constant. The notation

$$f(\lambda, \mu, \ldots) = O\{\phi(\lambda, \mu, \ldots)\}$$

means that $|f| < A|\phi|$ if $\lambda, \mu, \ldots$ are sufficiently near to certain limits.

2. LEMMA a. *Let $a_{m,n}$ denote any numbers, real or complex, such that, if*

$$s_{m,n} = \sum_{\mu=1}^{m} \sum_{\nu=1}^{n} a_{\mu,\nu},$$

then

$$|s_{m,n}| \leqslant G \quad (1 \leqslant m \leqslant M,\ 1 \leqslant n \leqslant N).$$

Let $b_{m,n}$ denote real numbers, let $0 \leqslant b_{m,n} \leqslant H$, and suppose that each of the expressions

$$b_{m,n} - b_{m+1,n} - b_{m,n+1} + b_{m+1,n+1},$$

$$b_{m,n} - b_{m,n+1}, \quad b_{m,n} - b_{m+1,n}$$

is of constant sign for all values of m and n considered. Then

$$\left| \sum_{m=1}^{M} \sum_{n=1}^{N} a_{m,n} b_{m,n} \right| \leqslant 5GH.$$

This follows from the partial summation formula for double sums thus. Putting $s_{m,0} = 0$, $s_{0,n} = 0$, we have

$$\sum_{m=1}^{M} \sum_{n=1}^{N} a_{m,n} b_{m,n} = \sum_{m=1}^{M} \sum_{n=1}^{N} (s_{m,n} - s_{m,n-1} - s_{m-1,n} + s_{m-1,n-1}) b_{m,n}$$

$$= \sum_{m=1}^{M-1} \sum_{n=1}^{N-1} s_{m,n}(b_{m,n} - b_{m,n+1} - b_{m+1,n} + b_{m+1,n+1})$$

$$+ \sum_{m=1}^{M-1} s_{m,N}(b_{m,N} - b_{m+1,N}) + \sum_{n=1}^{N-1} s_{M,n}(b_{M,n} - b_{M,n+1}) + s_{M,N} b_{M,N},$$

and the modulus of this does not exceed

$$G \sum_{m=1}^{M-1} \sum_{n=1}^{N-1} |b_{m,n} - b_{m,n+1} - b_{m+1,n} + b_{m+1,n+1}|$$

$$+ G \sum_{m=1}^{M-1} |b_{m,N} - b_{m+1,N}| + G \sum_{n=1}^{N-1} |b_{M,n} - b_{M,n+1}| + G b_{M,N}$$

$$= G|b_{1,1} - b_{1,N} - b_{M,1} + b_{M,N}| + G|b_{1,N} - b_{M,N}|$$

$$+ G|b_{M,1} - b_{M,N}| + G b_{M,N} \leqslant 5GH.$$

In particular *the result holds if* $b_{m,n} = b(m, n)$, *where each of the expressions*

$$\frac{\partial b(x, y)}{\partial x}, \quad \frac{\partial b(x, y)}{\partial y}, \quad \frac{\partial^2 b(x, y)}{\partial x \partial y}$$

is of constant sign for all values of x and y considered. For

$$b_{m,n} - b_{m,n+1} - b_{m+1,n} + b_{m+1,n+1} = \int_m^{m+1} \int_n^{n+1} \frac{\partial^2 b(x, y)}{\partial x \partial y} \, dx \, dy,$$

$$b_{m,n} - b_{m,n+1} = -\int_n^{n+1} \frac{\partial b(x, y)}{\partial y} \, dy, \qquad b_{m,n} - b_{m+1,n} = -\int_m^{m+1} \frac{\partial b(x, y)}{\partial x} \, dx.$$

For example, $b_{m,n} = (m^2 + an^2)^{-\frac{1}{2}}$ satisfies these conditions for $m > 0$, $n > 0$, since

$$\frac{\partial}{\partial x} \frac{1}{(x^2 + ay^2)^{\frac{1}{2}}} = -\frac{x}{(x^2 + ay^2)^{\frac{3}{2}}} < 0, \qquad \frac{\partial}{\partial y} \frac{1}{(x^2 + ay^2)^{\frac{1}{2}}} = -\frac{ay}{(x^2 + ay^2)^{\frac{3}{2}}} < 0,$$

$$\frac{\partial^2}{\partial x \partial y} \frac{1}{(x^2 + ay^2)^{\frac{1}{2}}} = \frac{3axy}{(x^2 + ay^2)^{\frac{5}{2}}} > 0.$$

3. LEMMA β. *Let $f(x, y)$ be a real function of x and y, defined for $a \leqslant x \leqslant b$, $\alpha \leqslant y \leqslant \beta$; let*

$$S = \sum_{m=a}^{b} \sum_{n=\alpha}^{\beta} e^{if(m,n)},$$

$$S_1 = \sum_{m=a}^{b-\mu} \sum_{n=\alpha}^{\beta-\nu} e^{i[f(m+\mu, n+\nu)-f(m,n)]},$$

and

$$S_2 = \sum_{m=a}^{b-\mu} \sum_{n=\alpha+\nu}^{\beta} e^{i[f(m+\mu, n-\nu)-f(m,n)]}.$$

Let ρ be a positive integer not exceeding $b-a$, and ρ' a positive integer not exceeding $\beta-\alpha$. Then

$$|S| < A\frac{(b-a)(\beta-\alpha)}{(\rho\rho')^{\frac{1}{2}}} + A\left\{\frac{(b-a)(\beta-\alpha)}{\rho\rho'}\sum_{\mu=1}^{\rho-1}\sum_{\nu=0}^{\rho'-1}|S_1|\right\}^{\frac{1}{2}} + A\left\{\frac{(b-a)(\beta-\alpha)}{\rho\rho'}\sum_{\mu=0}^{\rho-1}\sum_{\nu=1}^{\rho'-1}|S_2|\right\}^{\frac{1}{2}}.$$

We have, as in Lemma 3,

$$S = \frac{1}{\rho\rho'}\sum_m\sum_n\sum_{\mu=0}^{\rho-1}\sum_{\nu=0}^{\rho'-1} e^{if(m+\mu, n+\nu)},$$

where $e^{if(m,n)}$ denotes 0 if (m, n) lies outside the region of summation of S, and where m and n vary over all values for which the (μ, ν)-sum is not zero. Hence

$$|S| \leqslant \frac{1}{\rho\rho'}\sum_m\sum_n\left|\sum_\mu\sum_\nu e^{if(m+\mu, n+\nu)}\right|$$

$$\leqslant \frac{1}{\rho\rho'}\left\{\sum_m\sum_n 1\sum_m\sum_n\left|\sum_\mu\sum_\nu e^{if(m+\mu, n+\nu)}\right|^2\right\}^{\frac{1}{2}}$$

$$< \frac{A}{\rho\rho'}(b-a)^{\frac{1}{2}}(\beta-\alpha)^{\frac{1}{2}}\left\{\sum_m\sum_n\left|\sum_\mu\sum_\nu e^{if(m+\mu, n+\nu)}\right|^2\right\}^{\frac{1}{2}},$$

since m varies over $O(b-a)$ values, and n over $O(\beta-\alpha)$ values.

Now

$$\left|\sum_\mu\sum_\nu e^{if(m+\mu, n+\nu)}\right|^2 = \sum_\mu\sum_\nu\sum_{\mu'}\sum_{\nu'} e^{i[f(m+\mu, n+\nu)-f(m+\mu', n+\nu')]}.$$

Here the summand is 1 if $\mu = \mu'$ and $\nu = \nu'$, *i.e.* for $\rho\rho'$ terms. On summing with respect to m and n, we get from these terms a sum which is $O\{(b-a)(\beta-\alpha)\rho\rho'\}$.

Consider next the terms for which $\mu > \mu'$, $\nu \geqslant \nu'$. In these terms

$$f(m+\mu, n+\nu)-f(m+\mu', n+\nu') = f(k+\kappa, l+\lambda)-f(k, l),$$

for given values of k, l, κ, and λ, if

$$\mu' = 0, 1, \ldots, \rho-\kappa-1; \quad \mu = \kappa, \kappa+1, \ldots, \rho-1;$$

$$\nu' = 0, 1, \ldots, \rho'-\lambda-1; \quad \nu = \lambda, \lambda+1, \ldots, \rho'-1;$$

with consequent values of m and n in each case, *i.e.* $(\rho-\kappa)(\rho'-\lambda)$ terms. These terms therefore give the sum

$$\sum_{\kappa=1}^{\rho-1}\sum_{\lambda=0}^{\rho'-1}(\rho-\kappa)(\rho'-\lambda)\sum_{k}\sum_{l}e^{i[f(k+\kappa, l+\lambda)-f(k,l)]},$$

the modulus of which does not exceed

$$\rho\rho'\sum_{\kappa=1}^{\rho-1}\sum_{\lambda=0}^{\rho'-1}\left|\sum_{k}\sum_{l}e^{i[f(k+\kappa, l+\lambda)-f(k,l)]}\right| = \rho\rho'\sum_{\mu=1}^{\rho-1}\sum_{\nu=0}^{\rho'-1}|S_1|.$$

The terms with $\mu < \mu'$, $\nu \leqslant \nu'$ give a sum which is the conjugate of that just considered. The terms with $\mu \leqslant \mu'$, $\nu > \nu'$, or with $\mu \geqslant \mu'$, $\nu < \nu'$, lead in a similar way to the corresponding sums involving S_2. Hence

$$|S| < \frac{A}{\rho\rho'}(b-a)^{\frac{1}{2}}(\beta-\alpha)^{\frac{1}{2}}\{A(b-a)(\beta-\alpha)\rho\rho'+\rho\rho'\Sigma\Sigma|S_1|+\rho\rho'\Sigma\Sigma|S_2|\}^{\frac{1}{2}},$$

and the result follows.

4. LEMMA γ. *Let $f(x, y)$ be a real differentiable function of x and y. Let $f_x(x, y)$ have $O(1)$ maxima and minima, considered as a function of x, for each value of y considered, and let $f_y(x, y)$ satisfy a similar condition as a function of y, for each value of x considered. Let*

$$|f_x(x, y)| \leqslant \frac{3\pi}{2}, \quad |f_y(x, y)| \leqslant \frac{3\pi}{2}$$

for $a \leqslant x \leqslant b$, $\alpha \leqslant y \leqslant \beta$, where $b-a \leqslant l$, $\beta-\alpha \leqslant l$ $(l \geqslant 1)$. Then

$$\sum_{m=a}^{b}\sum_{n=\alpha}^{\beta}e^{if(m, n)} = \int_a^b\int_\alpha^\beta e^{if(x,y)}dx\,dy+O(l).$$

We observe that the result of Lemma 1 is still true if, in the condition $|f'(x)| \leqslant \frac{1}{2}$, the constant $\frac{1}{2}$ is replaced by $\frac{3}{4}$ (or by any number less than 1);

and that it is still true if $f'(x)$ is not monotonic, but has $O(1)$ maxima and minima; for, in this case, we can consider separately the $O(1)$ intervals where $f'(x)$ is monotonic, and add the results. We have, therefore,

$$\sum_{n=\alpha}^{\beta} e^{if(m,n)} = \int_{\alpha}^{\beta} e^{if(m,y)}\,dy + O(1).$$

Hence

$$\begin{aligned}\sum_{m=a}^{b}\sum_{n=\alpha}^{\beta} e^{if(m,n)} &= \sum_{m=a}^{b}\int_{\alpha}^{\beta} e^{if(m,y)}\,dy + \sum_{m=a}^{b} O(1)\\ &= \int_{\alpha}^{\beta}\Big\{\sum_{m=a}^{b} e^{if(m,y)}\Big\}\,dy + O(l)\\ &= \int_{\alpha}^{\beta}\Big\{\int_{a}^{b} e^{if(x,y)}\,dx + O(1)\Big\}\,dy + O(l)\\ &= \int_{\alpha}^{\beta} dy \int_{a}^{b} e^{if(x,y)}\,dx + O(l).\end{aligned}$$

5. In the following lemmas we suppose that $f(x, y)$ is a real function of x and y with continuous partial derivatives of as many orders as may be required (actually three). We suppose further that any curve defined by equating to zero a polynomial of given degree in these derivatives has $O(1)$ intersections with any other such curve, or with any straight line. This is true, for example, if, as in the case to be considered, all the curves are algebraic.

LEMMA 8. *Let $f(x, y)$ satisfy the above conditions in the rectangle $a \leqslant x \leqslant b$, $\alpha \leqslant y \leqslant \beta$, where $b-a \leqslant l$, $\beta-\alpha \leqslant l$. Let*

(i) $$r \leqslant |f_{xx}| < Ar, \quad r \leqslant |f_{yy}| < Ar, \quad |f_{xy}| < Ar,$$

and

(ii) $$|f_{xx}f_{yy} - f_{xy}^2| \geqslant r^2$$

throughout the rectangle. Then

$$\int_a^b\int_{\alpha}^{\beta} e^{if(x,y)}\,dx\,dy = O\left(\frac{1+|\log l|+|\log r|}{r}\right).$$

The conditions, of course, imply that f_{xx}, f_{yy}, and $f_{xx}f_{yy}-f_{xy}^2$ are of constant sign in the rectangle. The equation $f_x = \text{const.}$ defines x as a one-valued function of y, and $f_y = \text{const.}$ defines y as a one-valued function

of x. If $f_x = \text{const.}$, $f_y = v$,

$$\frac{dv}{dy} = \frac{f_{xx}f_{yy} - f_{xy}^2}{f_{xx}},$$

so that f_y is monotonic on $f_x = \text{const.}$ The curves $f_x = m\sqrt{r}$, $f_y = n\sqrt{r}$ therefore form a network, dividing the rectangle into regions $D_{m,n}$ in which

$$m\sqrt{r} \leqslant f_x < (m+1)\sqrt{r}, \quad n\sqrt{r} \leqslant f_y < (n+1)\sqrt{r}.$$

The total variation of either x or y in $D_{m,n}$ is $O(1/\sqrt{r})$; *e.g.* on the side $f_x = m\sqrt{r}$, let y change from y_1 to y_2. Then

$$\int_{n\sqrt{r}}^{(n+1)\sqrt{r}} dv = \int_{y_1}^{y_2} \left| \frac{f_{xx}f_{yy} - f_{xy}^2}{f_{xx}} \right| dy > Ar(y_2 - y_1),$$

so that $y_2 - y_1 = O(1/\sqrt{r})$. Since

$$\left| \frac{dx}{dy} \right| = \left| -\frac{f_{xy}}{f_{xx}} \right| < A$$

on this side, the x-variation is also $O(1/\sqrt{r})$. Similarly on the other sides.

Consider first the strip $0 \leqslant f_x \leqslant \sqrt{r}$ (supposing that any part of it belongs to the rectangle). For $n \geqslant 1$, put

$$\iint_{D_{0,n}} e^{if(x,y)}\,dx\,dy = \int dx \int e^{if(x,y)}\,dy.$$

The y-range consists of $O(1)$ intervals, and, since $f_y \geqslant n\sqrt{r}$, the integral over each of these intervals is $O\{1/(n\sqrt{r})\}$. Since the x-range is $O(1/\sqrt{r})$, the whole integral is $O\{1/(nr)\}$. Summing with respect to n, the upper limit of the sum is $O(l\sqrt{r})$, and the result stated follows for this part. Also the area of $D_{0,0}$ is $O(1/r)$, by the above remark, so that

$$\iint_{D_{0,0}} e^{if(x,y)}\,dx\,dy = O\left(\frac{1}{r}\right).$$

A similar argument applies to the part where $f_y < 0$.

Next consider the region $f_x \geqslant \sqrt{r}$. Let its left-hand and right-hand boundaries be $x = \chi(y)$ and $x = \omega(y)$. Then $\chi(y)$ and $\omega(y)$ are either constants or the solution of $f_x = \sqrt{r}$. Now

$$\int_{\chi(y)}^{\omega(y)} e^{if(x,y)}\,dx = \left[\frac{e^{if(x,y)}}{if_x}\right]_{\chi(y)}^{\omega(y)} + \frac{1}{i}\int_{\chi(y)}^{\omega(y)} \frac{f_{xx}}{f_x^2}\, e^{if(x,y)}\,dx.$$

The integrated term gives at the lower limit

$$\frac{1}{i}\int \frac{e^{if[\chi(y),\,y]}}{f_x\{\chi(y), y\}}\,dy = \frac{1}{i}\int \frac{e^{iF(y)}}{G(y)}\,dy,$$

say. If $\chi(y)$ is the solution of $f_x = \sqrt{r}$, then

$$|F'(y)| = |f_y - f_x f_{xy}/f_{xx}| \geqslant |f_y| - A\sqrt{r} > A\sqrt{r},$$

provided that $|f_y| > A\sqrt{r}$; $F'(y)$ has $O(1)$ maxima and minima, by the general conditions on $f(x, y)$; and $G(y) = \sqrt{r}$. Hence the integral is $O(1/r)$. For the part $|f_y| \leqslant A\sqrt{r}$, the interval of integration [corresponding to $O(1)$ regions $D_{m,n}$] is $O(1/\sqrt{r})$; hence this part also is $O(1/r)$.

If $\chi(y)$ is a constant, $F'(y) = f(y)$, and a similar argument* gives the same result. Also a similar proof holds for the upper limit.

Again, in $D_{m,n}$ $(m > 0,\ n \neq 0)$,

$$\int \frac{f_{xx}}{f_x^2} e^{if(x,y)} dy = O\left(\frac{r}{m^2 r} \frac{1}{|n|\sqrt{r}}\right) = O\left(\frac{1}{m^2 |n| \sqrt{r}}\right),$$

by the second mean-value theorem, both f_{xx} and $1/f_x^2$ having $O(1)$ maxima and minima. Hence

$$\iint_{D_{m,n}} \frac{f_{xx}}{f_x^2} e^{if(x,y)} dx\, dy = O\left(\frac{1}{m^2 |n| r}\right).$$

Also

$$\iint_{D_{m,0}} \frac{f_{xx}}{f_x^2} e^{if(x,y)} dx\, dy = \iint_{D_{m,0}} O\left(\frac{r}{m^2 r}\right) dx\, dy = O\left(\frac{1}{m^2 r}\right),$$

and similarly for $D_{m,-1}$. Summing with respect to m and n, the required result is obtained for this part.

Similar analysis applies to the region where $f_x < 0$, and the lemma follows.

LEMMA ϵ. *Let $f(x)$ satisfy the same conditions as in Lemma δ, but in a rectangle with sides inclined at a fixed angle to the axes, and of length $O(l)$. Then the result of Lemma δ still holds.*

The proof goes as before, except that, in the integrated term, we may now have $\chi(y) = \lambda y + \mu$, where λ and μ are constants. We have then to consider

$$\int \frac{e^{if(\lambda y+\mu,\, y)}}{f_x(\lambda y+\mu,\, y)} dy.$$

Here $$\left|\frac{d^2}{dy^2} f(\lambda y+\mu, y)\right| = \left|\lambda^2 f_{xx} + 2\lambda f_{xy} + f_{yy}\right| > Ar,$$

* Using the second mean-value theorem as in Lemma 6 of E. C. Titchmarsh, *Quart. J. of Math.*, 3 (1932), 133–141.

and the result follows as before, if $\lambda < \lambda_0$, where λ_0 depends on the various constants of the lemma.

If $\lambda \geqslant \lambda_0$, we have

$$|f_x(\lambda y+\mu, y)| = \left|\int_{x_0}^{\lambda y+\mu} f_{xx}(x, y)\,dx\right| \geqslant (\lambda y+\mu-x_0)\,r,$$

and also, in the region considered, $|f_x| \geqslant \sqrt{r}$. Hence

$$\int \frac{dy}{|f_x(\lambda y+\mu, y)|} \leqslant \frac{1}{\sqrt{r}} \int_{|\lambda y+\mu-x_0|\leqslant 1/\sqrt{r}} dy + \frac{1}{r}\int_{|\lambda y+\mu-x_0|>1/\sqrt{r}} \frac{dy}{|\lambda y+\mu-x_0|}$$

$$= O\left(\frac{1}{\lambda r}\right) + O\left\{\frac{1}{\lambda r}(\log r + \log|\lambda y_1+\mu-x_0|)\right\},$$

where $1/\sqrt{r} \leqslant |\lambda y_1+\mu-x_0| < O(\lambda l)$; and the result again follows.

LEMMA ζ. *The same result holds if, in Lemma* δ, *condition* (i) *is replaced by*

$$|f_{xx}| \leqslant \tfrac{1}{2}r, \quad |f_{yy}| \leqslant \tfrac{1}{2}r, \quad r \leqslant |f_{xy}| < Ar.$$

When we put $x\sqrt{2} = \xi+\eta$, $y\sqrt{2} = -\xi+\eta$, the integral becomes

$$\iint e^{i\phi(\xi,\eta)}\,d\xi\,d\eta,$$

where

$$\phi(\xi, \eta) = f\left(\frac{\xi+\eta}{\sqrt{2}}, \frac{-\xi+\eta}{\sqrt{2}}\right).$$

We have

$$\phi_{\xi\xi} = \tfrac{1}{2}(f_{xx}+f_{yy})-f_{xy}, \quad \phi_{\eta\eta} = \tfrac{1}{2}(f_{xx}+f_{yy})+f_{xy}, \quad \phi_{\xi\eta} = \tfrac{1}{2}(f_{xx}-f_{yy}),$$

and $\phi(\xi, \eta)$ satisfies the conditions formerly imposed on $f(x, y)$, with $\frac{1}{2}r$ instead of r. The result therefore follows from Lemma ϵ.

In general, it is condition (ii) of Lemma δ which is of most fundamental importance, it being possible to vary the other conditions considerably without altering the result. In the present application, however, f_{xx} and f_{yy} have opposite signs, and (ii) is an immediate consequence of the other conditions.

6. THEOREM 1. *If* $M \leqslant M' \leqslant 2M$, $N \leqslant N' \leqslant 2N$, *and* $L = \max(M, N)$, *then*

$$\sum_{m=M}^{M'} \sum_{n=N}^{N'} \frac{1}{(m^2+an^2)^{\frac{1}{2}+it}} = O\left(\frac{t\log t}{L}\right)$$

for

$$At^{\frac{1}{2}} < L < At.$$

Consider the sum $\sum_{M}^{M'} \sum_{N}^{N'} (m^2+an^2)^{it}$.

This is of the form $\Sigma\Sigma e^{if(m,n)}$, where

$$f(x, y) = t \log(x^2+ay^2),$$

$$f_x = \frac{2tx}{x^2+ay^2}, \quad f_y = \frac{2aty}{x^2+ay^2},$$

$$f_{xx} = \frac{2t(ay^2-x^2)}{(x^2+ay^2)^2}, \quad f_{yy} = \frac{2at(x^2-ay^2)}{(x^2+ay^2)^2}, \quad f_{xy} = -\frac{4atxy}{(x^2+ay^2)^2}.$$

We divide the rectangle of summation into squares $\Delta_{p,q}$, or parts of such squares, such that, in $\Delta_{p,q}$,

$$M+pl \leqslant m < M+(p+1)l, \quad N+ql \leqslant n < N+(q+1)l.$$

The variation of f_x in such a square does not exceed

$$l \max|f_{xx}|+l \max|f_{xy}| < Alt/L^2.$$

Let $$l = A_1 L^2/t,$$

where A_1 is a sufficiently small constant. Then $1 \leqslant l \leqslant L$ if

$$A_1^{-\frac{1}{2}} t^{\frac{1}{2}} \leqslant L \leqslant A_1^{-1} t.$$

Also, if A_1 is small enough, the variation of f_x in the square does not exceed π, and similarly the variation of f_y in the square does not exceed π. For each square we can therefore find integers P and Q such that, if

$$F(x, y) = f(x, y)-2\pi Px-2\pi Qy,$$

then $$|F_x| \leqslant \frac{3\pi}{2}, \quad |F_y| \leqslant \frac{3\pi}{2}$$

[the least favourable case occurs when f_x varies between $2\pi(P+\frac{1}{4})$ and $2\pi(P+\frac{3}{4})$, and similarly for f_y]. The other conditions of Lemma γ are clearly satisfied, and hence

$$\sum_{\Delta_{p,q}} \sum e^{if(m,n)} = \iint_{\Delta_{p,q}} e^{iF(x,y)}\,dx\,dy+O(l).$$

Since the number of squares is $O(L^2/l^2)$,

$$\sum_{M}^{M'} \sum_{N}^{N'} e^{if(m,n)} = \sum_{p} \sum_{q} \iint_{\Delta_{p,q}} e^{iF(x,y)}\,dx\,dy+O\left(\frac{L^2}{l}\right).$$

Consider, first, those squares at any point of which

$$|x-y\sqrt{a}| < A_2 L.$$

If A_1 and A_2 are small enough, the conditions of Lemma ζ are satisfied in such a square, with

$$r = \frac{At}{L^2}.$$

Hence, for these squares,

$$\iint_{\Delta_{p,q}} e^{iF(x,y)}\,dx\,dy = O\Big(\frac{L^2 \log t}{t}\Big).$$

In the remaining squares, the conditions of Lemma δ are satisfied, with $r = At/L^2$. Hence the same result holds for these squares.

Since there are $O(L^2/l^2)$ squares, we obtain altogether

$$\sum_{M}^{M'} \sum_{N}^{N'} e^{if(m,n)} = O\Big(\frac{L^4 \log t}{l^2 t}\Big) + O\Big(\frac{L^2}{l}\Big)$$

$$= O(t \log t) + O(t) = O(t \log t),$$

and the result then follows from Lemma α.

10. It is clear that Theorem 1 is of interest only if L is not too small. For smaller values of L we use the following result.

THEOREM 2. *If* $M \leqslant M' \leqslant 2M, N \leqslant N' \leqslant 2N$, *and* $L = \max(M,N) < At^{\frac{1}{3}}$, *then*

$$\sum_{m=M}^{M'} \sum_{n=N}^{N'} \frac{1}{(m^2+an^2)^{\frac{1}{4}+it}} = O(t^{\frac{1}{3}} \log^{\frac{1}{2}} t).$$

The number of terms in the sum is $O(L^2)$, and the modulus of each is $O(1/L)$. Hence the sum is $O(L) = O(t^{\frac{1}{3}})$ if $L \leqslant t^{\frac{1}{3}}$. If $L > t^{\frac{1}{3}}$, $M'-M < ALt^{-\frac{1}{3}}$, the number of terms is $O(L^2 t^{-\frac{1}{3}})$, and the sum is $O(Lt^{-\frac{1}{3}}) = O(t^{\frac{1}{3}})$. Similarly if $N'-N < ALt^{-\frac{1}{3}}$. We may, therefore, suppose that

$$L > t^{\frac{1}{3}}, \quad M'-M > ALt^{-\frac{1}{3}}, \quad N'-N > ALt^{-\frac{1}{3}}.$$

We now use Lemma β, with $\rho' = \rho$. If

$$S = \sum_{m=M}^{M'} \sum_{n=N}^{N'} (m^2+an^2)^{-it},$$

then

$$S_1 = \sum_{M}^{M'-\mu} \sum_{N}^{N'-\nu} \left\{\frac{(m+\mu)^2+a(n+\nu)^2}{m^2+an^2}\right\}^{-it} = \sum_{M}^{M''} \sum_{N}^{N''} e^{ig(m,n)},$$

say. Here

$$g(x, y) = t \log \frac{x^2+ay^2}{(x+\mu)^2+a(y+\nu)^2},$$

$$g_x(x, y) = 2t \left\{\frac{x}{x^2+ay^2} - \frac{x+\mu}{(x+\mu)^2+a(y+\nu)^2}\right\},$$

$$g_y(x, y) = 2at \left\{\frac{y}{x^2+ay^2} - \frac{y+\nu}{(x+\mu)^2+a(y+\nu)^2}\right\},$$

$$g_{xx}(x, y) = 2t \left\{\frac{ay^2-x^2}{(x^2+ay^2)^2} - \frac{a(y+\nu)^2-(x+\mu)^2}{\{(x+\mu)^2+a(y+\nu)^2\}^2}\right\},$$

$$g_{yy}(x, y) = 2at \left\{\frac{x^2-ay^2}{(x^2+ay^2)^2} - \frac{(x+\mu)^2-a(y+\nu)^2}{\{(x+\mu)^2+a(y+\nu)^2\}^2}\right\},$$

$$g_{xy}(x, y) = -4at \left\{\frac{xy}{(x^2+ay^2)^2} - \frac{(x+\mu)(y+\nu)}{\{(x+\mu)^2+a(y+\nu)^2\}^2}\right\}.$$

We observe that $g_{yy} = -ag_{xx}$, so that g_{xx} and g_{yy} have opposite signs for all values of x and y.

We have

$$g_{xx} = 2t \frac{\Phi(x, y, \mu, \nu)}{(x^2+ay^2)^2\{(x+\mu)^2+a(y+\nu)^2\}^2},$$

where

$$\Phi = (ay^2-x^2)\{(x+\mu)^2+a(y+\nu)^2\}^2 - (x^2+ay^2)^2\{a(y+\nu)^2-(x+\mu)^2\}$$

$$= 2(x^2+ay^2)\{x\mu(3ay^2-x^2)+ay\nu(ay^2-3x^2)\}+O(\lambda^2 L^4),$$

and $\lambda = \max(\mu, \nu)$. Hence

$$g_{xx} = O\left(\frac{t\lambda}{L^3}\right),$$

and a similar result clearly holds for g_{yy} and g_{xy}. We therefore divide up S_1 into squares $\Delta_{p,q}$, of side l, or parts of such squares, where

$$l = \frac{A_3 L^3}{t\rho}.$$

Then, if A_3 is sufficiently small, the variation of g_x and g_y in each square does not exceed π. For each square we can therefore find integers P

and Q such that, if

$$G(x, y) = g(x, y) - 2\pi Px - 2\pi Qy,$$

then $$|G_x(x, y)| \leqslant \frac{3\pi}{2}, \quad |G_y(x, y)| \leqslant \frac{3\pi}{2}.$$

Hence, by Lemma γ,

$$\sum_{\Delta_{p,q}} \sum e^{ig(m,n)} = \iint_{\Delta_{p,q}} e^{iG(x,y)}\,dx\,dy + O(l)$$

(assuming that $1 \leqslant l \leqslant L$); and so, since there are $O(L^2/l^2)$ squares,

$$S_1 = \sum_p \sum_q \iint_{\Delta_{p,q}} e^{iG(x,y)}\,dx\,dy + O\left(\frac{L^2}{l}\right).$$

We have also

$$g_{xy} = -4at\,\frac{\Psi(x, y, \mu, \nu)}{(x^2+ay^2)^2\{(x+\mu)^2+a(y+\nu)^2\}^2},$$

where

$$\begin{aligned}\Psi &= xy\{(x+\mu)^2+a(y+\nu)^2\}^2-(x+\mu)(y+\nu)(x^2+ay^2)^2\\ &= (x^2+ay^2)(-\nu x^3+3\mu x^2 y+3a\nu xy^2-a\mu y^3)+O(\lambda^2 L^4).\end{aligned}$$

Let $$\phi(x, y) = \mu x(3ay^2-x^2)+a\nu y(ay^2-3x^2),$$

$$\psi(x, y) = \nu x(3ay^2-x^2)-\mu y(ay^2-3x^2),$$

so that $$\Phi = 2(x^2+ay^2)\,\phi+O(\lambda^2 L^4),$$

$$\Psi = (x^2+ay^2)\,\psi+O(\lambda^2 L^4).$$

Then

$$\phi^2+a\psi^2 = (\mu^2+a\nu^2)\{x^2(3ay^2-x^2)^2+ay^2(ay^2-3x^2)^2\} > A\lambda^2 L^6.$$

We consider the following cases.

(i) Suppose that $$|\phi| > A_4\lambda L^3$$

throughout the square $\Delta_{p,q}$. Then

$$|\Phi| > A\lambda L^5,$$

$$|g_{xx}| > \frac{A\lambda t}{L^3},$$

and similarly for g_{yy}. We therefore obtain, by Lemma δ,

$$\iint_{\Delta_{p,q}} e^{iG(x,y)}\,dx\,dy = O\left(\frac{L^3 \log t}{\lambda t}\right).$$

The number of such squares is $O(L^2/l^2)$, so that their total contribution is

$$O\left(\frac{L^5 \log t}{\lambda t l^2}\right) = O\left(\frac{\rho^2 t \log t}{\lambda L}\right).$$

(ii) Suppose that $$|\phi| \leqslant A_4 \lambda L^3$$

at some point of the square. Then

$$\begin{aligned} |\phi| &< A_4 \lambda L^3 + l(\max|\phi_x| + \max|\phi_y|) \\ &< A_4 \lambda L^3 + Al\lambda L^2 < 2A_4 \lambda L^3 \end{aligned}$$

throughout the square, if l/L is small enough. Hence

$$\Phi < 2A_4 \lambda L^3 (x^2 + ay^2) + O(\lambda^2 L^4) < 3A_4 \lambda L^3 (x^2 + ay^2),$$

if λ/L is small enough. Also

$$a\psi^2 > A\lambda^2 L^6 - A_4{}^2 \lambda^2 L^6 > A\lambda^2 L^6,$$

if A_4 is small enough. Hence

$$|\psi| > A\lambda L^3,$$

and so $$|\Psi| > A\lambda L^5 + O(\lambda^2 L^4) > A\lambda L^5,$$

if λ/L is small enough.

If A_4 is small enough, the conditions of Lemma ζ are satisfied, with

$$r = \frac{A\lambda t}{L^3},$$

and we therefore obtain the same result as in case (i).

Combining the above results, we have

$$S_1 = O\left(\frac{\rho^2 t \log t}{\lambda L}\right).$$

Now

$$\sum_{\mu=1}^{\rho-1} \sum_{\nu=0}^{\rho-1} \frac{1}{\lambda} < 2 \sum_{\mu=1}^{\rho-1} \sum_{\nu=0}^{\mu} \frac{1}{\mu} = 2 \sum_{\mu=1}^{\rho-1} \frac{\mu+1}{\mu} \leqslant 4(\rho-1).$$

Hence $$\sum_{\mu=1}^{\rho-1} \sum_{\nu=0}^{\rho-1} |S_1| = O\left(\frac{\rho^3 t \log t}{L}\right).$$

A similar result clearly holds for S_2. Hence, by Lemma β, with $\rho' = \rho$, we have

$$S = O\left(\frac{L^2}{\rho}\right) + O(L^{\frac{1}{2}} \rho^{\frac{1}{2}} t^{\frac{1}{2}} \log^{\frac{1}{2}} t).$$

Let $\rho = [Lt^{-\frac{1}{3}}]$. Then

$$S = O(Lt^{\frac{1}{3}} \log^{\frac{1}{2}} t).$$

The result now follows from Lemma α.

We have supposed in the proof that $1 \leqslant l \leqslant L$, *i.e.* that

$$1 \leqslant \frac{A_3 L^3}{t\rho} \leqslant L,$$

i.e. $$1 \leqslant AL^2 t^{-\frac{2}{3}} \leqslant L,$$

i.e. $$At^{\frac{1}{3}} \leqslant L \leqslant At^{\frac{2}{3}},$$

and this is the condition assumed.

11. We can now prove the main theorem. It follows from a theorem of Potter* that, for $\sigma \geqslant \frac{1}{2}$,

$$Z(s) = \sum\sum_{m^2+an^2 \leqslant ct^2} \frac{1}{(m^2+an^2)^s} + O(1),$$

where c is a constant which exceeds a lower limit depending only on a. Now

$$\sum\sum_{m^2+an^2 \leqslant ct^2} \frac{1}{(m^2+an^2)^{\frac{1}{2}+it}} = \sum_p \sum_q \left\{ \sum_{\substack{2^{p-1} \leqslant m < 2^p \\ m^2+an^2 \leqslant ct^2}} \sum_{2^{q-1} \leqslant n < 2^q} \frac{1}{(m^2+an^2)^{\frac{1}{2}+it}} \right\},$$

where p and q both run through $O(\log t)$ values. The inner sum is of the form considered in Theorems 1 and 2, except where the rectangle of summation is cut by the elliptic boundary. This occurs only for sums throughout which $\max(m, n) > At$.

Putting $L = \max(2^{p-1}, 2^{q-1})$, we apply Theorem 2 to the inner sum if $L < At^{\frac{2}{3}}$, and Theorem 1 if $L \geqslant At^{\frac{2}{3}}$. In the former case the inner sum is $O(t^{\frac{1}{3}} \log^{\frac{1}{2}} t)$, and in the latter case it is

$$O\left(\frac{t \log t}{L}\right) = O(t^{\frac{1}{3}} \log t).$$

The total contribution of these sums to $Z(\frac{1}{2}+it)$ is therefore $O(t^{\frac{1}{3}} \log^3 t)$.

The above analysis does not apply, at any rate directly, to sums over regions cut by the elliptic boundary; but for these a comparatively simple argument is sufficient.

---

* H. S. A. Potter, *Approximate equations for the Epstein zeta-function*, Theorem 2, *infra*, 501–515.

Consider one of these regions throughout which $x < Ay$, so that $y > At$. It is easily seen that, if c exceeds a certain value depending only on a, the conditions of Lemma 1 are satisfied by the sum

$$\Sigma (m^2+an^2)^{-it},$$

so that $$\Sigma(m^2+an^2)^{-it} = \int (m^2+ay^2)^{-it}\,dy + O(1).$$

The integral is of the form $\int e^{if(y)}\,dy$, where

$$f(y) = -t\log(m^2+ay^2), \qquad f'(y) = -\frac{2aty}{(m^2+ay^2)}.$$

Hence $f'(y) < -A$, and so the integral is bounded. Hence

$$\Sigma\Sigma(m^2+an^2)^{-it} = \sum_m O(1) = O(t),$$

and, by Lemma a,

$$\Sigma\Sigma(m^2+an^2)^{-\frac{1}{2}-it} = O(1).$$

Hence the total contribution of these sums to $Z(\frac{1}{2}+it)$ is $O(\log^2 t)$. This completes the proof.

APPROXIMATE EQUATIONS FOR THE EPSTEIN ZETA-FUNCTION

By H. S. A. POTTER.

[Received 8 December, 1932.—Read 15 December, 1932.]

1. *Introduction.*

Suppose that

$$\phi = \phi(x, y) = ax^2+bxy+cy^2, \tag{1.01}$$

where a, b, c are real and

$$a > 0, \quad \Delta = 4ac-b^2 > 0. \tag{1.02}$$

Then the series

$$\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty}{}' \phi(m, n)^{-s} \tag{1.1}$$

defines an analytic function $Z(s)$ regular for $\sigma > 1$. This function $Z(s)$ is, perhaps, the most interesting particular case of the Epstein zeta-function*. Epstein† and Mordell‡ have proved that $Z(s)$, like $\zeta(s)$, can be continued analytically throughout the s-plane and that it satisfies the functional equation

$$Z(s) = \left(\frac{2\pi}{\sqrt{\Delta}}\right)^{2s-1} \frac{\Gamma(1-s)}{\Gamma(s)} Z(1-s). \tag{1.11}$$

In § 2 I prove an approximate equation for $Z(s)$ and give two other forms of the equation. These formulae are analogous to the well-

* Epstein, **1**, **2**.

† Epstein, **1**.

‡ Mordell, **1**. His statement of the functional equation should be corrected by changing $\sqrt{\Delta}$ into $1/\sqrt{\Delta}$.

known formula*

$$\zeta(s) - \sum_{n \leqslant x} n^{-s} + \frac{x^{1-s}}{1-s} = O(x^{-\sigma})$$

uniformly in the region $\sigma \geqslant \sigma_0 > 0$, $|t| \leqslant 2\pi x/\gamma$ $(\gamma > 1)$.

Sections 3–4 are devoted to a proof of the approximate functional equation for $Z(s)$. The method is one of "real" analysis, and is, in fact, modelled on the first proof† by Hardy and Littlewood of the corresponding result for $\zeta(s)$. The equation for $Z(s)$, however, in many ways corresponds more closely to the approximate functional equation for $\zeta^2(s)$‡ than to that for $\zeta(s)$. But my proof differs in one important respect from that of Hardy and Littlewood. Their proof is based on the series §

$$\sum_{n=1}^{\infty} \frac{\sin 2n\pi x}{n\pi},$$

but the corresponding series for $Z(s)$, namely

$$\sqrt{x} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty}{}' \frac{J_1[4\pi\sqrt{\{x\phi(m, n)/\Delta\}}]}{\sqrt{\phi(m, n)}},$$

is not by any means so simple. Hence I have based my proof on the integrated series

$$\frac{\sqrt{\Delta}}{2\pi} x \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty}{}' \frac{J_2[4\pi\sqrt{\{x\phi(m, n)/\Delta\}}]}{\phi(m, n)},$$

which is absolutely and uniformly convergent in any interval $0 \leqslant x \leqslant A$.

2. *Equations for* $Z(s)$.

In the following sections let us write for shortness,

$$\phi = \phi(m, n),$$

and

$$\sum_{\phi} = \lim_{X \to \infty} \sum_{\phi < X}.$$

As usual, a dash attached to a summation sign means that the term for

---

* Hardy and Littlewood, 1, Lemma 2.
† Hardy and Littlewood, 1.
‡ Hardy and Littlewood, 2.
§ Hardy and Littlewood, 1, Lemma 1.

which $m = n = 0$ is to be omitted. We begin by proving the fundamental formula* used.

2.1. LEMMA 1. *Let*

$$R(u) = \sum_{\phi \leqslant u} 1, \quad R_1(x) = \int_0^x R(u)\,du,$$

and

$$\psi(u) = R(u) - \frac{2\pi}{\sqrt{\Delta}}u, \quad \psi_1(x) = \int_0^x \psi(u)\,du.$$

Then

$$\psi_1(x) = \frac{\sqrt{\Delta}}{2\pi}\, x \sum_{\phi}{}' \frac{J_2\{4\pi\sqrt{(x\phi/\Delta)}\}}{\phi}. \tag{2.11}$$

Let α, β $(\alpha \leqslant \beta)$ be the roots of

$$x - \phi(u, v) = 0$$

considered as a quadratic in v. By the Fourier† summation formula,

$$\Phi(x, u) = \sum_{\alpha \leqslant n \leqslant \beta} \{x - \phi(u, n)\} = \sum_{n=-\infty}^{\infty} \int_\alpha^\beta \cos 2\pi n v \{x - \phi(u, v)\}\,dv, \tag{2.12}$$

where $x > 0$, $-2\sqrt{(cx/\Delta)} \leqslant u \leqslant 2\sqrt{(cx/\Delta)}$. Applying the summation formula again, we have

$$\sum_{|m| \leqslant 2\sqrt{(cx/\Delta)}} \Phi(x, m) = \sum_{m=-\infty}^{\infty} \int_{-2\sqrt{(cx/\Delta)}}^{2\sqrt{(cx/\Delta)}} \cos 2\pi m u \Phi(x, u)\,du. \tag{2.13}$$

The left-hand side of (2.13) is

$$\sum_{\phi \leqslant x} (x - \phi) = \sum_{\phi \leqslant x} \int_\phi^x du = \int_0^x \sum_{\phi \leqslant u} 1\,du = R_1(x).$$

On the other hand, the series for $\Phi(x, u)$ in (2.12) converges uniformly in u, for, on integrating by parts, we have

$$\left| \int_\alpha^\beta \cos 2\pi n v \{x - \phi(u, v)\}\,dv \right| = \left| \frac{1}{2\pi n} \int_\alpha^\beta \sin 2\pi n v (bu + 2cv)\,dv \right| < A\,\frac{\sqrt{x}}{n^2},$$

---

* The formula is a particular case of a much more general one proved by Landau, **1**, and Wilton, **1**. Hardy, **1**, **2**, has suggested two other proofs supposing the coefficients a, b, c of $\phi(m, n)$ to be integers, but this restriction is unnecessary.

† The lemma can, of course, be proved by Poisson's summation formula for two variables (*cf.* Bochner, **1**, § 46). This method has been used for the particular form $x^2 + y^2$ by Mordell, **2**, and Bochner, **2**. The present proof follows that in Landau's *Vorlesungen über Zahlentheorie*, 2 (1927), Satz 508.

by the second mean-value theorem. Hence, by (2.12) and (2.13), after an inversion which is now justified,

$$R_1(x) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \int_{-2\sqrt{(cx/\Delta)}}^{2\sqrt{(cx/\Delta)}} \cos 2\pi mu\, du \int_{\alpha}^{\beta} \cos 2\pi nv\{x-\phi(u, v)\}\, dv$$

$$= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \iint_{\phi(u,v)\leqslant x} \cos 2\pi mu \cos 2\pi nv\{x-\phi(u, v)\}\, du\, dv$$

$$= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \iint_{\phi(u,v)\leqslant x} \cos 2\pi(mu+nv)\{x-\phi(u, v)\}\, du\, dv$$

on rearranging the order of the terms. We may do this, since $\sum_{n=-\infty}^{\infty}$ here means $\lim_{\lambda\to\infty} \sum_{n=-\lambda}^{\lambda}$. Now change to new variables U, V defined by

$$U+iV = \sqrt{a}u + \frac{b+i\sqrt{\Delta}}{2\sqrt{a}}\, v.$$

We have
$$\frac{\partial(U, V)}{\partial(u, v)} = \tfrac{1}{2}\sqrt{\Delta}$$

and
$$mu+nv = MU+NV,$$

where

$$M = \frac{m}{\sqrt{a}}, \quad N = -\frac{bm}{\sqrt{(a\Delta)}} + 2n\sqrt{\left(\frac{a}{\Delta}\right)}.$$

Hence

$$R_1(x) = \frac{2}{\sqrt{\Delta}} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \iint_{U^2+V^2\leqslant x} \cos 2\pi(MU+NV)(x-U^2-V^2)\, dU\, dV$$

$$= \frac{\pi}{\sqrt{\Delta}}\, x^2 + \frac{\sqrt{\Delta}}{2\pi}\, x \sum_{\phi}{}' \frac{J_2\{4\pi\sqrt{(x\phi/\Delta)}\}}{\phi},$$

the integral being evaluated as in Landau's *Vorlesungen über Zahlentheorie*, 2 (1927), Satz 503. The series is absolutely convergent, so that the implied rearrangement is justified. Equation (2.11) now follows, since

$$\psi_1(x) = R_1(x) - \frac{\pi}{\sqrt{\Delta}}\, x^2.$$

2.2. LEMMA 2. *If* $\sigma > -\frac{1}{4}$, $s \neq 1$, $x > 0$, *and*

(2.201)
$$\xi = \xi(x, m, n) = 4\pi\sqrt{\left(\frac{x\phi(m, n)}{\Delta}\right)},$$

*then**

$$(2.21)\quad Z(s)-\sum_{\phi\leqslant x}{}'\phi^{-s}+\frac{2\pi}{\sqrt{\Delta}}\frac{x^{1-s}}{1-s}+\psi(x)\,x^{-s}$$

$$=2s\left(\frac{4\pi}{\sqrt{\Delta}}\right)^{2s-1}\sum_{\phi}{}'\phi^{s-1}\int_{\xi}^{\infty}u^{-2s}J_1(u)\,du.$$

The functional equation (1.11) for $Z(s)$ may easily be deduced from (2.21). For, choose σ so that $-\frac{1}{4}<\sigma<0$ and let $x\to 0$, then (2.21) gives†

$$Z(s)=2s\left(\frac{4\pi}{\sqrt{\Delta}}\right)^{2s-1}Z(1-s)\int_0^{\infty}u^{-2s}J_1(u)\,du=\left(\frac{2\pi}{\sqrt{\Delta}}\right)^{2s-1}\frac{\Gamma(1-s)}{\Gamma(s)}Z(1-s).$$

The values assumed by $\phi(m, n)$ as m and n run through all integer values form an enumerable nowhere-dense set. Thus we may arrange its members in order of magnitude and represent them by the sequence [$\phi(0, 0)$ being omitted]

$$0<\lambda_1\leqslant\lambda_2\leqslant\ldots\leqslant\lambda_n\leqslant\lambda_{n+1}\leqslant\ldots.$$

It is clear that $$R(v)=n,$$

if $\lambda_n\leqslant v<\lambda_{n+1}$. Let us write

$$R(x)=N,$$

and suppose that $\sigma>1$. Then

$$s\int_x^{\infty}v^{-s-1}R(v)\,dv=R(x)\,x^{-s}+\sum_{n=N+1}^{\infty}\lambda_n^{-s}.$$

Hence $$s\int_x^{\infty}v^{-s-1}\psi(v)\,dv=\frac{2\pi}{\sqrt{\Delta}}\frac{x^{1-s}}{1-s}+\psi(x)\,x^{-s}+\sum_{n=N+1}^{\infty}\lambda_n^{-s}.$$

On the other hand,

$$s\int_x^{\infty}v^{-s-1}\psi(v)\,dv=-s\psi_1(x)\,x^{-s-1}+s(s+1)\int_x^{\infty}v^{-s-2}\psi_1(v)\,dv,$$

and $$\sum_{n=N+1}^{\infty}\lambda_n^{-s}=Z(s)-\sum_{\phi\leqslant x}{}'\phi^{-s},$$

* This equation is included in Lemmas 1 and 2 of the paper by Jones and Ingham, 1.

† The integral is evaluated in Watson's *Theory of Bessel functions*, 391.

so that

$$(2.22)\quad Z(s)-\sum_{\phi\leqslant x}{}'\phi^{-s}+\frac{2\pi}{\sqrt{\Delta}}\frac{x^{1-s}}{1-s}+\psi(x)x^{-s}$$
$$=-s\psi_1(x)x^{-s-1}+s(s+1)\int_x^\infty v^{-s-2}\psi_1(v)\,dv.$$

We now use the formula for $\psi_1(v)$ proved in Lemma 1. The series is absolutely and uniformly convergent in any finite interval; also

$$\int_x^\infty |v^{-s-2}\psi_1(v)|\,dv=\int_x^\infty O\left(v^{-\sigma-1}\sum_\phi{}'\frac{(v\phi)^{-\frac{1}{2}}}{\phi}\right)dv=\int_x^\infty O(v^{-\sigma-\frac{3}{2}})\,dv,$$

which is convergent for $\sigma>-\frac{1}{4}$. Hence (2.22), proved for $\sigma>1$, also holds for $\sigma>-\frac{1}{4}$ and we may insert the series for $\psi_1(v)$ and integrate term by term. This gives, after an obvious change of variable,

$$(2.23)\quad s(s+1)\int_x^\infty v^{-s-2}\psi_1(v)\,dv$$
$$=4s(s+1)\left(\frac{4\pi}{\sqrt{\Delta}}\right)^{2s-1}\sum_\phi{}'\phi^{s-1}\int_\xi^\infty u^{-2s-1}J_2(u)\,du,$$

provided that $\sigma>-\frac{1}{4}$. Now

$$\frac{d}{du}\{u^2J_2(u)\}=u^2J_1(u),$$

so that

$$2(s+1)\phi^{s-1}\int_\xi^\infty u^{-2s-1}J_2(u)\,du$$
$$=\left(\frac{4\pi}{\sqrt{\Delta}}\right)^{-2s}x^{-s}\frac{J_2\{4\pi\sqrt{(x\phi/\Delta)}\}}{\phi}+\phi^{s-1}\int_\xi^\infty u^{-2s}J_1(u)\,du,$$

if $\sigma>-\frac{1}{4}$. Substituting in (2.23) and recalling (2.11) and (2.22), we obtain (2.21).

2.3. THEOREM 1. *Suppose that* $\sigma\geqslant\sigma_0>-\frac{1}{4}$, $|s-1|\geqslant\delta>0$,

$$(2.301)\qquad |t|\leqslant\frac{2\pi}{\gamma}\sqrt{\left(\frac{\lambda x}{\Delta}\right)},$$

where

$$\lambda=\min\{\phi(m,n)\}$$

for integer values of m, n, *and* $\gamma>1$. *Then*

$$(2.31)\qquad Z(s)-\sum_{\phi\leqslant x}{}'\phi^{-s}+\frac{2\pi}{\sqrt{\Delta}}\frac{x^{1-s}}{1-s}+\psi(x)x^{-s}=O(x^{\frac{1}{2}-\sigma})$$

uniformly in s.

Since $$J_1(u) = \left(\frac{2}{\pi u}\right)^{\frac{1}{2}} \cos\left(u - \frac{3\pi}{4}\right) + O(u^{-\frac{3}{2}}),$$

we have, in (2.21),

$$(2.32)\quad \int_\xi^\infty u^{-2s} J_1(u)\, du$$

$$= \frac{e^{-\frac{1}{4}\pi i}}{\sqrt{(2\pi)}} \int_\xi^\infty u^{-2s-\frac{1}{2}} e^{iu}\, du + \frac{e^{\frac{1}{4}\pi i}}{\sqrt{(2\pi)}} \int_\xi^\infty u^{-2s-\frac{1}{2}} e^{-iu}\, du + \int_\xi^\infty O(u^{-2\sigma-\frac{3}{2}})\, du$$

$$= \frac{e^{-\frac{1}{4}\pi i}}{\sqrt{(2\pi)}} I_1 + \frac{e^{\frac{1}{4}\pi i}}{\sqrt{(2\pi)}} I_2 + O(x^{-\sigma-\frac{1}{2}} \phi^{-\sigma-\frac{1}{2}}),$$

say. Now

$$I_1 = \int_\xi^\infty u^{-2\sigma-\frac{1}{2}} e^{iw}\, du,$$

where $w = u - 2t \log u$, so that

$$\frac{dw}{du} = 1 - \frac{2t}{u} \geqslant 1 - \frac{\sqrt{\Delta}|t|}{2\pi\sqrt{(x\phi)}} \geqslant 1 - \frac{\sqrt{\Delta}|t|}{2\pi\sqrt{(\lambda x)}} \geqslant 1 - \frac{1}{\gamma} > 0.$$

Hence $$I_1 = \int_{u=\xi}^\infty \frac{u^{-2\sigma-\frac{1}{2}}}{1 - 2t/u} e^{iw}\, dw.$$

Here the real factor decreases steadily as u increases, if $\sigma > -\frac{1}{4}$; so we may apply the second mean-value theorem to the real and imaginary parts and obtain

$$(2.321)\quad I_1 = O(\xi^{-2\sigma-\frac{1}{2}}) = O(x^{-\sigma-\frac{1}{2}} \phi^{-\sigma-\frac{1}{2}}).$$

Similarly

$$(2.322)\quad I_2 = O(x^{-\sigma-\frac{1}{2}} \phi^{-\sigma-\frac{1}{2}}).$$

Collecting our results from (2.21), (2.32), (2.321), and (2.322), we have

$$Z(s) - \sum_{\phi \leqslant x}' \phi^{-s} + \frac{2\pi}{\sqrt{\Delta}} \frac{x^{1-s}}{1-s} + \psi(x) x^{-s} = O(|t| x^{-\sigma-\frac{1}{2}} \sum_\phi' \phi^{-\frac{1}{2}}) = O(x^{\frac{1}{2}-\sigma}),$$

since $|t| < Ax^{\frac{1}{2}}$ in the given region.

2.4. THEOREM 2. *If the conditions of Theorem* 1 *are satisfied, then*

$$(2.41)\quad Z(s) = \sum_{\phi \leqslant x}' \phi^{-s} - \frac{2\pi}{\sqrt{\Delta}} \frac{x^{1-s}}{1-s} + O(x^{\frac{1}{2}-\sigma})$$

uniformly in s.

For $\psi(x)$ is simply the remainder function in the problem concerning the lattice points of an ellipse and it is easily proved that

$$\psi(x) = O(x^{\frac{1}{2}}).$$

2.5. THEOREM 3. *If the conditions of Theorem* 1 *are satisfied, there is an absolute constant* $\theta \leqslant \frac{1}{3}$ *such that*

$$Z(s) = \sum_{\phi \leqslant x}' \phi^{-s} - \frac{2\pi}{\sqrt{\Delta}} \frac{x^{1-s}}{1-s} + O(x^{\theta-\sigma}) \tag{2.51}$$

uniformly in s.

For

$$\psi(x) = O(x^{\theta}),$$

where Landau* has proved that we may take $\theta = \frac{1}{3}$.

3. *The approximate functional equation.*

3.1. THEOREM 4. *If* A *is a positive constant, and*

$$-\tfrac{1}{8} \leqslant \sigma \leqslant 1\tfrac{1}{8}, \quad x > A, \quad y > A, \quad 4\pi^2 xy = \Delta t^2, \tag{3.101}$$

$$X = X(s) = \left(\frac{2\pi}{\sqrt{\Delta}}\right)^{2s-1} \frac{\Gamma(1-s)}{\Gamma(s)}, \tag{3.102}$$

then

$$Z(s) = \sum_{\phi \leqslant x}' \phi^{-s} + X \sum_{\phi \leqslant y}' \phi^{s-1} + O\left\{x^{\frac{1}{2}-\sigma}\left(\frac{x+y}{|t|}\right)^{\frac{1}{2}} \log|t|\right\}. \tag{3.11}$$

It will only be necessary to prove the theorem for the strip $-\frac{1}{8} \leqslant \sigma \leqslant \frac{1}{2}$. For then

$$Z(1-s) = \sum_{\phi \leqslant x}' \phi^{s-1} + X_1 \sum_{\phi \leqslant y}' \phi^{-s} + O\left\{x^{\sigma-\frac{1}{2}}\left(\frac{x+y}{|t|}\right)^{\frac{1}{2}} \log|t|\right\},$$

where $X_1 = X(1-s) = X^{-1}$, provided that $-\frac{1}{8} \leqslant 1-\sigma \leqslant \frac{1}{2}$. Substituting for $Z(1-s)$ by means of the functional equation (1.11), we obtain

$$Z(s) = X \sum_{\phi \leqslant x}' \phi^{s-1} + \sum_{\phi \leqslant y}' \phi^{-s} + O\left\{y^{\frac{1}{2}-\sigma}\left(\frac{x+y}{|t|}\right)^{\frac{1}{2}} \log|t|\right\},$$

provided that $\frac{1}{2} \leqslant \sigma \leqslant 1\frac{1}{8}$, and this is (3.11) with x and y interchanged.

---

* Landau, 1.

3.2. LEMMA 3. *Suppose that the A's are positive constants, that*

$$-\tfrac{1}{8} \leqslant \sigma \leqslant \tfrac{1}{2}, \quad t > A, \quad \xi > A, \tag{3.201}$$

and that

$$I = I(\xi, s) = \int_{\xi}^{\infty} u^{-\frac{1}{2}-2s} \cos(u - \tfrac{3}{4}\pi)\, du. \tag{3.21}$$

Then

$$I = -\Gamma(\tfrac{1}{2}-2s) \sin \pi s + O\left(\frac{\xi^{\frac{1}{2}-2\sigma}}{t}\right) \quad (\xi < At < 2t), \tag{3.22}$$

$$I = -\Gamma(\tfrac{1}{2}-2s) \sin \pi s + O\left(\frac{\xi^{\frac{3}{2}-2\sigma}}{t(2t-\xi)}\right) \quad (At < \xi < 2t), \tag{3.23}$$

$$I = O\left(\frac{\xi^{\frac{1}{2}-2\sigma}}{\xi - 2t}\right) \quad (2t < \xi < At), \tag{3.24}$$

$$I = O(\xi^{-\frac{1}{2}-2\sigma}) \quad (2t < At < \xi), \tag{3.25}$$

and

$$I = O(\xi^{-\frac{1}{2}-2\sigma} t^{\frac{1}{2}}) \tag{3.26}$$

in any case.

This is just another form of the Hardy-Littlewood lemma*.

3.3. LEMMA 4. *Let*

$$K(u) = J_1(u) - \left(\frac{2}{\pi u}\right)^{\frac{1}{2}} \cos(u - \tfrac{3}{4}\pi). \tag{3.31}$$

Then

$$\int_{\xi}^{\infty} u^{-2s} K(u)\, du = O(t^{-\frac{1}{2}} \xi^{-\frac{1}{2}-2\sigma}), \tag{3.32}$$

provided that the conditions (3.201) *of Lemma* 3 *are satisfied.*

For

$$\begin{aligned} \int_{\xi}^{\infty} u^{-2s} K(u)\, du &= \frac{1}{2s}\, \xi^{1-2s} K(\xi) + \frac{1}{2s} \int_{\xi}^{\infty} u^{-2s} \frac{d}{du} \{uK(u)\}\, du \\ &= O(t^{-1} \xi^{-\frac{1}{2}-2\sigma}) + \frac{1}{2s} I. \end{aligned} \tag{3.33}$$

---

* Hardy and Littlewood, **1**, Lemma 12. Their lemma deals with

$$\int_{\xi}^{\infty} u^{-s'} \begin{matrix} \cos \\ \sin \end{matrix} u\, du,$$

where $0 < \sigma' < 1$. It is, however, easily seen to be true throughout $\frac{1}{4} \leqslant \sigma' \leqslant \frac{3}{2}$ (*cf.* Hardy and Littlewood, **2**, 85), and the present result is obtained by putting $s' = 2s + \frac{1}{2}$.

But

$$\frac{d}{du}\{uK(u)\} = uJ_0(u) + \left(\frac{2u}{\pi}\right)^{\frac{1}{2}} \sin(u-\tfrac{3}{4}\pi) - \tfrac{1}{2}\left(\frac{2}{\pi u}\right)^{\frac{1}{2}} \cos(u-\tfrac{3}{4}\pi)$$

$$= -\tfrac{3}{8}\left(\frac{2}{\pi u}\right)^{\frac{1}{2}} \cos(u-\tfrac{3}{4}\pi) + O(u^{-\frac{3}{2}}),$$

when we substitute for $J_0(u)$ its well-known asymptotic expansion. Thus

$$(3.34) \qquad I = -\tfrac{3}{8}\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_\xi^\infty u^{-\frac{1}{2}-2s} \cos(u-\tfrac{3}{4}\pi)\, du + \int_\xi^\infty O(u^{-\frac{3}{2}-2\sigma})\, du$$

$$= O(\xi^{-\frac{1}{2}-2\sigma} t^{\frac{1}{2}})$$

by (3.26). The result follows from (3.33) and (3.34).

3.4. LEMMA 5. *We have*

$$(3.41) \qquad \sum_{\phi < By}{}' \phi^{-\eta} = O(y^{1-\eta}) \quad (\eta \neq 1),$$

$$= O(\log y) \quad (\eta = 1),$$

$$(3.42) \qquad \sum_{\sqrt{(By)} \leqslant \sqrt{\phi} < \sqrt{y}-A} \frac{1}{\phi^{\frac{1}{2}}(y^{\frac{1}{2}} - \phi^{\frac{1}{2}})} = O(y^{\frac{1}{2}} \log y),$$

$$(3.43) \qquad \sum_{\sqrt{y}-A \leqslant \sqrt{\phi} \leqslant \sqrt{y}+A} \phi^{-\eta} = O(y^{\frac{1}{2}-\eta}) \qquad (\eta \neq 1),$$

$$= O(y^{-\frac{1}{2}} \log y) \quad (\eta = 1),$$

$$(3.44) \qquad \sum_{\sqrt{y}+A < \sqrt{\phi} \leqslant \sqrt{(Cy)}} \frac{1}{\phi^{\frac{1}{2}}(\phi^{\frac{1}{2}} - y^{\frac{1}{2}})} = O(y^{-\frac{1}{2}} \log y),$$

and

$$(3.45) \qquad \sum_{Cy < \phi} \phi^{-\eta} = O(y^{1-\eta}) \quad (\eta > 1).$$

For* the number of values of $\sqrt{\phi}$ between N and $N+1$ is $O(N)$. Thus

$$\sum_{\phi < By}{}' \phi^{-\eta} = \sum_{N < \sqrt{(By)}} \sum_{N \leqslant \sqrt{\phi} < N+1} \phi^{-\eta}$$

$$= \sum_{N < \sqrt{(By)}} O(N^{1-2\eta})$$

$$= O(y^{1-\eta}) \quad (\eta < 1),$$

$$= O(\log y) \quad (\eta = 1).$$

This is (3.41), and (3.43) and (3.45) may be proved similarly.

---

* This elementary proof is due to Prof. Titchmarsh. I thank him for his very helpful criticisms and suggestions.

Also

$$\sum_{\sqrt{(By)} \leqslant \sqrt{\phi} < \sqrt{y}-A} \frac{1}{\phi^{\frac{1}{4}}(y^{\frac{1}{4}}-\phi^{\frac{1}{4}})} = O\left(\frac{1}{y^{\frac{1}{4}}} \sum_{\sqrt{(By)} \leqslant N < \sqrt{y}-A} \sum_{N \leqslant \sqrt{\phi} < N+1} \frac{1}{y^{\frac{1}{4}}-(N+1)}\right)$$

$$= O\left(y^{\frac{1}{4}} \sum_{\sqrt{(By)} \leqslant N < \sqrt{y}-A} \frac{1}{y^{\frac{1}{4}}-(N+1)}\right)$$

$$= O(y^{\frac{1}{4}} \log y).$$

We may prove (3.44) in a similar manner.

3.5. LEMMA 6. *We have, if* $t > A$,

$$(3.51) \qquad -2s\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \left(\frac{4\pi}{\sqrt{\Delta}}\right)^{2s-1} \Gamma(\tfrac{1}{2}-2s) \sin \pi s = X(s)+O(t^{-2\sigma}).$$

Consider

$$f(s) = X(s)+2s\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \left(\frac{4\pi}{\sqrt{\Delta}}\right)^{2s-1} \Gamma(\tfrac{1}{2}-2s) \sin \pi s$$

$$= X(s)\left\{1-\left(\frac{-2s}{\frac{1}{2}-2s}\right) \frac{(2\pi)^{\frac{1}{2}} 2^{2s-1} \Gamma(\frac{3}{2}-2s)}{\{\Gamma(1-s)\}^2}\right\}$$

$$= X(s)\{1-e^{g(s)}\}.$$

Using Stirling's theorem, we have

$$g(s) = \log\left(\frac{-2s}{\frac{1}{2}-2s}\right)+\tfrac{1}{2}\log(2\pi)+(2s-1)\log 2+\log\Gamma(\tfrac{3}{2}-2s)-2\log\Gamma(1-s)$$

$$= -\log\left\{1+O\left(\frac{1}{t}\right)\right\}+(2s-1)\log 2+(1-2s)\log(\tfrac{3}{2}-2s)$$

$$-(\tfrac{3}{2}-2s)-2\{(\tfrac{1}{2}-s)\log(1-s)-(1-s)\}+O\left(\frac{1}{t}\right)$$

$$= (1-2s)\log\left\{\frac{\frac{3}{2}-2s}{2(1-s)}\right\}+\tfrac{1}{2}+O\left(\frac{1}{t}\right)$$

$$= (1-2s)\log\left(1-\frac{1}{4(1-s)}\right)+\tfrac{1}{2}+O\left(\frac{1}{t}\right)$$

$$= \frac{2s-1}{4(1-s)}+\tfrac{1}{2}+O\left(\frac{1}{t}\right) = O\left(\frac{1}{t}\right).$$

Also $$X(s) = O(t^{1-2\sigma}),$$

so that $$f(s) = O\left(t^{1-2\sigma}\,\frac{1}{t}\right) = O(t^{-2\sigma}).$$

4. *Proof of Theorem* 4.

4.1. For simplicity suppose that $t > 0$. We have to reconsider the series arising from Lemma 2. This is

$$\begin{aligned}(4.11)\quad & s\sum_{\phi}{}' \phi^{s-1}\int_{\xi}^{\infty} u^{-2s}J_1(u)\,du \\ &= \sqrt{\left(\frac{2}{\pi}\right)}\sum_{\phi}{}' s\phi^{s-1}\int_{\xi}^{\infty} u^{-\frac{1}{2}-2s}\cos(u-\tfrac{3}{4}\pi)\,du \\ &\qquad + \sum_{\phi}{}' s\phi^{s-1}\int_{\xi}^{\infty} u^{-2s}K(u)\,du \\ &= \sqrt{\left(\frac{2}{\pi}\right)}(\Sigma_1+\Sigma_2+\Sigma_3+\Sigma_4+\Sigma_5)\,s\phi^{s-1} \\ &\qquad \times\int_{\xi}^{\infty} u^{-\frac{1}{2}-2s}\cos(u-\tfrac{3}{4}\pi)\,du + S_6 \\ &= S_1+S_2+S_3+S_4+S_5+S_6,\end{aligned}$$

where the ranges of summation are $\phi < By$ in Σ_1, $\sqrt{(By)} \leqslant \sqrt{\phi} < \sqrt{y}-A$ in Σ_2, $\sqrt{y}-A \leqslant \sqrt{\phi} \leqslant \sqrt{y}+A$ in Σ_3, $\sqrt{y}+A < \sqrt{\phi} \leqslant \sqrt{(Cy)}$ in Σ_4, and $Cy < \phi$ in Σ_5. Here B and C are constants such that $0 < B < 1 < C$, and

$$\xi = 4\pi\sqrt{\left(\frac{x\phi}{\Delta}\right)} = 2t\sqrt{\left(\frac{\phi}{y}\right)}.$$

Let us consider S_6 first. We have, by Lemma 4,

$$\begin{aligned}(4.12)\quad S_6 &= O(t^{\frac{1}{2}}x^{-\frac{1}{2}-\sigma}\sum_{\phi}{}'\phi^{-\frac{5}{4}}) = O(x^{-\sigma}y^{\frac{1}{4}}) \\ &= O(x^{\frac{1}{4}-\sigma}y^{\frac{1}{4}}).\end{aligned}$$

In S_1, $\phi < By$, so that

$$\xi = 2t\sqrt{\left(\frac{\phi}{y}\right)} < 2\sqrt{Bt} < 2t.$$

Hence, by (3.22),

$$\begin{aligned}(4.13)\quad S_1 &= -s\sqrt{\left(\frac{2}{\pi}\right)}\Gamma(\tfrac{1}{2}-2s)\sin\pi s\sum_{\phi<By}{}'\phi^{s-1}+S_{12} \\ &= S_{11}+S_{12},\end{aligned}$$

say, where

$$S_{12} = O\left(t \sum_{\phi < By}{}' \phi^{\sigma-1} \frac{(x\phi)^{\frac{1}{2}-\sigma}}{t}\right)$$

$$= O\left(x^{\frac{1}{2}-\sigma} \sum_{\phi < By}{}' \phi^{-\frac{1}{2}}\right) = O(x^{\frac{1}{2}-\sigma} y^{\frac{1}{2}}).$$

4.2. Now consider S_5. We have

$$\xi = 2t\sqrt{\left(\frac{\phi}{y}\right)} > 2\sqrt{Ct} > 2t,$$

so that, by (3.25) and (3.45),

$$S_5 = O\left(t \sum_{Cy < \phi} \phi^{\sigma-1}(x\phi)^{-\frac{1}{2}-\sigma}\right) \tag{4.21}$$

$$= O\left(tx^{-\frac{1}{2}-\sigma} \sum_{Cy < \phi} \phi^{-\frac{1}{2}}\right) = O(tx^{-\frac{1}{2}-\sigma} y^{-\frac{1}{2}})$$

$$= O(x^{\frac{1}{2}-\sigma} y^{\frac{1}{2}}).$$

Again, by (3.26) and (3.43),

$$S_3 = O\left(t \sum_{\sqrt{y}-A \leqslant \sqrt{\phi} \leqslant \sqrt{y}+A} \phi^{\sigma-1}(x\phi)^{-\frac{1}{2}-\sigma} t^{\frac{1}{2}}\right) \tag{4.22}$$

$$= O(t^{\frac{1}{2}} x^{-\frac{1}{2}-\sigma} y^{-\frac{1}{2}})$$

$$= O(x^{\frac{1}{2}-\sigma}).$$

4.3. Finally consider S_2 and S_4. In S_2, $\sqrt{(By)} \leqslant \sqrt{\phi} < \sqrt{y}-A$, so that $2\sqrt{Bt} \leqslant \xi < 2t$. Hence, by (3.23) and (3.42),

$$S_2 = -s\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \Gamma(\tfrac{1}{2}-2s) \sin \pi s \sum_{\sqrt{(By)} \leqslant \sqrt{\phi} < \sqrt{y}-A} \phi^{s-1} + S_{22} \tag{4.31}$$

$$= S_{21} + S_{22},$$

say, where

$$S_{22} = O\left(t \sum_{\sqrt{(By)} \leqslant \sqrt{\phi} < \sqrt{y}-A} \phi^{\sigma-1} \frac{(x\phi)^{\frac{1}{2}-\sigma}}{t\{2t - 2t\sqrt{(\phi/y)}\}}\right) \tag{4.32}$$

$$= O\left(\frac{x^{\frac{1}{2}-\sigma} y^{\frac{1}{2}}}{t} \sum_{\sqrt{(By)} \leqslant \sqrt{\phi} < \sqrt{y}-A} \frac{1}{\phi^{\frac{1}{2}}(y^{\frac{1}{2}} - \phi^{\frac{1}{2}})}\right)$$

$$= O\left(\frac{x^{\frac{1}{2}-\sigma}}{t} y^{\frac{1}{2}} \log y\right) = O(x^{\frac{1}{2}-\sigma} y^{\frac{1}{2}} \log t).$$

In S_4, $2t<\xi<2\sqrt{Ct}$. Hence, by (3.24) and (3.44),

$$S_4 = O\Big(t \sum_{\sqrt{y}+A<\sqrt{\phi}\leqslant\sqrt{(Cy)}} \phi^{\sigma-1}\frac{(x\phi)^{\frac{1}{4}-\sigma}}{2t\sqrt{(\phi/y)-2t}}\Big) \tag{4.33}$$

$$= O\Big(x^{\frac{1}{4}-\sigma}y^{\frac{1}{4}} \sum_{\sqrt{y}+A<\sqrt{\phi}\leqslant\sqrt{(Cy)}} \frac{1}{\phi^{\frac{3}{4}}(\phi^{\frac{1}{2}}-y^{\frac{1}{2}})}\Big)$$

$$= O(x^{\frac{1}{4}-\sigma}y^{\frac{1}{4}}\log t).$$

4.4. Collecting our results from (4.11) to (4.33) we have, by Lemma 2,

$$Z(s) - \sum_{\phi\leqslant x}{}' \phi^{-s} + \frac{2\pi}{\sqrt{\Delta}}\frac{x^{1-s}}{1-s} + \psi(x)x^{-s} \tag{4.41}$$

$$= 2\Big(\frac{4\pi}{\sqrt{\Delta}}\Big)^{2s-1}(S_{11}+S_{21}) + O(x^{\frac{1}{4}-\sigma}y^{\frac{1}{4}}\log t) + O(x^{\frac{1}{2}-\sigma}),$$

where

$$2\Big(\frac{4\pi}{\sqrt{\Delta}}\Big)^{2s-1}(S_{11}+S_{21}) = -2s\Big(\frac{2}{\pi}\Big)^{\frac{1}{2}}\Big(\frac{4\pi}{\sqrt{\Delta}}\Big)^{2s-1}\Gamma(\tfrac{1}{2}-2s)\sin\pi s \sum_{\sqrt{\phi}<\sqrt{y}-A}{}' \phi^{s-1}.$$

By Lemma 6, the error involved in replacing the factor outside this sum by $X(s)$ is

$$O\Big(t^{-2\sigma}\sum_{\phi\leqslant y}{}' \phi^{\sigma-1}\Big) = O(t^{-2\sigma}y^{\sigma}\log y) = O(x^{-\sigma}\log t),$$

and, if we replace $\sqrt{y}-A$ by $\sqrt{y}$, the error is

$$O\Big(t^{1-2\sigma}\sum_{\sqrt{y}-A<\sqrt{\phi}\leqslant\sqrt{y}}\phi^{\sigma-1}\Big) = O(t^{1-2\sigma}y^{\sigma-\frac{1}{2}}\log y)$$

$$= O(x^{\frac{1}{2}-\sigma}\log t).$$

Also
$$\frac{2\pi}{\sqrt{\Delta}}\frac{x^{1-s}}{1-s} = O\Big(\frac{x^{1-\sigma}}{t}\Big) = O(x^{\frac{1}{2}-\sigma}),$$

and
$$\psi(x)x^{-s} = O(x^{\frac{1}{2}-\sigma}).$$

Thus
$$Z(s) = \sum_{\phi\leqslant x}{}' \phi^{-s} + X\sum_{\phi\leqslant y}{}' \phi^{s-1} + O(x^{\frac{1}{4}-\sigma}M\log t),$$

where $M=\max(x^{\frac{1}{4}}, y^{\frac{1}{4}})$.

If $x\leqslant y$,
$$M = y^{\frac{1}{4}} = Ax^{\frac{1}{4}}\Big(\frac{y}{t}\Big)^{\frac{1}{4}},$$

while, if $x\geqslant y$,
$$M = x^{\frac{1}{4}} < Ax^{\frac{1}{4}}\Big(\frac{x}{t}\Big)^{\frac{1}{4}}.$$

In any case,
$$M < Ax^{\frac{1}{4}}\Big(\frac{x+y}{t}\Big)^{\frac{1}{4}}.$$

This completes the proof of the theorem.

References.

S. Bochner, **1**. "Vorlesungen über Fouriersche Integrale" (Leipzig, 1932).

————, **2**. "Die Poissonsche Summationsformel in mehreren Veränderlichen" *Math. Annalen*, 106 (1932), 56–63.

P. Epstein, **1**. "Zur Theorie allgemeiner Zetafunktionen", *Math. Annalen*, 56 (1903), 615–644.

————, **2**. "Zur Theorie allgemeiner Zetafunktionen II", *Math. Annalen*, 63 (1907), 205–216.

G. H. Hardy, **1**. "On the expression of a number as the sum of two squares", *Quart. J. of Math.*, 46 (1915), 263–283.

————, **2**. "The lattice points of a circle", *Proc. Royal Soc.* (A), 107 (1925), 623–635.

G. H. Hardy and J. E. Littlewood, **1**. "The zeros of Riemann's zeta-function on the critical line", *Math. Zeitschrift*, 10 (1921), 283–317.

————, **2**. "The approximate functional equation for $\zeta(s)$ and $\zeta^2(s)$", *Proc. London Math. Soc.* (2), 29 (1929), 81–97.

J. E. Jones and A. E. Ingham, **1**. "On the calculation of certain crystal potential constants, and on the cubic crystal of least potential energy", *Proc. Royal Soc.* (A), 107 (1925), 636–653.

E. Landau, **1**. "Zur analytischen Zahlentheorie der quadratischen Formen", *Berliner Sitzungsberichte* (1915), 458–476.

L. J. Mordell, **1**. "The zeta-functions arising from quadratic forms, and their functional equations", *Quart. J. of Math.* (Oxford), 1 (1930), 77–101.

————, **2**, "Poisson's summation formula in several variables and applications", *Proc. Camb. Phil. Soc.*, 25 (1929), 412–420.

J. R. Wilton, **1**. "The lattice points of an n-dimensional ellipsoid", *Journal London Math. Soc.*, 2 (1927), 227–233.

THEOREMS CONCERNING CESÀRO MEANS OF POWER SERIES

By G. H. HARDY *and* J. E. LITTLEWOOD.

[Received 18 June, 1932.—Read 19 June, 1932.]

1. *Introduction.*

1.1. We suppose throughout this paper that $z = re^{i\theta}$, where $r \leqslant 1$, that

$$f(z) = \Sigma c_n z^n = \sum_0^\infty c_n z^n$$

is a power series convergent for $r < 1$, that $\lambda > 0$, and that

$$M_\lambda = M_\lambda(r) = M_\lambda(f) = M_\lambda(r, f) = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{i\theta})|^\lambda d\theta\right)^{1/\lambda}.$$

If $M_\lambda(r)$ is bounded, we say that $f(z)$ belongs to the (complex) class L^λ. It is well known that $f(z)$ then possesses a "boundary function" $f(e^{i\theta})$, of the (ordinary) Lebesgue class L^λ, with which it has certain standard relations. These are (i) that

$$f(z) = f(re^{i\theta}) \to f(e^{i\theta}),$$

when $r \to 1$, for almost all θ; (ii) that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{i\theta}) - f(e^{i\theta})|^\lambda d\theta \to 0,$$

i.e. that $f(re^{i\theta})$ "converges strongly" to $f(e^{i\theta})$; and (iii) that

$$M_\lambda(r) \to M_\lambda(1) = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(e^{i\theta})|^\lambda d\theta\right)^{1/\lambda}.$$

When $\lambda > 1$, all this is true also (with the appropriate changes of notation) for harmonic functions $u(r, \theta)$.

1.2. We denote by $s_n^k(f, z)$ the Cesàro sum, of order $k > -1$, formed from the series $\Sigma c_n z^n$, and by $\sigma_n^k(f, z)$ the corresponding mean. The

formulae defining s_n^k and σ_n^k are

$$\sigma_n^k(f, z) = \frac{s_n^k(f, z)}{A_n^k} = \frac{1}{A_n^k} \sum_{\nu=0}^{n} A_{n-\nu}^k c_\nu z^\nu,$$

where $$A_n^k = \binom{n+k}{n} = \frac{\Gamma(n+k+1)}{\Gamma(n+1)\,\Gamma(k+1)}$$

$\Big($so that $$A_n^k \sim \frac{n^k}{\Gamma(k+1)}$$

when $n \to \infty\Big)$. We may also define $s_n^k(f, z)$ by

$$\Sigma s_n^k(f, z)\, u^n = \frac{f(zu)}{(1-u)^{k+1}},$$

where $\rho = |u| < 1$. We may omit either or both of the arguments of s_n^k or σ_n^k when no ambiguity results.

The properties of the means σ_n^k are very well known when $\lambda \geqslant 1$ and $k > 0$. In particular it is known (i) that

$$\sigma_n^k(e^{i\theta}) \to f(e^{i\theta}), \tag{1.2.1}$$

when $n \to \infty$, for almost all θ; (ii) that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |\sigma_n^k(e^{i\theta})|^\lambda d\theta \leqslant A M_\lambda^\lambda(1), \tag{1.2.2}$$

where A depends only on λ and k; (iii) that $\sigma_n^k(e^{i\theta})$ converges strongly to $f(e^{i\theta})$, *i.e.* that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |\sigma_n^k(e^{i\theta}) - f(e^{i\theta})|^\lambda d\theta \to 0; \tag{1.2.3}$$

and (iv) that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |\sigma_n^k(e^{i\theta})|^\lambda d\theta \to M_\lambda^\lambda(1). \tag{1.2.4}$$

No one seems to have attacked the corresponding problems for $\lambda < 1$, and we propose to consider them here.

The problems raised by (1.2.2), (1.2.3), and (1.2.4) we solve completely; we show that these results are true when

$$k > \frac{1}{\lambda} - 1, \tag{1.2.5}$$

but false (for appropriate f) when

$$k \leqslant \frac{1}{\lambda} - 1. \tag{1.2.6}$$

We cannot give so definite a solution of the remaining problem. We show that (1.2.1) is true when

$$k > \left[\frac{1}{\lambda}\right],$$

but it is obviously unlikely that this is the best possible result†.

1.3. We may clear the ground by three preliminary remarks:

(i) Whatever λ may be, (1.2.4) is an elementary corollary of (1.2.3). In fact, (1.2.4) follows from (1.2.3) by Minkowski's inequality

$$\left(\int |g+h|^\lambda dx\right)^{1/\lambda} \leqslant \left(\int |g|^\lambda dx\right)^{1/\lambda} + \left(\int |h|^\lambda dx\right)^{1/\lambda}$$

when $\lambda \geqslant 1$, and by the associated inequality

$$\int |g+h|^\lambda dx \leqslant \int |g|^\lambda dx + \int |h|^\lambda dx$$

when $\lambda < 1$.

(ii) We may prove all our theorems on the assumption that $f(z)$ is *wurzelfrei*, *i.e.* has no zeros inside the unit circle. Any f of L^λ is the sum of two *wurzelfrei* functions f_1 and f_2, for which‡

$$M_\lambda(f_s) \leqslant 2M_\lambda(f) \quad (s = 1, 2),$$

and (i), (ii), and (iii), for f, follow by combination of the corresponding results for f_1 and f_2.

(iii) When $\lambda > 1$ there are corresponding results for harmonic functions $u(r, \theta)$, $f(e^{i\theta})$ being replaced by $u(1, \theta)$ and $M_\lambda{}^\lambda(1)$ by

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |u(1, \theta)|^\lambda d\theta.$$

All this breaks down when $0 < \lambda < 1$. A harmonic function of L^λ (*i.e.* a function for which

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |u(r, \theta)|^\lambda d\theta \tag{1.3.1}$$

† The difference between the two bounds for k is sometimes small and at other times nearly 1.

‡ See Hardy and Littlewood (2), 207.

is bounded) has generally no "boundary function"; the integral (1.3.1) does not usually tend to a limit; and it is improbable that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|\sigma_n^k(1,\theta)|^\lambda d\theta,$$

the integral corresponding to (1.2.2), is bounded†.

2. *The boundedness of* $\int|\sigma_n^k|^\lambda d\theta$.

2.1. THEOREM 1. *Suppose that* $0<\lambda<1$, *that* $f(z)$ *is* L^λ, *and that*

$$k>\frac{1}{\lambda}-1. \tag{2.1.1}$$

Then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|\sigma_n^k(e^{i\vartheta})|^\lambda d\theta\leqslant AM_\lambda^\lambda(1)=\frac{A}{2\pi}\int_{-\pi}^{\pi}|f(e^{i\theta})|^\lambda d\theta, \tag{2.1.2}$$

where $A=A(\lambda, k)$ *depends only on* λ *and* k.

We may suppose (as we explained in § 1.3) that f is *wurzelfrei*. We take

$$r=1,\quad z=e^{i\theta},\quad u=\rho e^{i\phi},\quad \rho<1.$$

We define an integer l by

$$\frac{1}{l}<\lambda\leqslant\frac{1}{l-1}, \tag{2.1.3}$$

so that

$$l=\left[\frac{1}{\lambda}\right]+1\geqslant 2,$$

and α by

$$k=l-1+l\alpha,\quad \alpha=\frac{k-l+1}{l}. \tag{2.1.4}$$

It is plain that $\alpha>-1$. Also $\alpha<0$ if

$$k<l-1=\left[\frac{1}{\lambda}\right],$$

† In view of the breakdown, for $\lambda<1$, of the orthodox relations between pairs of conjugate harmonic functions. See our paper 5.

which is always true if k is near enough to the lower bound assigned by (2.1.1). Hence

$$-1 < \alpha < 0$$

in the cases with which we are primarily concerned.

We write $$f = g^l;$$

since f is *wurzelfrei*, g is regular. Then g is L^p, where

$$p = l\lambda > 1. \tag{2.1.5}$$

From $$\frac{f(zu)}{(1-u)^{k+1}} = \left\{\frac{g(zu)}{(1-u)^{\alpha+1}}\right\}^l$$

we deduce

$$s_n{}^k(f) = \sum_{\nu_1+\nu_2+\ldots+\nu_l=n} s_{\nu_1}^{\alpha}(g)\, s_{\nu_2}^{\alpha}(g) \ldots s_{\nu_l}^{\alpha}(g), \tag{2.1.6}$$

and it is on this identity that all our work depends.

2.2. We write

$$S_\nu = |s_\nu{}^\alpha(g)|. \tag{2.2.1}$$

It follows from (2.1.6) and Hölder's inequality that

$$|s_n{}^k(f)| \leqslant \Sigma S_\nu{}^l,$$

where the summation is $(l-1)$-fold, ν having one of the values $\nu_1, \nu_2, \ldots$ and the summation being then over $\nu_1+\nu_2+\ldots = n$†. When we allow for this, we obtain

$$|s_n{}^k(f)| \leqslant An^{l-2} \sum_{\nu=0}^{n} S_\nu{}^l = An^{l-2} P, \tag{2.2.2}$$

say, where A (as always in the sequel) is a number of the type prescribed by the theorem.

It follows from (2.1.3) and (2.1.5) that

$$1 < p \leqslant 1+\lambda < 2 \tag{2.2.3}$$

and that

$$l \leqslant \frac{l\lambda}{l\lambda-1} = \frac{p}{p-1} = p'. \tag{2.2.4}$$

† Thus, for example, $\Sigma 1$ is asymptotic to a multiple of n^{l-1}, and not to a multiple of n.

Hence, by Hölder's inequality,

$$\sum_0^n S_\nu^{\,l} \leqslant n^{(p'-l)/p'} \left(\sum_0^n S_\nu^{p'}\right)^{l/p'}. \tag{2.2.5}$$

Also S_ν is the modulus of the coefficient of u^ν in

$$\frac{g(zu)}{(1-u)^{\alpha+1}} = \frac{g(\rho e^{i\phi+i\theta})}{(1-\rho e^{i\phi})^{\alpha+1}},$$

and (since $p<2$) we may apply Hausdorff's theorem to this function. We thus obtain

$$(\Sigma S_\nu^{p'} \rho^{p'\nu})^{p/p'} \leqslant \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{|g(\rho e^{i\phi+i\theta})|^p}{|1-\rho e^{i\phi}|^{(\alpha+1)p}}\, d\phi = J(\theta), \tag{2.2.6}$$

say.

2.3. Taking

$$\rho = 1-\frac{1}{n}$$

in (2.2.6), we obtain

$$\left(\sum_0^n S_\nu^{p'}\right)^{p/p'} \leqslant AJ(\theta).$$

Combining this with (2.2.5), and observing that $p=l\lambda$ and that

$$\frac{p'-l}{p'}\lambda = \lambda\left(1-l+\frac{l}{p}\right) = -l\lambda+\lambda+1,$$

we obtain

$$P^\lambda = \left(\sum_0^n S_\nu^{\,l}\right)^\lambda \leqslant An^{-l\lambda+\lambda+1} J(\theta).$$

Hence

$$\begin{aligned}
\int_{-\pi}^{\pi} P^\lambda\, d\theta &\leqslant An^{-l\lambda+\lambda+1} \int_{-\pi}^{\pi} J(\theta)\, d\theta \qquad (2.3.1)\\
&\leqslant An^{-l\lambda+\lambda+1} \int_{-\pi}^{\pi} d\theta \int_{-\pi}^{\pi} \frac{|g(\rho e^{i\phi+i\theta})|^{l\lambda}}{|1-\rho e^{i\phi}|^{(\alpha+1)l\lambda}}\, d\phi\\
&= An^{-l\lambda+\lambda+1} \int_{-\pi}^{\pi} \frac{d\phi}{|1-\rho e^{i\phi}|^{(k+1)\lambda}} \int_{-\pi}^{\pi} |g(\rho e^{i\phi+i\theta})|^{l\lambda}\, d\theta\\
&\leqslant An^{-l\lambda+\lambda+1} M_\lambda{}^\lambda(1) \int_{-\pi}^{\pi} \frac{d\phi}{|1-\rho e^{i\phi}|^{(k+1)\lambda}}.
\end{aligned}$$

Since $(k+1)\lambda > 1$, in virtue of (2.1.1), the integral here is less than

$$An^{(k+1)\lambda-1}.$$

It follows that

$$(2.3.2)\quad \int_{-\pi}^{\pi} P^\lambda\, d\theta \leqslant A n^{-l\lambda+\lambda+1} M_\lambda{}^\lambda(1)\, n^{(k+1)\lambda-1} = A n^{k\lambda-(l-2)\lambda} M_\lambda{}^\lambda(1).$$

Finally, combining this with (2.2.2), we obtain

$$\int_{-\pi}^{\pi} |s_n{}^k(f)|^\lambda\, d\theta \leqslant A n^{k\lambda} M_\lambda{}^\lambda(1),$$

which is equivalent to (2.1.2)

3. *Strong convergence.*

3.1. THEOREM 2. *If the conditions of Theorem* 1 *are satisfied, then*

$$(3.1.1)\quad \frac{1}{2\pi}\int_{-\pi}^{\pi} |\sigma_n{}^k(e^{i\theta})-f(e^{i\theta})|^\lambda\, d\theta \to 0.$$

THEOREM 3. *Under the same conditions,*

$$(3.1.2)\quad \frac{1}{2\pi}\int_{-\pi}^{\pi} |\sigma_n{}^k(e^{i\theta})|^\lambda\, d\theta \to M_\lambda{}^\lambda(1) = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(e^{i\theta})|^\lambda\, d\theta.$$

The second of these theorems is, as we remarked in §1.3 (i), an elementary corollary of the first, and we need not refer to it further.

We have two proofs of Theorem 2, the first of which is direct and proceeds on the same lines as those of Theorem 1. The second is shorter (when Theorem 1 has been proved), but its basis lies rather deeper, and we postpone it to §4.

We begin by observing that

$$s_n{}^k(f, z)-A_n{}^k f(z)$$

is the coefficient of u^n in

$$\frac{g^l(zu)-g^l(z)}{(1-u)^{k+1}} = \prod_\omega \left\{\frac{g(zu)-\omega g(z)}{(1-u)^{\alpha+1}}\right\},$$

where ω runs through

$$\omega_1 = 1, \quad \omega_2 = e^{2\pi i/l}, \quad \ldots,$$

the l-th roots of unity. Hence

$$s_n{}^k(f, z)-A_n{}^k f(z) = \sum_{\nu_1+\ldots+\nu_l=n} \{s_{\nu_1}^\alpha(g)-\omega_1 A_{\nu_1}^\alpha g\}\ldots\{s_{\nu_l}^\alpha(g)-\omega_l A_{\nu_l}^\alpha g\},$$

and so

$$|s_n^k(f)-A_n^k f| \leqslant (\Sigma|s_\nu^\alpha(g)-A_\nu^\alpha g|^l)^{1/l} \prod_{\omega\neq 1} (\Sigma|s_\nu^\alpha(g)-\omega A_\nu^\alpha g|^l)^{1/l}.$$

In each of these sums the range of summation is multiple, as in § 2.2. When we allow for this, as there, we obtain

$$|s_n^k(f)-A_n^k(f)| \leqslant An^{l-2} P_0^{1/l} P_1^{1/l} \dots P_{l-1}^{1/l}, \tag{3.1.3}$$

where

$$P_0 = \sum_{\nu=0}^{n} |s_\nu^\alpha(g)-A_\nu^\alpha g|^l, \tag{3.1.4}$$

and each of $P_1, P_2, \dots, P_{l-1}$ is majorized by

$$Q = \sum_{\nu=0}^{n} \{|s_\nu^\alpha(g)|+|A_\nu^\alpha||g|\}^l. \tag{3.1.5}$$

We shall prove that

$$\int_{-\pi}^{\pi} Q^\lambda d\theta \leqslant An^{k\lambda-(l-2)\lambda} M_\lambda^\lambda(1), \tag{3.1.6}$$

and that

$$\int_{-\pi}^{\pi} P_0^\lambda d\theta = o(n^{k\lambda-(l-2)\lambda}) \tag{3.1.7}$$

when $n\to\infty$. These results correspond to (2.3.2); and it is plain after § 2.3 that, when they are proved, we can deduce

$$\int_{-\pi}^{\pi} |s_n^k(f)-A_n^k f|^\lambda d\theta = o(n^{k\lambda}),$$

which is (3.1.1).

3.2. Now

$$\begin{aligned}
\int_{-\pi}^{\pi} Q^\lambda d\theta &= \int_{-\pi}^{\pi} \left\{\sum_0^n (S_\nu+|A_\nu^\alpha||g|)^l\right\}^\lambda d\theta \\
&\leqslant A\int_{-\pi}^{\pi} \left\{\sum_0^n (S_\nu^l+|A_\nu^\alpha|^l|g|^l)\right\}^\lambda d\theta \\
&\leqslant A\int_{-\pi}^{\pi} \left(\sum_0^n S_\nu^l\right)^\lambda d\theta + A\left(\sum_0^n |A_\nu^\alpha|^l\right)^\lambda \int_{-\pi}^{\pi} |g|^{l\lambda} d\theta \\
&\leqslant An^{k\lambda-(l-2)\lambda} M_\lambda^\lambda(1) + AM_\lambda^\lambda(1)\left(\sum_0^n |A_\nu^\alpha|^l\right)^\lambda,
\end{aligned}$$

by (2.3.1). Also $$A_\nu{}^\alpha = O(\nu^\alpha)$$

and $$la = k-l+1 > \frac{1}{\lambda}-l \geqslant -1,$$

by (2.1.3). Hence

$$\left(\sum_0^n |A_\nu{}^\alpha|^l\right)^\lambda = O(n^{al\lambda+\lambda}) = O(n^{k\lambda-(l-2)\lambda});$$

and this completes the proof of (3.1.5).

It remains only to prove (3.1.4). For this, we observe that, since $s_\nu{}^\alpha(g)-A_\nu{}^\alpha g$ is the coefficient of u^ν in

$$\frac{g(zu)-g(z)}{(1-u)^{\alpha+1}},$$

the same argument which led to (2.3.1) in §2.3 will lead here to

$$(3.2.1)\quad \int_{-\pi}^{\pi} P_0{}^\lambda\, d\theta \leqslant An^{-l\lambda+\lambda+1}\int_{-\pi}^{\pi} d\theta \int_{-\pi}^{\pi} \frac{|g(\rho e^{i\phi+i\theta})-g(e^{i\theta})|^{l\lambda}}{|1-\rho e^{i\phi}|^{(k+1)\lambda}}\, d\phi.$$

The integral here is

$$(3.2.2)\quad \int_{|\phi|\geqslant\delta} \frac{d\phi}{|1-\rho e^{i\phi}|^{(k+1)\lambda}} \int_{-\pi}^{\pi} |g(\rho e^{i\phi+i\theta})-g(e^{i\theta})|^{l\lambda}\, d\theta + \int_{|\phi|\leqslant\delta} \ldots d\phi \int_{-\pi}^{\pi} \ldots d\theta$$
$$= I_1+I_2,$$

say. It is plain that I_1 is bounded, for any fixed δ, when $n\to\infty$ and $\rho\to 1$; and *a fortiori*

$$(3.2.3)\quad I_1 = o(n^{(k+1)\lambda-1}).$$

The inner integral in I_2 does not exceed

$$A\int_{|\phi|\leqslant\delta} |g(\rho e^{i\phi+i\theta})-g(\rho e^{i\theta})|^{l\lambda}\, d\theta + A\int_{-\pi}^{\pi} |g(\rho e^{i\theta})-g(e^{i\theta})|^{l\lambda}\, d\theta.$$

The second term tends to zero when $\rho\to 1$, and the first tends to zero, uniformly in ρ, when $\delta\to 0$†. It follows that

$$(3.2.4)\quad I_2 < \epsilon n^{(k+1)\lambda-1}$$

if δ is sufficiently small and n sufficiently large. Hence

$$(3.2.5)\quad I_1+I_2 = o(n^{(k+1)\lambda-1});$$

† See F. Riesz (7), 95.

and (3.1.4) follows from (3.2.1), (3.2.2), and (3.2.5). This completes the proof of Theorem 2.

4. *An alternative proof of Theorem 2.*

4.1. Our second proof of Theorem 2 is formally simpler, but depends on theorems one of which lies rather deeper while others are presumably not the best of their kind.

THEOREM 4. *If*

$$k > \left[\frac{1}{\lambda}\right] = l-1, \tag{4.1.1}$$

then

$$\sigma_n{}^k(f, e^{i\theta}) \to f(e^{i\theta}) \tag{4.1.2}$$

for almost all θ.

If f is *wurzelfrei*, we have, by (2.1.6),

$$\sigma_n{}^k(f) = \frac{1}{A_n{}^k} \sum_{\nu_1+\ldots+\nu_l=n} s_{\nu_1}^{\alpha}(g) \ldots s_{\nu_l}^{\alpha}(g). \tag{4.1.3}$$

Now $k > l-1$ and so $\alpha > 0$. Hence

$$\sigma_\nu{}^\alpha(g) \to g(e^{i\theta}),$$

or

$$s_\nu{}^\alpha(g) \sim \frac{\nu^\alpha}{\Gamma(\alpha+1)} g(e^{i\theta}),$$

for almost all θ. It follows from well known elementary theorems† that

$$\Sigma s_{\nu_1}^{\alpha}(g) \ldots s_{\nu_l}^{\alpha}(g) \sim \frac{g^l}{\{\Gamma(\alpha+1)\}^l} \Sigma \nu_1{}^\alpha \ldots \nu_l{}^\alpha$$

$$\sim \frac{n^{l\alpha+l-1}}{\Gamma(l\alpha+l-1)} g^l = \frac{n^k}{\Gamma(k+1)} f,$$

and $\sigma_n{}^k(f) \to f$, for the same θ.

The extension to general f follows in the usual manner.

THEOREM 5. *If* k *satisfies* (4.1.1), *and*

$$\sigma^* = \sigma^*(f, \theta) = \operatorname*{Max}_n |\sigma_n{}^k(f, e^{i\theta})|, \tag{4.1.4}$$

† See Knopp (6).

then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\sigma^{*\lambda}\,d\theta \leqslant AM_\lambda^\lambda(1)=\frac{A}{2\pi}\int_{-\pi}^{\pi}|f(e^{i\theta})|^\lambda\,d\theta. \tag{4.1.5}$$

It follows from (2.1.6), when we allow again for the multiple summation, that

$$\begin{aligned}|\sigma_n^k(f)| &\leqslant An^{-k}.n^{l-1}.\operatorname*{Max}_{\nu\leqslant n}|s_\nu^\alpha(g)|^l\\ &\leqslant An^{-k+l-1+l\alpha}\operatorname*{Max}_{\nu\leqslant n}|\sigma_\nu^\alpha(g)|^l=A\operatorname*{Max}_{\nu\leqslant n}|\sigma_\nu^\alpha(g)|^l.\end{aligned}$$

Since the right-hand side increases with n, this involves

$$\sigma_n^*=\operatorname*{Max}_{\nu\leqslant n}|\sigma_\nu^k(f)|\leqslant A\operatorname*{Max}_{\nu\leqslant n}|\sigma_\nu^\alpha(g)|^l;$$

and so

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi}\sigma_n^{*\lambda}\,d\theta &\leqslant \frac{A}{2\pi}\int_{-\pi}^{\pi}\operatorname*{Max}_{\nu\leqslant n}|\sigma_\nu^\alpha(g)|^{l\lambda}\,d\theta\\ &\leqslant AM_\lambda^\lambda(1),\end{aligned}$$

by Theorem 21 of our paper **3**. Since σ_n^* increases with n, this in its turn involves (4.1.5).

The proofs of Theorems 4 and 5 demand that α shall be positive, and so that k shall satisfy (4.1.1). We can none the less deduce Theorem 2 (with the better bound for k) by combining them with Theorem 1.

Suppose first that k satisfies (4.1.1). Then

$$|\sigma_n^k(f)-f|^\lambda$$

is majorized by

$$(\sigma^*+|f|)^\lambda,$$

an integrable function independent of n; and $\sigma_n^k(f)-f\to 0$ for almost all θ. It follows by Lebesgue's criterion that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|\sigma_n^k(f)-f|^\lambda\,d\theta\to 0$$

for *some* k (in fact, for $k>l-1$). Hence there is a polynomial $P(e^{i\theta})$ satisfying

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|f-P|^\lambda\,d\theta<\epsilon; \tag{4.1.6}$$

for $\sigma_n^k(f)$ is such a polynomial when n is sufficiently large.

We now suppose k subject only to (1.2.5). Applying Theorem 1 to $f-P$, we obtain

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|\sigma_n^k(f)-\sigma_n^k(P)|^\lambda\,d\theta<A\epsilon \tag{4.1.7}$$

for all n. From (4.1.6) and (4.1.7) we deduce

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|\sigma_n^k(f)-f|^\lambda\,d\theta<A\epsilon+\frac{1}{2\pi}\int_{-\pi}^{\pi}|\sigma_n^k(P)-P|^\lambda\,d\theta.$$

Since $\sigma_n^k(P)\to P$ uniformly, this proves Theorem 2.

5. *Negative results.*

5.1. We have to show finally that our bound (2.1.1) for k cannot be improved.

THEOREM 6. *If*

$$k\leqslant\frac{1}{\lambda}-1, \tag{5.1.1}$$

then the result of Theorem 1 *is sometimes false.*

The proof is easy when

$$k<\frac{1}{\lambda}-1. \tag{5.1.2}$$

The function $$f(z)=(1-z)^{-(1/\lambda)+\delta}$$

is L^λ if $\delta>0$, and c_n is (exactly) of order

$$\frac{1}{\lambda}-1-\delta.$$

We shall prove that, if (2.1.2) is true, then

$$c_n=O(n^k). \tag{5.1.3}$$

This will give a contradiction if k satisfies (5.1.2) and δ is sufficiently small.

Suppose, first, that

$$k\neq\frac{1}{\lambda}-p$$

for any integral p. Since

$$s_n^{k+1}=s_0^k+s_1^k+\ldots+s_n^k,$$

the truth of (2.1.2) for $k=l$ involves its truth for $k=l+1$. It is, therefore, true for a k for which

$$\frac{1}{\lambda}-2<k<\frac{1}{\lambda}-1, \tag{5.1.4}$$

and we may suppose that k satisfies this condition.

Taking $z=e^{i\theta}$, as in §2.1, we have

$$\Sigma c_n z^n u^n = (1-u)^{k+1}\Sigma s_n{}^k(z)\, u^n,$$

and so

$$c_n z^n = \sum_0^n (-1)^\nu \binom{k+1}{\nu} s_{n-\nu}^k(z).$$

Hence

$$\int |c_n z^n|^\lambda d\theta \leqslant \sum_0^n \left|\binom{k+1}{\nu}\right|^\lambda \int |s_{n-\nu}^k(z)|^\lambda d\theta.$$

But

$$\binom{k+1}{\nu} = O(\nu^{-k-2}),$$

and so

$$|c_n|^\lambda = \frac{1}{2\pi}\int |c_n z^n|^\lambda d\theta = O\left\{n^{k\lambda}\sum_0^n \nu^{-(k+2)\lambda}\right\} = O(n^{k\lambda}),$$

which is (5.1.3).

A slight change is required when k differs from $1/\lambda$ by an integer. We may then suppose that

$$k=\frac{1}{\lambda}-2.$$

We start from

$$s_n(z) = \sum_0^n (-1)^\nu \binom{k}{\nu} s_{n-\nu}^k(g)$$

and show, as above, that

$$\int |s_n(z)|^\lambda d\theta = O(n^{k\lambda}).$$

From this and

$$c_n z^n = s_n(z) - s_{n-1}(z)$$

we deduce (5.1.3).

5.2. The case of equality in (5.1.1) is rather more delicate. We use an example suggested by one which we have used for a somewhat similar purpose elsewhere†. We shall pass rather lightly over some details of

† Hardy and Littlewood (**4**).

the argument which are a little tedious but do not involve any point of principle.

We take

$$f(z) = (N+1)^{-(p\lambda-1)/\lambda}\left(\frac{1-z^{N+1}}{1-z}\right)^p = (N+1)^{-\beta}\left(\frac{1-z^{N+1}}{1-z}\right)^p,$$

where $p\lambda > 1$, $p > 2$, and N is large. Then

$$\int_{-\pi}^{\pi} |f|^\lambda d\theta \leqslant (N+1)^{-p\lambda+1}\int_{-\pi}^{\pi}\left|\frac{\sin\frac{1}{2}(N+1)\theta}{\sin\frac{1}{2}\theta}\right|^{p\lambda} d\theta$$

$$< AN^{-p\lambda+1}\int_0^{1/N} N^{p\lambda}\,d\theta + AN^{-p\lambda+1}\int_{1/N}^{\pi}\theta^{-p\lambda}\,d\theta < A,$$

where the A are independent of N.

The Cesàro sum formed from $f(z)$, with $z = e^{i\theta}$, of order

$$k = \frac{1}{\lambda} - 1$$

and rank N, is

$$s_N{}^k = A(N+1)^{-\beta}\sum_0^N (-1)^n \binom{N-n+k}{N-n}\binom{-p}{n} e^{ni\theta}.$$

We shall replace the second binomial coefficient by its asymptotic value $(-1)^n An^{p-1}$, and the first by

$$A(N-n)^k;$$

the second substitution amounts to passing from a Cesàro sum to the corresponding Rieszian sum. The substitutions may be justified by arguments similar to those which we use below to justify replacing a sum by the corresponding integral. Thus $s_N{}^k$ is replaced by

$$t_N{}^k = A(N+1)^{-\beta}\sum_0^N (N-n)^k n^{p-1} e^{ni\theta}.$$

We shall show that, if $\delta > 0$ is chosen appropriately,

$$\int_{N^{-1}}^{N^{-1+\delta}} |t_N{}^k|^\lambda d\theta > A\delta N^{k\lambda}\log N = A\delta N^{1-\lambda}\log N. \tag{5.2.1}$$

We replace $t_N{}^k$ by the corresponding integral

$$u_N{}^k = A(N+1)^{-\beta}\int_0^N (N-x)^k x^{p-1} e^{xi\theta}\,dx.$$

To justify this we observe that, when $n < x < n+1$,

$$(N-x)^k x^{p-1} e^{xi\theta} - (N-n)^k n^{p-1} e^{ni\theta}$$

is the sum of two terms of orders

$$O(N^{k+p-1}\theta), \quad O(N^{k+p-2})$$

respectively. The contributions of these are of orders

$$O(N^\sigma), \quad O(N^\tau),$$

where

$$\sigma = 1-2\lambda+\delta(1+\lambda),$$

$$\tau = 1-2\lambda+\delta.$$

Each of these is less than $1-\lambda$ if δ is small enough, and the error involved in replacing series by integral is then negligible.

Now

$$u_N{}^k = A(N+1)^{-\beta} N^{k+p} e^{Ni\theta} \int_0^1 y^k (1-y)^{p-1} e^{-Nyi\theta}\, dy.$$

The asymptotic behaviour of the last integral, for large positive $N\theta$, is known; the integral behaves like

$$A\omega(N\theta)^{-k-1},$$

where $|\omega| = 1$†. Hence, finally, we obtain

$$\int_{N^{-1}}^{N^{-1+\delta}} |u_N{}^k|^\lambda\, d\theta > AN^\varpi \int_{N^{-1}}^{N^{-1+\delta}} \frac{d\theta}{\theta} = A\delta N^\varpi \log N,$$

where

$$\varpi = \lambda(-\beta+k+p-k-1) = \lambda(-\beta+p-1) = 1-\lambda.$$

Replacing u_n first by t_n and then by s_n, we conclude that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |s_n{}^k|^\lambda\, d\theta > A\delta N^{1-\lambda} \log N = A\delta N^{k\lambda} \log N;$$

and this completes the proof.

---

† See, for example, Hardy and Littlewood (1), 216–217. There is a second asymptotic term of order

$$(N\theta)^{-p},$$

here negligible because $p > 1/\lambda = k+1$.

References.

1. G. H. Hardy and J. E. Littlewood, "The allied series of a Fourier series", *Proc. London Math. Soc.* (2), 24 (1924), 211–246.

2. ———, "Some new properties of Fourier constants", *Math. Annalen*, 97 (1926), 159–209.

3. ———, "A maximal theorem with function-theoretic applications", *Acta Math.*, 54 (1930), 81–116.

4. ———, "An additional note on Parseval's theorem", *Math. Zeitschrift*, 34 (1931), 634–636.

5. ———, "Some properties of conjugate functions", *Journal für Math.*, 167 (1931), 405–423.

6. K. Knopp, "Über Summen der Form $a_0 b_n + a_1 b_{n-1} + \ldots + a_n b_0$", *Rend. di Palermo*, 32 (1911), 95–110.

7. F. Riesz, "Über die Randwerte einer analytischen Funktion", *Math. Zeitschrift*, 18 (1923), 87–95.

ON THE MODULI OF REGULAR FUNCTIONS

By H. FRAZER.

[Received 26 August, 1932.—Read 10 November, 1932.]

1. In the first part of this paper we generalize the following result due to Fejér and F. Riesz*:

If $f(z)$ is regular inside and on a circle Γ, and D is any diameter of Γ, then

$$\int_D |f(z)|^2 |dz| \leqslant \tfrac{1}{2} \int_\Gamma |f(z)|^2 |dz|, \tag{1}$$

and, further, the constant $\frac{1}{2}$ is the best possible constant.

We deal with the case of n diameters $D_0, D_1, \ldots, D_{n-1}$ of Γ, and such that the angle between consecutive radii is π/n. We show that, *if $f(z)$ is regular inside and on Γ, and if $\lambda > 0$, then*

$$\sum_{r=0}^{n-1} \left\{ \int_{D_r} |f(z)|^\lambda |dz| \right\} \leqslant \frac{1}{2 \sin(\frac{1}{2}\pi/n)} \int_\Gamma |f(z)|^\lambda |dz|, \tag{2}$$

and, further, the constant $\frac{1}{2}\operatorname{cosec}(\frac{1}{2}\pi/n)$ is the best possible constant. (1) is the particular case $n = 1$ of (2).†

In § 3 we show that (2) implies results for regular polygons, and, in particular, that, *if $f(z)$ is regular inside and on a square S, and D_1, D_2 are the two diagonals of S, and if $\lambda > 0$, then*

$$\int_{D_1 + D_2} |f(z)|^\lambda |dz| \leqslant \frac{1}{\sqrt{2}} \int_S |f(z)|^\lambda |dz|. \tag{3}$$

From (3) we obtain a generalization of a theorem due to R. M. Gabriel‡, concerning functions regular in a rectangle. The theorem in question states that, *if $f(z)$ is regular inside and on a rectangle R, and C is any convex curve inside R, and if $\lambda > 0$, then*

$$\int_C |f(z)|^\lambda |dz| \leqslant A \int_R |f(z)|^\lambda |dz|. \tag{4}$$

---

* L. Fejér and F. Riesz, "Über einige funktionentheoretische Ungleichungen," *Math. Zeitschrift*, 11 (1921), 305–314.

† The result for general $\lambda > 0$ is, as we show later, implicit in the case $\lambda = 2$.

‡ R. M. Gabriel, "Some further results concerning the integrals of the moduli of regular functions along curves of certain types," *Proc. Camb. Phil. Soc.*, 25 (1929), 139–144.

The generalization consists in replacing the convex curve C by a type of curve of which the convex curve is a particular case. We consider a rectangle R of length n times the breadth, n being any positive integer, and imagine R as a sequence of n squares by supposing $n-1$ lines drawn parallel to the breadth and lying in R. We say that a curve C is "periodically convex" in R, if that portion C_r of C lying in the r-th square $(1 \leqslant r \leqslant n)$ is convex*. We show that, *if $f(z)$ is regular inside and on a rectangle R, and C is any "periodically convex" curve lying in R, and if $\lambda > 0$, then*

$$\int_C |f(z)|^\lambda |dz| \leqslant A \int_R |f(z)|^\lambda |dz|, \tag{5}$$

where A is an absolute constant.

In § 4 we are again concerned with regular functions in a regular polygon. It has been shown by R. M. Gabriel that, *if $f(z)$ is regular inside and on a circle Γ, and C is any convex curve inside Γ, and if $\lambda > 0$, then*

$$\int_C |f(z)|^\lambda |dz| \leqslant 2 \int_\Gamma |f(z)|^\lambda |dz|. \tag{6}$$†

Attempts to obtain a generalized form of this theorem with the circle Γ replaced by a convex curve Γ surrounding C have not been successful, but it has been shown by the same writer that, *if the circle Γ is replaced by an n-sided convex polygon Γ surrounding C, then*

$$\int_C |f(z)|^\lambda |dz| \leqslant A(n) \int_\Gamma |f(z)|^\lambda |dz|. \tag{7}$$‡

We prove a theorem of the "periodically convex" type for circles by means of which, together with (6) and (7), we are able to show that the constant $A(n)$ may be replaced by an absolute constant A in the case where the n-sided convex polygon Γ is regular.

We conclude this paper by mentioning a further generalization of the result due to Fejér and Riesz.

We denote by A an absolute constant not necessarily the same throughout.

* We observe that a "periodically convex" curve is not necessarily simple or closed, that a convex curve is a particular case of this more general curve, and that, for $n = 1$, the "periodically convex" curve is necessarily convex. The term is also used in this paper to denote the same type of curve in an infinite strip.

† R. M. Gabriel, "Some results concerning the integrals of the moduli of regular functions along curves of certain types", *Proc. London Math. Soc.* (2), 28 (1928), 121–127.

‡ R. M. Gabriel, *Proc. Camb. Phil. Soc., loc. cit.*

2. THEOREM 2.1. *If $f(z)$ is regular inside and on a circle Γ, and $D_0, D_1, D_2, \ldots, D_{n-1}$ are n diameters of Γ such that the angle between consecutive radii is π/n, and if $\lambda > 0$, then*

$$\sum_{r=0}^{n-1} \left\{ \int_{D_r} |f(z)|^\lambda |dz| \right\} \leqslant \frac{1}{2\sin(\frac{1}{2}\pi/n)} \int_\Gamma |f(z)|^\lambda |dz|.$$

We may take Γ to be the circle $|z| = 1$, and the diameter D_0 to be on the real axis. The n diameters subdivide Γ into $2n$ equal arcs, which we shall denote by $\Gamma_0, \Gamma_1, \ldots, \Gamma_{2n-1}$, and which are understood to be described in the positive direction, Γ_0 commencing at the point $z = 1$. We shall write

$$\Gamma_0 + \Gamma_1 + \ldots + \Gamma_{n-1} = K_1; \quad \Gamma_n + \Gamma_{n+1} + \ldots + \Gamma_{2n-1} = K_2;$$

so that K_1 and K_2 are the parts of Γ on either side of the real axis. By $\bar{f}(z)$ we shall mean the conjugate of $f(z)$, and by $\mathfrak{f}(\bar{z})$ we shall mean $\bar{f}(\bar{z})$, where $\bar{z}$ is the conjugate of z*. By a well-known theorem $\mathfrak{f}(\bar{z})$ is regular inside and on Γ. Then

$$\sum_{r=0}^{n-1} \left\{ \int_{D_r} |f(z)|^2 |dz| \right\} = \sum_{r=0}^{n-1} \left\{ \int_{D_r} f(z)\bar{f}(z)\,|dz| \right\}$$

$$= \sum_{r=0}^{n-1} \left\{ \int_{D_r} \left[\sum_{s=0}^{n-1} F_s(z) \right] \left[\sum_{s=0}^{n-1} \bar{F}_s(z) \right] |dz| \right\},$$

where

$$F_s(z) = \frac{1}{n} \sum_{t=0}^{n-1} \omega^{-st} f(\omega^t z) = \frac{1}{n} \sum_{t=0}^{n-1} e^{-2\pi st i/n} f(e^{2\pi t i/n} z),$$

and where $\bar{F}_s(z)$ is the conjugate of $F_s(z)$†, so the sum under consideration may be written

$$\sum_{r=0}^{n-1} \left\{ \int_{D_r} \left(\sum_{s=0}^{n-1} |F_s(z)|^2 \right) |dz| \right\} + \sum_{r=0}^{n-1} \left\{ \int_{D_r} \left(\sum_{\substack{s,\,u=0 \\ s \neq u}}^{n-1} F_s(z)\,\bar{F}_u(z) \right) |dz| \right\}$$

$$= I + J, \quad \text{say}$$

$$= I + \frac{1}{n^2} \sum_{r=0}^{n-1} \left\{ \int_{D_r} \left(\sum_{\substack{s,\,u=0 \\ s \neq u}}^{n-1} \left[\sum_{t=0}^{n-1} \omega^{(u-s)t} |f(\omega^t z)|^2 \right] \right) |dz| \right\}$$

$$+ \frac{1}{n^2} \sum_{r=0}^{n-1} \left\{ \int_{D_r} \left(\sum_{\substack{s,\,u=0 \\ s \neq u}}^{n-1} \left[\sum_{\substack{p,\,q=0 \\ p \neq q}}^{n-1} \omega^{qu-ps} f(\omega^p z)\,\bar{f}(\omega^q z) \right] \right) |dz| \right\}$$

$$= I + J_1 + J_2.$$

---

* If $f(z) = a_0 + a_1 z + \ldots$, then $\bar{f}(z) = \bar{a}_0 + \bar{a}_1 \bar{z} + \ldots$, and $\mathfrak{f}(\bar{z}) = \bar{a}_0 + \bar{a}_1 z + \ldots$.

† So that $F_s(z) = \sum_{\nu=0}^{\infty} a_{\nu n+s} z^{\nu n+s}$, and $\bar{F}_s(z) = \sum_{\nu=0}^{\infty} \bar{a}_{\nu n+s} \bar{z}^{\nu n+s}$. For brevity we shall write $\mathfrak{F}_s(\bar{z})$ for the conjugate of $F_s(\bar{z})$, so that $\mathfrak{F}_s(\bar{z}) = \sum_{\nu=0}^{\infty} \bar{a}_{\nu n+s} z^{\nu n+s}$. We observe that $\mathfrak{F}_s(\bar{z})$ is regular inside and on Γ.

Now

$$J_1 = \frac{1}{n^2}\sum_{r=0}^{n-1}\left\{\int_{D_r}\left[n(n-1)|f(z)|^2 - n\sum_{t=1}^{n-1}|f(\omega^t z)|^2\right]|dz|\right\}$$

$$= 0,$$

since $$\int_{D_r}|f(z)|^2|dz| = \int_{D_0}|f(\omega^{\frac{1}{2}r}z)|^2|dz| = \ldots;$$

also

$$J_2 = \frac{1}{n^2}\sum_{r=0}^{n-1}\left[\int_{D_r}\left\{\sum_{\substack{p,\,q=0\\ p\neq q}}^{n-1}\left[f(\omega^p z)\bar{f}(\omega^q z)\left(\sum_{\substack{s,\,u=0\\ s\neq u}}^{n-1}\omega^{qu-ps}\right)\right]\right\}|dz|\right]$$

$$= 0,$$

since $$\sum_{\substack{s,\,u=0\\ s\neq u\\ p\neq q}}^{n-1}\omega^{qu-ps} = 0.$$

Hence

$$\sum_{r=0}^{n-1}\left\{\int_{D_r}|f(z)|^2|dz|\right\} = I = \sum_{r=0}^{n-1}\left\{\int_{D_r}\left(\sum_{s=0}^{n-1}|F_s(z)|^2\right)|dz|\right\}$$

$$= \sum_{r=0}^{n-1}\left\{\int_{D_r}\left(\sum_{s=0}^{n-1}F_s(z)\,\bar{F}_s(z)\right)|dz|\right\}$$

$$= \sum_{r=0}^{n-1}\left\{\int_{D_r}\left(\sum_{s=0}^{n-1}\omega^{-rs}F_s(z)\,\bar{\mathfrak{F}}_s(\bar{z})\right)|dz|\right\};$$

we get the last expression because on D_r we have

$$\bar{F}_s(z) = \sum_{\nu=0}^{\infty}\bar{a}_{\nu n+s}\bar{z}^{\nu n+s} = \bar{z}^s\sum_{\nu=0}^{\infty}\bar{a}_{\nu n+s}\bar{z}^{\nu n} = z^{-s}\sum_{\nu=0}^{\infty}\bar{a}_{\nu n+s}z^{\nu n}$$

$$= e^{-2\pi sri/n}z^s\sum_{\nu=0}^{\infty}\bar{a}_{\nu n+s}z^{\nu n}$$

$$= \omega^{-rs}\sum_{\nu=0}^{\infty}\bar{a}_{\nu n+s}z^{\nu n+s} = \omega^{-rs}\mathfrak{F}_s(\bar{z}).$$

Resuming, we have

$$\sum_{r=0}^{n-1}\left\{\int_{D_r}|f(z)|^2|dz|\right\}$$

$$= \left|\sum_{r=0}^{n-1}\left\{\omega^{-\frac{1}{2}r}\int_{D_r}\left(\sum_{s=0}^{n-1}\omega^{-rs}F_s(z)\,\mathfrak{F}_s(\bar{z})\right)dz\right\}\right|$$

$$= \tfrac{1}{2}\left|\sum_{s=0}^{n-1}\left[\sum_{u=0}^{n-1}\left\{\int_{\Gamma_u}F_s(z)\,\mathfrak{F}_s(\bar{z})\left(\sum_{p=u+1}^{n-1}-\sum_{p=0}^{u}\right)\omega^{-p(s+\frac{1}{2})}dz\right\}\right.\right.$$

$$\left.\left.+\sum_{u=n}^{2n-1}\left\{\int_{\Gamma_u}F_s(z)\,\mathfrak{F}_s(\bar{z})\left(\sum_{p=0}^{u-n}-\sum_{p=u-n+1}^{n-1}\right)\omega^{-p(s+\frac{1}{2})}dz\right\}\right]\right|$$

(by Cauchy's theorem)

$$= \tfrac{1}{2}\left|\sum_{s=0}^{n-1}\left[\sum_{u=0}^{2n-1}\left\{\int_{\Gamma_u} F_s(z)\,\mathfrak{F}_s(\bar{z})\,\frac{2\omega^{-(u+1)(s+\frac{1}{2})}}{1-\omega^{-(s+\frac{1}{2})}}\,dz\right\}\right]\right| = K,$$

since
$$\left(\sum_{p=u+1}^{n-1} - \sum_{p=0}^{u}\right)\omega^{-p(s+\frac{1}{2})} = \frac{2\omega^{-(u+1)(s+\frac{1}{2})} - \omega^{-n(s+\frac{1}{2})} - 1}{1-\omega^{-(s+\frac{1}{2})}}$$

$$= \frac{2\omega^{-(u+1)(s+\frac{1}{2})}}{1-\omega^{-(s+\frac{1}{2})}}$$

$$= \left(\sum_{p=0}^{u-n} - \sum_{p=u-n+1}^{n-1}\right)\omega^{-p(s+\frac{1}{2})}.$$

Now

$$K \leqslant \tfrac{1}{2}\sum_{s=0}^{n-1}\left[\sum_{u=0}^{2n-1}\left\{\int_{\Gamma_u} |F_s(z)||\mathfrak{F}_s(\bar{z})|\,\frac{1}{|\sin\{\pi(s+\frac{1}{2})/n\}|}\,|dz|\right\}\right]$$

$$= \tfrac{1}{2}\sum_{s=0}^{n-1}\frac{1}{|\sin\{\pi(s+\frac{1}{2})/n\}|}\left\{\int_{\Gamma} |F_s(z)||\mathfrak{F}_s(\bar{z})||dz|\right\}$$

$$\leqslant \frac{1}{2\sin(\frac{1}{2}\pi/n)}\sum_{s=0}^{n-1}\left\{\int_{K_1+K_2} |F_s(z)||\mathfrak{F}_s(\bar{z})||dz|\right\}$$

$$= \frac{1}{2\sin(\frac{1}{2}\pi/n)}\sum_{s=0}^{n-1} I_s,$$

and

$$I_s \leqslant \left(\int_{K_1} |F_s(z)|^2|dz|\right)^{\frac{1}{2}}\left(\int_{K_1} |\mathfrak{F}_s(\bar{z})|^2|dz|\right)^{\frac{1}{2}}$$

$$+\left(\int_{K_2} |F_s(z)|^2|dz|\right)^{\frac{1}{2}}\left(\int_{K_2} |\mathfrak{F}_s(\bar{z})|^2|dz|\right)^{\frac{1}{2}}$$

$$= \left(\int_{K_1} |F_s(z)|^2|dz|\right)^{\frac{1}{2}}\left(\int_{K_2} |F_s(z)|^2|dz|\right)^{\frac{1}{2}}$$

$$+\left(\int_{K_2} |F_s(z)|^2|dz|\right)^{\frac{1}{2}}\left(\int_{K_1} |F_s(z)|^2|dz|\right)^{\frac{1}{2}}$$

$$\leqslant \tfrac{1}{2}\left(\int_{K_1} |F_s(z)|^2|dz| + \int_{K_2} |F_s(z)|^2|dz|\right)$$

$$+\tfrac{1}{2}\left(\int_{K_2} |F_s(z)|^2|dz| + \int_{K_1} |F_s(z)|^2|dz|\right)$$

$$= \int_{K_1} |F_s(z)|^2|dz| + \int_{K_2} |F_s(z)|^2|dz|;$$

therefore

$$\sum_{r=0}^{n-1}\left\{\int_{D_r}|f(z)|^2|dz|\right\} \leqslant \frac{1}{2\sin(\frac{1}{2}\pi/n)}\sum_{s=0}^{n-1}\left\{\int_{\Gamma}|F_s(z)|^2|dz|\right\}$$

$$= \frac{1}{2\sin(\frac{1}{2}\pi/n)}\int_{\Gamma}\sum_{s=0}^{n-1}|F_s(z)|^2|dz|$$

$$= \frac{1}{2\sin(\frac{1}{2}\pi/n)}\int_{\Gamma}\left|\sum_{s=0}^{n-1}F_s(z)\right|^2|dz|$$

(the cross terms vanishing in the same way as in J_1, J_2)

$$= \frac{1}{2\sin(\frac{1}{2}\pi/n)}\int_{\Gamma}|f(z)|^2|dz|.$$

The theorem is thus proved for the case $\lambda = 2$. The extension to general $\lambda > 0$ follows immediately if $f(z)$ has no zeros inside or on Γ, for $f^{\frac{1}{2}\lambda}(z)$ is regular inside and on Γ. Now suppose that $f(z)$ has zeros at the points $z_1, z_2, \ldots, z_k$ (k being necessarily finite), multiple zeros counting multiply; we may suppose that $f(z)$ has no zeros on Γ. Then

$$f_1(z) = f(z)\Big/\prod_{r=1}^{k}\left\{\frac{z-z_r}{1-z\bar{z}_r}\right\}$$

is regular, and has no zeros inside or on Γ; also

$$|f(z)| = |f_1(z)|$$

on Γ, and

$$|f(z)| < |f_1(z)|$$

inside Γ. Therefore

$$\sum_{r=0}^{n-1}\left\{\int_{D_r}|f(z)|^\lambda|dz|\right\} \leqslant \sum_{r=0}^{n-1}\left\{\int_{D_r}|f_1(z)|^\lambda|dz|\right\} \leqslant \frac{1}{2\sin(\frac{1}{2}\pi/n)}\int_{\Gamma}|f_1(z)|^\lambda|dz|$$

$$= \frac{1}{2\sin(\frac{1}{2}\pi/n)}\int_{\Gamma}|f(z)|^\lambda|dz|.$$

We now proceed to show that the constant $\frac{1}{2}\operatorname{cosec}(\frac{1}{2}\pi/n)$ in Theorem 2.1 is the best possible constant. If $f(z)$ is regular inside and on Γ, so is its derivative $f'(z)$, and hence

$$\sum_{r=0}^{n-1}\left\{\int_{D_r}|f'(z)||dz|\right\} \leqslant \frac{1}{2\sin(\frac{1}{2}\pi/n)}\int_{\Gamma}|f'(z)||dz|.$$

It follows that, if $\phi(z)$ is regular and schlicht in $|z| \leqslant 1$, and $\zeta = \phi(z)$

represents the unit circle Γ in the z plane conformally on the curve Γ' in the ζ plane, and the diameters $D_0, D_1, \ldots, D_{n-1}$ of Γ (as defined in Theorem 2.1) on the curves $D_0', D_1', \ldots, D_{n-1}'$ in the ζ plane; and if $\bar{\Gamma}'$ is the length of Γ', and $\bar{D}_r'$ is the length of D_r' $(0 \leqslant r \leqslant n-1)$, then

$$\sum_{r=0}^{n-1} \bar{D}_r' \leqslant \frac{1}{2 \sin(\frac{1}{2}\pi/n)} \bar{\Gamma}'.$$

Now the transformation

$$\zeta = \int_0^z \frac{dz}{(1-z^{2n})^{1/n}} \tag{8}$$

represents the interior of the circle Γ on the interior of the regular convex polygon Γ' of $2n$ sides in the ζ plane, with the diameters $D_0, D_1, \ldots, D_{n-1}$ of Γ corresponding to the diagonals $D_0', D_1', \ldots, D_{n-1}'$ of Γ' passing through the centre of Γ'. Now, if Γ_ρ' is the transform in the ζ plane of $|z| = \rho < 1$ in the z plane, by a theorem due to Hardy*, we have,

$$\bar{\Gamma}_\rho' \leqslant \bar{\Gamma}';$$

hence, using an obvious notation, we have, by Theorem 2.1,

$$\sum_{r=0}^{n-1} \bar{D}_{r\rho}' \leqslant \frac{1}{2 \sin(\frac{1}{2}\pi/n)} \bar{\Gamma}_\rho' \leqslant \frac{1}{2 \sin(\frac{1}{2}\pi/n)} \bar{\Gamma}'.$$

Since (as can be shown by a simple calculation)

$$\sum_{r=0}^{n-1} \bar{D}_{r\rho}' \to \frac{1}{2 \sin(\frac{1}{2}\pi/n)} \bar{\Gamma}'$$

as $\rho \to 1-0$, it follows that the function

$$f(z) = \left[1 - \left(\frac{z}{R}\right)^{2n}\right]^{-1/\lambda n} \quad (R > 1;\ \lambda > 0)$$

is such that

$$\frac{\sum_{r=0}^{n-1} \left\{ \int_{D_r} |f(z)|^\lambda |dz| \right\}}{\int_\Gamma |f(z)|^\lambda |dz|} \to \frac{1}{2 \sin(\frac{1}{2}\pi/n)}$$

as $R \to 1+0$; that is, the constant $\frac{1}{2}\operatorname{cosec}(\frac{1}{2}\pi/n)$ of Theorem 2.1 is the best possible constant.

---

* G. H. Hardy, "The mean value of the modulus of an analytic function", *Proc. London Math. Soc.* (2), 14 (1915), 269–277.

In the remaining portion of this section we deal with subharmonic functions.

THEOREM 2.2. *If $U(z)$ is subharmonic, positive, and continuous inside and on a circle Γ, and if $D_0, D_1, \ldots, D_{n-1}$ are n diameters of Γ such that the angles between consecutive radii is π/n, then*

$$\sum_{r=0}^{n-1} \left\{ \int_{D_r} U^\lambda(z)\,|dz| \right\} \leqslant \frac{1}{\sin(\frac{1}{2}\pi/n)} \int_\Gamma U^\lambda(z)\,|dz|$$

if $\lambda \geqslant 2$, and

$$\sum_{r=0}^{n-1} \left\{ \int_{D_r} U^\lambda(z)\,|dz| \right\} \leqslant \frac{An}{\lambda - 1} \int_\Gamma U^\lambda(z)\,|dz|$$

if $1 < \lambda < 2$.

This follows by the methods of a previous paper* applied to Theorem 2.1.

Fejér and Riesz in their paper (*loc. cit.*) mention that their theorem implies that if, with fixed $\lambda > 0$,

$$\int_{-\pi}^{\pi} |f(re^{i\theta})|^\lambda d\theta \quad (r < 1)$$

is bounded, then

$$\int_{-r}^{r} |f(\rho e^{i\theta})|^\lambda d\rho \quad (r < 1)$$

is bounded. They state that the corresponding result is not true for real harmonic functions, and, in fact, by means of conjugate functions they deduce from (1) that

$$\int_{-\pi}^{\pi} u^2(r, \theta)\, d\theta < A \quad (r < 1)$$

implies that, for $r < 1$, then

$$\int_{-r}^{r} u^2(\rho, \theta)\, d\rho$$

is bounded; on the other hand, they show by means of an example that

* H. Frazer, "Some inequalities concerning the integrals of positive subharmonic functions along curves of certain types", *Journal London Math. Soc.*, 6 (1931), 113–117. The result for $\lambda \geqslant 2$ is new, but the one for $1 < \lambda < 2$ can be obtained more simply from the results of the paper quoted.

the boundedness of $\int_{-\pi}^{\pi} |u(r, \theta)|\, d\theta \quad (r<1)$

does not imply that $\int_{-r}^{r} |u(\rho, \theta)|\, d\rho \quad (r<1)$

is bounded.

From Theorem 2.2 it follows that if, with fixed $\lambda > 1$,

$$\int_{-\pi}^{\pi} |u(r, \theta)|^{\lambda} d\theta \quad (r<1)$$

is bounded, then $\int_{-r}^{r} |u(\rho, \theta)|^{\lambda} d\rho \quad (r<1)$

is bounded. We conclude this section by showing how we can obtain this result simply, by means of analysis which does not refer to complex number theory and without appealing to the theory of conjugate functions. For if $U(r, \psi)$ is subharmonic, positive, and continuous inside and on the circle $|z|=1$, and if $u(\rho, \theta)$ is the Poisson integral of $U(1, \psi)$, then for $\lambda > 1$, and for fixed r such that $\frac{1}{2} < r < 1$, we have

$$\begin{aligned}\int_0^r u^{\lambda}(\rho, 0)\, d\rho &= \left(\int_0^{\frac{1}{2}} + \int_{\frac{1}{2}}^{r}\right) u^{\lambda}(\rho)\, d\rho \\ &\leqslant A \int_{-\pi}^{\pi} U(\psi)\, d\psi + \int_{\frac{1}{2}}^{r} u^{\lambda}(\rho)\, d\rho.\end{aligned}$$

Now

$$\begin{aligned}\int_{\frac{1}{2}}^{r} u^{\lambda}(\rho)\, d\rho &= \int_{\frac{1}{2}}^{r} u^{\lambda-1}(\rho) \left\{\frac{1}{2\pi} \int_{-\pi}^{\pi} U(\psi)\, \frac{1-\rho^2}{1-2\rho\cos\psi+\rho^2}\, d\psi\right\} d\rho \\ &\leqslant A \int_{1-r}^{\frac{1}{2}} u^{\lambda-1}(t) \left\{\int_0^{\pi} [U(\psi)+U(-\psi)]\, \frac{t}{t^2+\psi^2}\, d\psi\right\} dt \\ &= A \int_{1-r}^{\frac{1}{2}} u^{\lambda-1}(t) \left\{\left(\int_0^{t} + \int_t^{\pi}\right) [U(\psi)+U(-\psi)]\, \frac{t}{t^2+\psi^2}\, d\psi\right\} dt \\ &\leqslant A \int_{1-r}^{\frac{1}{2}} u^{\lambda-1}(t) \left\{\frac{1}{t} \int_0^{t} [U(\psi)+U(-\psi)]\, d\psi\right\} dt \\ &\qquad + A \int_{1-r}^{\frac{1}{2}} u^{\lambda-1}(t) \left\{\int_0^{\pi} \frac{U(\psi)+U(-\psi)}{t+\psi}\, d\psi\right\} dt \\ &\leqslant A(\lambda) \left[\int_{1-r}^{\frac{1}{2}} u^{\lambda}(t)\, dt\right]^{(\lambda-1)/\lambda} \left[\int_0^{\pi} [U(\psi)+U(-\psi)]^{\lambda}\, d\psi\right]^{1/\lambda} \\ &\leqslant A(\lambda) \left[\int_{1-r}^{\frac{1}{2}} u^{\lambda}(t)\, dt\right]^{(\lambda-1)/\lambda} \left[\int_{-\pi}^{\pi} U^{\lambda}(\psi)\, d\psi\right]^{1/\lambda}.\end{aligned}$$

Hence $$\int_0^r u^\lambda(\rho)\,d\rho \leqslant A(\lambda)\int_{-\pi}^{\pi} U^\lambda(e^{i\psi})\,d\psi.$$

The required result now follows without difficulty.

3. LEMMA 3.1*. *Suppose that C_1 and C_2 are curves lying in a domain D, and that a transformation regular and schlicht in D puts C_1 on C_1', C_2 on C_2', and D on D'. If*

$$\int_{C_1}|f(z)|^\lambda|dz| \leqslant B\int_{C_2}|f(z)|^\lambda|dz| \quad (\lambda>0)$$

for all $f(z)$ regular in D, then

$$\int_{C_1'}|f(z)|^\lambda|dz| \leqslant B\int_{C_2'}|f(z)|^\lambda|dz| \quad (\lambda>0)$$

for all $f(z)$ regular in D'.

THEOREM 3.2. *If $f(z)$ is regular inside and on a regular polygon Γ of $2n$ sides, and if $D_0, D_1, \ldots, D_{n-1}$ are the diagonals of Γ passing through the centre of Γ, and if $\lambda>0$, then*

$$\sum_{r=0}^{n-1}\left(\int_{D_r}|f(z)|^\lambda|dz|\right) \leqslant \frac{1}{2\sin(\frac{1}{2}\pi/n)}\int_\Gamma|f(z)|^\lambda|dz|.$$

This follows from Theorem 2.2 and (8) by Lemma 3.1.

LEMMA 3.3.† *If $f(z)$ is regular inside and on a simple closed curve K symmetrical about a straight line B, then*

$$\int_B|f(z)|^2|dz| \leqslant \tfrac{1}{2}\int_K|f(z)|^2|dz|.$$

THEOREM 3.4. *If $f(z)$ is regular inside and on a rectangle R, and C is any "periodically convex" curve lying in R, and if $\lambda>0$, then*

$$\int_C|f(z)|^\lambda|dz| \leqslant A\int_R|f(z)|^\lambda|dz|.$$

We may take R to be the rectangle formed by joining the points $(0, 0)$, $(na, 0)$, (na, ia), $(0, ia)$, which we shall denote by R_0, R_n, R_n', R_0'.

* R. M. Gabriel, *Proc. London Math. Soc.*, *loc. cit.*

† R. M. Gabriel, *Proc. London Math. Soc.*, *loc. cit.*

Subdivide $R_0 R_n$ into n equal parts by points $R_r (1 \leqslant r \leqslant n-1)$, and similarly subdivide $R_0' R_n'$ by points R_r'. In order to prove our theorem for the case $\lambda = 2$, it will suffice to show that

$$\sum_{r=1}^{n-1} \left\{ \int_{R_r R_r'} |f(z)|^2 |dz| \right\} \leqslant A \int_R |f(z)|^2 |dz|, \tag{9}$$

for then

$$\begin{aligned} \int_C |f(z)|^2 |dz| &\leqslant \sum_{r=1}^{n} \left\{ \int_{C_r} |f(z)|^2 |dz| \right\} \\ &\leqslant A \sum_{r=1}^{n-1} \left\{ \int_{R_r R_r'} |f(z)|^2 |dz| \right\} + A \int_R |f(z)|^2 |dz| \\ &\leqslant A \int_R |f(z)|^2 |dz| \end{aligned}$$

by (4). Now, by Lemma 3.3,

$$\begin{aligned} \sum_{r=1}^{n-1} \left\{ \int_{R_r R_r'} |f(z)|^2 |dz| \right\} &\leqslant \sum_{r=1}^{n-1} \left\{ \tfrac{1}{2} \int_{R_r R_{r-1}' + R_{r-1}' R_{r+1} + R_{r+1}' R_r} |f(z)|^2 |dz| \right\} \\ &= \sum_{r=1}^{n-1} \left\{ \tfrac{1}{2} \int_{R_r R_{r-1}' + R_{r-1}' R_r} |f(z)|^2 |dz| \right\} + A \int_R |f(z)|^2 |dz| \\ &\leqslant \frac{1}{\sqrt{2}} \left\{ \sum_{r=1}^{n-1} \int_{R_r R_r'} |f(z)|^2 |dz| \right\} + A \int_R |f(z)|^2 |dz| \end{aligned}$$

by (3). That is,

$$\begin{aligned} \sum_{r=1}^{n-1} \left\{ \int_{R_r R_r'} |f(z)|^2 |dz| \right\} &\leqslant \frac{\sqrt{2}\, A}{\sqrt{2}-1} \int_R |f(z)|^2 |dz| \\ &= A \int_R |f(z)|^2 |dz|. \end{aligned}$$

The theorem for $\lambda = 2$ now follows. The extension to general $\lambda > 0$ follows by a method similar to that for a circle, and presents no difficulty.

Our last theorem has an obvious extension to the case of an infinite strip*.

THEOREM 3.5. *If $f(z)$ is regular inside and on the boundary L of the infinite strip $\alpha \leqslant x \leqslant \beta$, and if $f(z) = o(1)$ uniformly in $\alpha \leqslant x \leqslant \beta$, as $|y| \to \infty$;*

* It is possible to prove Theorem 3.5 more directly, without appealing to Theorem 3.4, by a consideration of Poisson's integral for the strip. In fact, it was a consideration of Theorem 3.5 proved in this way that led us to suspect the existence of Theorem 3.4.

if, further, C is a "periodically convex" curve lying in the strip, and if $\lambda > 0$, then

$$\int_C |f(z)|^\lambda |dz| \leqslant A \int_L |f(z)|^\lambda |dz|.$$

4. In the present section our main purpose is to prove

THEOREM 4.1. *If $f(z)$ is regular inside and on a regular polygon Γ, and if C is any convex curve inside Γ, and if $\lambda > 0$, then*

$$\int_C |f(z)|^\lambda |dz| \leqslant A \int_\Gamma |f(z)|^\lambda |dz|.$$

In order to simplify the presentation of the proof we shall suppose throughout that $n \geqslant 16$, where n is the number of sides of Γ. The result for $n < 16$ follows from (7). We shall also suppose for the sake of simplicity that n is even. A consideration of the argument will reveal the fact that a simple and obvious modification enables us to obtain a similar result for n odd.

LEMMA 4.2*. *If $U(re^{i\theta})$ is subharmonic, positive, and continuous in $0 \leqslant r \leqslant 1$, and if $u(re^{i\theta})$ is the Poisson integral of $U(e^{i\theta})$, then*

$$U(re^{i\theta}) \leqslant u(re^{i\theta})$$

in $0 \leqslant r < 1$.

THEOREM 4.3. *Suppose that $U(z)$ is subharmonic, positive, and continuous inside and on the circle $|z| = 1$, and that $D_0, D_1, \ldots, D_{n-1}$ are n diameters of the circle such that the angle between consecutive radii is π/n. If $\{C_r\}$ denotes the sequence of $2n$ circles lying in the unit circle, and with centres at the points of intersection of the n diameters with the circle $|z| = 1-\pi/n$, then*

$$\int_{\{C_r\}} U(z)|dz| \leqslant A \int_{|z|=1} U(z)|dz|.$$

We may take D_0 to be on the real axis. Let $\{C_r'\}$ denote the series of $2n$ circles concentric with $\{C_r\}$ and lying strictly inside $|z| = 1$, and such that C_r' is of radius $\rho_r < \pi/n$. Let $u(z)$ be the Poisson integral of $U(e^{i\theta})$. Since $u(z)$ is a potential function, we have

$$\int_{C_r'} u(z)|dz| = 2\pi\rho_r u\left[\left(1+\frac{\pi}{n}\right)e^{\pi i/n}\right];$$

* This is a particular case of a lemma due to F. Riesz. See F. Riesz, "Sur les fonctions subharmoniques et leur rapport à la théorie du potentiel", *Acta Math.*, 48 (1926), 329–343.

hence

$$\int_{C_{r'}} u(z)|dz| = 2\pi\rho_r \left\{ \frac{1}{2\pi}\int_{-\pi}^{\pi} U(e^{i\psi}) \frac{1-(1-\pi/n)^2}{1-2(1-\pi/n)\cos(\psi-r\pi/n)+(1-\pi/n)^2}\, d\psi \right\}$$

$$\leqslant \frac{A}{n^2}\int_{-\pi}^{\pi} \frac{U(e^{i\psi})}{1/n^2+(\psi-r\pi/n)^2}\, d\psi;$$

whence it follows that

$$\int_{\{C_{r'}\}} u(z)|dz| \leqslant \frac{A}{n^2}\sum_{r=0}^{2n-1}\left\{\int_{-\pi}^{\pi} \frac{U(e^{i\psi})}{(1/n^2)+(\psi-r\pi/n)^2}\, d\psi\right\}$$

$$\leqslant A\int_{-\pi}^{\pi} U(e^{i\psi})\, d\psi.$$

Applying Lemma 4.2, we have

$$\int_{\{C_{r'}\}} U(z)|dz| \leqslant \int_{\{C_{r'}\}} u(z)|dz| \leqslant A\int_{|z|=1} U(z)|dz|.$$

The theorem now follows by continuity.

THEOREM 4.4. *With the notation of the last theorem let $\mathfrak{D}$ denote those portions of the n diameters lying in the circles $\{C_r\}$, and let $f(z)$ be regular inside and on $|z|=1$; then, for $\lambda>0$,*

$$\int_{\mathfrak{D}} |f(z)^{\lambda}|dz| \leqslant A\int_{|z|=1} |f(z)|^{\lambda}|dz|.$$

This follows from Theorem 4.3 by (6).

LEMMA 4.5. *If $f(z)$ is regular inside and on any convex polygon Γ, and L is a chord of Γ dividing the polygon into two portions Γ_1, Γ_2; if, further, either Γ_1 or Γ_2 is composed of not more than two linear segments, then, for $\lambda>0$,*

$$\int_L |f(z)|^{\lambda}|dz| \leqslant A\int_{\Gamma} |f(z)|^{\lambda}|dz|.$$

This follows by an obvious modification of the argument used by Gabriel in his proof of (7).

LEMMA 4.6. *There is a regular and schlicht transformation representing the inside of the circle $|z|=1$ conformally on the inside of a regular polygon Γ of $2n$ sides, and such that*

(a) *the n diameters $D_0, D_1, \ldots, D_{n-1}$, as defined in Theorem 4.3 (with D_0 on the real axis), transform into the n diagonals of Γ passing through the*

centre of the polygon, and

(*b*) *any circle lying strictly inside* $|z|=1$ *is transformed into a convex curve inside the polygon.*

Consider the transformation

$$\zeta = \phi(z) = \int_0^z \frac{dz}{(1-z^{2n})^{1/n}}.$$

(*a*) holds by the argument of § 2. It remains to consider (*b*). We merely observe that

$$\Re z \frac{\phi''(z)}{\phi'(z)} > -1$$

for $|z|<1$; whence (*b*) follows*.

LEMMA 4.7. *If* $f(z)$ *is regular inside and on a regular polygon* Γ *of* $2n$ *sides, and if* Γ' *is the polygon of* $2n$ *sides whose vertices are the mid-points of the sides of* Γ, *and if* $\lambda>0$, *then*

$$\int_{\Gamma'} |f(z)|^\lambda |dz| \leqslant A \int_\Gamma |f(z)|^\lambda |dz|.$$

This follows without difficulty from Theorem 4.3, Theorem 4.4, Lemma 4.5, Lemma 4.6, and Lemma 3.1.

We now proceed to prove Theorem 4.1 in the case in which the number of sides of the regular polygon is even. We first observe that, by Lemma 4.7 and (7), the result follows in the case where the convex curve C is the largest circle C' that can be inscribed in Γ. Now if C is any convex curve inside Γ, then C is wholly contained in this circle C' and in the $2n$ triangles comprising the domain $\Gamma-\Gamma'$. Hence, by (6) and (7), we have

$$\int_C |f(z)|^\lambda |dz| \leqslant A \int_{C'} |f(z)|^\lambda |dz| + A \int_{\Gamma'} |f(z)|^\lambda |dz| \leqslant A \int_\Gamma |f(z)|^\lambda |dz|.$$

A consideration of the arguments of the present section reveals the fact that the theorem is equally true for n odd.

5. A simple modification of the proof of Theorem 2.1 enables us to prove the following further generalization.

THEOREM 5.1. *If* $f(z)$ *is regular inside and on a circle* Γ, *and if* $R_0, R_1, \ldots, R_{n-1}$ *are* n *radii of* Γ *such that the angle between consecutive*

* See Pólya and Szegö, *Aufgaben und Lehrsätze aus der Analysis*, 1 (1925), 105 (108), 145 (318).

radii is $2\pi/n$, *and if* $\lambda > 0$, *then*

$$\sum_{r=0}^{n-1}\left\{\int_{R_r}|f(z)|^\lambda|dz|\right\} \leqslant \frac{1}{2\sin(\pi/n)}\int_\Gamma|f(z)|^\lambda|dz| \quad (n \geqslant 2),$$

and, further, the constant $\frac{1}{2}\operatorname{cosec}(\pi/n)$ *is the best possible constant.*

We proceed as before, but, instead of introducing the function $\mathfrak{F}_s(z)$ and then applying Cauchy's theorem, we appeal to the following well-known inequalities:

If the a's are real and non-negative, and if $\alpha > 0$, we have

$$\sum_{r=0}^{n}\sum_{s=0}^{n}\frac{a_r a_s}{r+s+\alpha} \leqslant \pi\sum_{t=0}^{n}a_t^2$$

if α is an integer, and

$$\sum_{r=0}^{n}\sum_{s=0}^{n}\frac{a_r a_s}{r+s+\alpha} \leqslant \frac{\pi}{|\sin(\alpha\pi)|}\sum_{t=0}^{n}a_t^2$$

if α is not an integer.

We can show that the constant in Theorem 5.1 is best possible, in much the same way as before, by considering the transformation

$$\zeta = \int_0^z \frac{dz}{(1-z^n)^{2/n}}.$$

We conclude this paper by observing that Theorems 2.1 and 3.2 are particular cases of a more general theorem that can be proved by the method of Theorem 2.1.

THEOREM 5.2. *If* $f(z)$ *is regular inside and on a simple closed curve* Γ *which is symmetrical about each of* n *intersecting straight lines* $L_0, L_1, \ldots, L_{n-1}$ *and such that the angle between any two consecutive lines is* π/n, *then*

$$\sum_{r=0}^{n-1}\left\{\int_{L_r}|f(z)|^2|dz|\right\} \leqslant \frac{1}{2\sin(\frac{1}{2}\pi/n)}\int_\Gamma|f(z)|^2|dz|.$$

A BOUNDARY-VALUE PROBLEM FOR HARMONIC FUNCTIONS

By W. G. L. SUTTON.

[Received 17 September, 1932.—Read 10 November, 1932.]

1. Let n be the inner normal and s the arc of a simple closed plane curve C of continuous curvature and length l. It is well-known that the function u, harmonic throughout the interior S of C, continuous in $S+C$, and satisfying on C a linear boundary condition involving only u, $\partial u/\partial n$, $\partial u/\partial s$, is representable as the potential of a simple layer on C whose density satisfies an integral equation of the second kind. If, however, the boundary condition involves $\partial^2 u/\partial s^2$, this representation fails since it does not yield a convergent expression for this derivate. In this paper we give another method which reduces a class of such problems to a symmetric integral equation of the first kind.

2. We consider the determination of u subject to the boundary condition

$$\frac{\partial u}{\partial n}+L[u]=f, \tag{1}$$

where

$$L[u]\equiv\frac{\partial}{\partial s}\left(p\frac{\partial u}{\partial s}\right)-qu$$

and $p(s)$, $p'(s)$, $q(s)$, $f(s)$ are continuous on C. We assume further that

$$\int_C q\,ds=\kappa\neq 0. \tag{2}$$

Now let $G(M, P)$ be a Green's function of the second kind for C such that, for P on C, M in $S+C$,

$$\frac{\partial G}{\partial n_P}=\frac{2\pi}{\kappa}q(P) \tag{3}*$$

and

$$\int_C G(M, P)q(P)\,ds_P=0. \tag{4}$$

* Thus $\int_C \frac{\partial G}{\partial n}\,ds=2\pi$, as required for the existence of $G(M, P)$.

It is known* that $G(M, P)$ is uniquely determined, that

$$G(M, P) = G(P, M) \tag{5}$$

for all pairs of different points M, P in $S+C$, and that

$$|G(M, P)| < 2\left|\log \frac{1}{MP}\right| + \text{constant} \tag{6}$$

for M, P in $S+C$.

Suppose now that u is any solution of problem (1) with $\partial u/\partial n$ continuous on C. Then, necessarily, $L[u]$ is continuous on C and

$$\int_C \frac{\partial u}{\partial n}\, ds = 0, \tag{7}$$

which, by (1), yields

$$\int_C (qu+f)\, ds = 0. \tag{8}$$

Further, we have, for all points M in $S+C$,

$$\begin{aligned} u(M) &= \frac{1}{2\pi}\int_C \left\{u(P)\frac{\partial G}{\partial n_P} - G(M, P)\frac{\partial u}{\partial n_P}\right\} ds_P \\ &= \frac{1}{\kappa}\int_C qu\, ds + \frac{1}{2\pi}\int_C \{L[u(P)] - f(P)\}\, G(M, P)\, ds_P. \end{aligned}$$

By (2) and (4), the solution of this equation is indeterminate to the extent of an arbitrary additive constant. By (2) we can adjust the constant to satisfy (8) so that the solution sought must satisfy the equation

$$u(M) = \frac{1}{2\pi}\int_C L[u(P)]\, G(M, P)\, ds_P - \frac{1}{2\pi}\int_C f(P)\left\{G(M, P) + \frac{2\pi}{\kappa}\right\} ds_P. \tag{9}$$

Conversely, let u be a solution of (9) with $L[u]$ continuous on C. Then

$$\int_C q(P)\, u(P)\, ds_P = \frac{1}{2\pi}\int_C q(P)\, ds_P \int_C \{L[u(P')] - f(P')\}\, G(P, P')\, ds_{P'} - \int_C f(P)\, ds_P.$$

By (6) we may change the order of integration. Doing this and using (5), we find that the double integral vanishes by (4). Hence (8) holds, and hence

$$\int_C \{L[u(P)] - f(P)\}\, ds_P = 0.$$

---

* *E.g.*, Goursat, *Cours d'analyse*, 3, § 613.

Hence, by known results, u is harmonic in S, continuous in $S+C$, and satisfies (1) on C, that is, u is the solution of problem (1).

Since u is uniquely determined by its boundary values $u(s)$ we have the following result:

The boundary value of any solution of problem (1) *satisfies the integro-differential equation*

$$u(s) = \frac{1}{2\pi}\int_C L[u(t)]\, G(s,t)\, dt + F(s), \tag{10}$$

where

$$F(s) = -\frac{1}{2\pi}\int_C f(t)\left\{G(s,t) + \frac{2\pi}{\kappa}\right\} dt.$$

Conversely, every solution of (10) *with* $u(s)$, $u'(s)$, $u''(s)$ *continuous on* C *is the boundary value of a solution of* (1).

3. Bounitsky* and others have reduced equations of type (10) to integral equations by repeated partial integrations. This method fails here, since $G(s,t)$ is logarithmically infinite for $t=s$, and instead we proceed in the following manner.

Assume that $p \neq 0$ on C and consider the differential system

$$\left.\begin{aligned} L[v(s)] &= 0 \\ v(0) &= v(l) \\ v'(0) &= v'(l) \end{aligned}\right\} \quad (0 \leqslant s \leqslant l) \tag{11}$$

Two cases arise:

(i) If the equations (11) are incompatible, then† the Green's function $K(s,t)$ for (11) exists and is symmetric. Writing now

$$L[u(s)] = -\phi(s) \tag{12}$$

and applying Green's formula to $u(s)$ and $K(s,t)$, we find in the usual way that

$$u(s) = \int_0^l K(s,t)\,\phi(t)\, dt. \tag{13}$$

Eliminating u from (10) by means of (12) and (13), we find that ϕ must satisfy

$$\int_C \left\{K(s,t) + \frac{1}{2\pi}G(s,t)\right\}\phi(t)\, dt = F(s), \tag{14}$$

---

* *Bull. des Sci. Math.*, 32 (1908), 14–31.

† *E.g.*, Courant and Hilbert, *Methoden der math. Phys.*, 1 (1924), Ch. V, § 10 (1), (3).

an integral equation of the first kind with symmetric nucleus. Conversely, let $\phi(s)$ be any solution of (14) continuous on C, and define $u(s)$ by (13). By differentiating (13) we deduce (12) and thence from (14) reproduce (10). Since also u satisfies the periodic boundary conditions, it is a solution of the original problem.

(ii) If the equations (11) are compatible, let $K(s, t)$ be the Green's function "in the wider sense"* adjusted so that

$$\int_0^l K(s, t)\psi(t)\,dt = 0, \tag{15}$$

where $\psi(s)$ is the solution of (11) continuous in $0 \leqslant s \leqslant l$, and normalized so that

$$\int_0^l \psi^2\,ds = 1. \tag{16}$$

Applying Green's formula to $u(s)$ and $\psi(s)$, we deduce that

$$\int_0^l \phi(s)\psi(s)\,ds = 0, \tag{17}$$

and, applying the same formula to $u(s)$ and $K(s, t)$, that

$$u(s) = \int_0^l K(s, t)\phi(t)\,dt + \psi(s)\int_0^l u(t)\psi(t)\,dt. \tag{18}$$

Substituting in (18) the value of u given by (10) in the form

$$u(s) = -\frac{1}{2\pi}\int_0^l G(s, t)\phi(t)\,dt + F(s), \tag{19}$$

we obtain the equation for ϕ:

$$\int_0^l \left\{K(s, t) + \frac{1}{2\pi}G(s, t) + \frac{1}{2\pi}\Gamma(s, t)\right\}\phi(t) = H(s), \tag{20}$$

where $$\Gamma(s, t) \equiv -\psi(s)\int_0^l G(\rho, t)\psi(\rho)\,d\rho,$$

and $$H(s) \equiv F(s) - \psi(s)\int_0^l F(t)\psi(t)\,dt.$$

The nucleus of (20) is not in general symmetric, but since, from (17), we have

$$\int_0^l \Gamma(t, s)\phi(t)\,dt = 0, \tag{21}$$

---

* Courant and Hilbert, *loc. cit.*

we may replace (20) by the equivalent equation

$$\frac{1}{2\pi}\int_0^l \{2\pi K(s,t)+G(s,t)+\Gamma(s,t)+\Gamma(t,s)\}\phi(t) = H(s), \tag{22}$$

whose nucleus is symmetric.

Conversely, let $\phi(s)$ be a solution of (22) continuous on C. We first show that in general the condition (17) is satisfied. To prove this, multiply (22) by $\psi(s)$ and integrate with respect to s from 0 to l. By (16) the right-hand side vanishes and the contributions from $G(s,t)$ and $\Gamma(s,t)$ cancel each other, while that from $K(s,t)$ vanishes by (15) and the symmetry of $K(s,t)$. We are left with

$$G\int_0^l \phi(t)\psi(t)\,dt = 0,$$

where

$$G \equiv \int_0^l\int_0^l G(\rho,s)\psi(\rho)\psi(s)\,d\rho\,ds.$$

Since $G \neq 0$ in general, the result follows.

Defining now u in terms of ϕ by (19), we get from equation (22)

$$u(s) = \int_0^l K(s,t)\phi(t)\,dt - \frac{\psi(s)}{2\pi}\int_0^l\int_0^l G(\rho,t)\psi(\rho)\phi(t)\,d\rho\,dt + \psi(s)\int_0^l F(t)\psi(t)\,dt.$$

Using the symmetry of $G(\rho,t)$ and interchanging ρ and t in the double integral, we get a result which by (19) reduces to (18). From (18) by differentiation we get (12), which with (19) gives (10) as required. Since also, from (18), u satisfies the periodic conditions it is a solution of the original problem.

The reduction to a symmetric integral equation of the first kind has, therefore, been established under the restriction (2) and, when the equations (11) are compatible, the further condition $G \neq 0$.

INDEX.

INDEX OF PAPERS PUBLISHED IN THIS VOLUME ARRANGED ACCORDING TO NAMES OF AUTHORS.

[*The numbers refer to pages.*]

Bailey, W. N. Transformations of well-poised hypergeometric series, 235.
Baker, W. E. Conics satisfying eight conditions, 96.
Dixon, A. C. On the solving nuclei of integral equations whose nuclei are homogeneous and of degree -1, 369.
Edge, W. L. Cayley's problem of the in-and-circumscribed triangle, 142.
Frazer, H. On the moduli of regular functions, 532.
Hall, P. A contribution to the theory of groups of prime-power order, 29.
Hardy, G. H., and Littlewood, J. E. Theorems concerning Cesàro means of power series, 516.
Hodge, W. V. D. A Dirichlet problem for harmonic functionals, with applications to analytic varieties, 257.
Milne, W. P. A special type of quintic symmetroid (II), 207.
———— Relation between the quadritangent hyperplanes of Noether's canonical curve for $p = 5$ and the 5-tangent conics of the plane quintic curve, 217.
Page, A. On the representations of a number as a sum of squares and products (*First paper*), 241.
———— On the representations of a number as a sum of squares and products (*Second paper*), 470.
Potter, H. S. A. Approximate equations for the Epstein zeta-function, 501.
Room, T. G. The freedoms of determinantal manifolds, 1.
Sutton, W. G. L. A boundary-value problem for harmonic functions, 547.
Szegö, G. Über einige asymptotische Entwicklungen der Legendreschen Funktionen, 427.
Titchmarsh, E. C. On Epstein's zeta-function, 485.
Todd, J. A. Conics in space, and their representation by points in space of nineteen dimensions, 172.
Whittaker, J. M. On Lidstone's series and two-point expansions of analytic functions, 451.
Wilton, J. R. An extended form of Dirichlet's divisor problem, 391.
Wright, E. Maitland. Asymptotic partition formulae: (II) Weighted partitions, 117.
Young, Alfred. On quantitative substitutional analysis (*Seventh paper*), 304.

INDEX ACCORDING TO SUBJECTS.

[*The numbers refer to pages.*]

Algebra : Young, A., 304.
Analytic Functions : Dixon, A. C., 369; Frazer, H., 532; Potter, H. S. A., 501; Szegö, G., 427; Titchmarsh, E. C., 485; Whittaker, J. M., 451.
Asymptotic Expansions : Szegö, G., 427; Wright, E. M., 117.
Dirichlet Series : Potter, H. S. A., 501; Titchmarsh, E. C., 485.
Functions of a Complex Variable : Dixon, A. C., 369; Frazer, H., 532; Potter, H. S. A., 501; Szegö, G., 427; Titchmarsh, E. C., 485; Whittaker, J. M., 451.
Functions of a Real Variable : Hardy, G. H., and Littlewood, J. E., 516.
Geometry : Baker, W. E., 96; Edge, W. L., 142; Hodge, W. V. D., 257; Milne, W. P., 207, 217; Room, T. G., 1; Todd, J. A., 172.
Groups, Theory of : Hall, P., 29.
Harmonic Functions : Hodge, W. V. D., 257; Sutton, W. G. L., 547.
Hypergeometric Series : Bailey, W. N., 235.
Integral Calculus : Hodge, W. V. D., 257; Sutton, W. G. L., 547.
Integral Equations : Dixon, A. C., 369; Hodge, W. V. D., 257; Sutton, W. G. L., 547.
Legendre Functions : Szegö, G., 427.
Mathematical Physics : Sutton, W. G. L., 547.
Numbers, Theory of : Page, A., 241, 470; Wilton, J. R., 391.
Partitions : Wright, E. M., 117.
Power Series : Hardy, G. H., and Littlewood, J. E., 516.
Series, Theory of : Bailey, W. N., 235; Hardy, G. H., and Littlewood, J. E., 516.
Substitutional Analysis : Young, A., 304.
Summability : Hardy, G. H., and Littlewood, J. E., 516.
Zeta-function, Theory of : Potter, H. S. A. 501; Titchmarsh, E. C., 485.

ADDENDA AND CORRIGENDA

SUR LES FORMULES D'INVERSION DE FOURIER ET DE HANKEL

M. PLANCHEREL.

[*Proc. London Math. Soc.* (2), 24 (1926), 62–70.]

P. 67, line 4. For "*intégrales*" read "*intégrables*".

P. 70. In each factor on the right-hand side of the last formula, for "ξ^{ν}" read "$\xi^{\nu+\frac{1}{2}}$", so that the formula reads

$$\int_0^\infty (\xi^{\nu+\frac{1}{2}} e^{-\alpha^2\xi^2})(\xi^{\nu+\frac{1}{2}} e^{-\beta^2\xi^2})\,d\xi = \frac{1}{(2\alpha^2.2\beta^2)^{\nu+1}} \int_0^\infty (\xi^{\nu+\frac{1}{2}} e^{-(\xi^2/4\alpha^2)})(\xi^{\nu+\frac{1}{2}} e^{-(\xi^2/4\beta^2)})\,d\xi.$$

THE COMPLETE SYSTEM OF THE BINARY (2, 1, 1) FORM

C. W. GILHAM.

[*Proc. London Math. Soc.* (2), 32 (1931), 259–272.]

Since the publication of this paper it has been found that four of the sixty-five concomitants, namely

$$(fJ_2, Q_4)^3, \quad (fJ_3, Q_4)^3, \quad (fJ_2, P_6)^4, \quad (fJ_3, P_6)^4,$$

are reducible. Details of the reductions will appear in a subsequent volume of the *Proceedings*.

BIBL.
LONDIN.
UNIV.